Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Weak-type regularity for the Bergman projection over N-dimensional classical Hartogs triangles

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Abstract
In this paper, we study the weak-type regularity of the Bergman projection on n-dimensional classical Hartogs triangles. We extend the results of Huo–Wick on the 2-dimensional classical Hartogs triangle to the n-dimensional classical Hartogs triangle and show that the Bergman projection is of weak type at the upper endpoint of L^q-boundedness but not of weak type at the lower endpoint of L^q-boundedness.

Keywords: Weak-type; Bergman projection; n-dimensional classical Hartogs triangle; Bergman kernel

1 Introduction

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain and dV the Lebesgue measure on Ω. Denote by $L^2(\Omega)$ the space of square-integrable functions and $A^2(\Omega)$ the subspace of the square-integrable holomorphic functions. It is easy to verify that $A^q(\Omega)$ is a closed subspace of $L^q(\Omega)$ for any $1 \le q < \infty$ by the mean value formula and the Hölder inequality. Considering the case $q = 2$, there exists an orthogonal projection $\mathbf{P}_\Omega$ from $L^2(\Omega)$ onto $A^2(\Omega)$ which can be represented as an integral operator

$$\mathbf{P}_\Omega(f)(z) = \int_\Omega f(w)K_\Omega(z,w)\,dV(w)$$

for any f in $L^2(\Omega)$, where $K_\Omega(z,w)$ satisfies $K_\Omega(w,z) = \overline{K_\Omega(z,w)}$, which is called the Bergman kernel function. Moreover, by Riesz representation theorem, the function $K_\Omega(z,w)$ is unique. The orthogonal projection $\mathbf{P}_\Omega$ from $L^2(\Omega)$ onto $A^2(\Omega)$ is called the *Bergman projection*. Let $\mathbf{P}^+_\Omega$ be defined by

$$\mathbf{P}^+_\Omega(f)(z) = \int_\Omega \left|K_\Omega(z,w)\right|f(w)\,dV(w),$$

which is called the absolute Bergman projection; see [11]. The theory of Bergman spaces can be dated back to [2] in the early 1950s, where the first systematic treatment of the subspace of the square-integrable holomorphic functions on Ω was given. Since then, a

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lot of papers in this area have appeared. An important problem in Bergman space theory is to study the mapping properties of $\mathbf{P}$, i.e., which functional spaces or classes are preserved by $\mathbf{P}$. The boundedness of $\mathbf{P}$ on $L^2(\Omega)$ can be easily deduced from the definition of $\mathbf{P}$. We naturally consider the question of the boundedness of $\mathbf{P}$ on $L^q(\Omega)$ for $1 < q < \infty$, which is not an easy problem to solve. As far as we know, the first to characterize the L^q-boundedness were Zaharjuta and Judovič (see [23]). By using of the estimates of the Bergman kernel, many authors have reached the conclusion that the Bergman projection is bounded on the L^q space for all $1 < q < \infty$ on a large class of smooth pseudoconvex domains of finite type, including all finite-type domains in $\mathbb{C}^2$, finite-type convex domains, strongly pseudoconvex domains, and finite-type domains with locally diagonalizable Levi form. See [4, 10, 12, 17–19] for more details. Nevertheless, it is worth noting that the Bergman projection is not L^q bounded for all $1 < q < \infty$ on the domains with serious singularities at boundaries in general; see [6]. But the Bergman projection is L^q-bounded on strongly pseudoconvex domains with C^2 boundary; see [15].

Let T be a linear operator on $L^q(\Omega)$. If there exists a constant $c > 0$ such that

$$\left|\left\{z \in \Omega : \left|Tf(z)\right| > \lambda\right\}\right| \leq c\frac{\|f\|^q_{L^q(\Omega)}}{\lambda^q}$$

for any $f \in L^q(\Omega)$ and any $\lambda > 0$, then we say that T is of weak-type (q,q). This paper focuses on the weak-type regularity of the Bergman projection for n-dimensional classical Hartogs triangles. Let $\mathbb{D}$ be the unit disk and define the n-dimensional classical Hartogs triangle $\mathbb{H}^n$ $(n \geq 2)$ as follows:

$$\mathbb{H}^n := \left\{(z_1,\dots,z_n) \in \mathbb{D}^n : |z_1| < \cdots < |z_n| < 1\right\}.$$

In general, there exist two ways to obtain the L^q-regularity of the Bergman projection. One is to choose a proper test function by Schur's lemma; see [28]. The other is to use the weak-type estimate of the Bergman projection to obtain the L^q-boundedness. Both techniques are very effective in getting the L^q-regularity. Unfortunately, we cannot get the weak-type regularity at the endpoints of L^q-boundedness from the Schur's test. Thus this paper mainly adopts the second method.

The L^q-boundedness of the Bergman projection on Hartogs triangles has been studied for many years by different authors. It follows from the work of Deng–Huang–Zhao–Zheng [8] that the Bergman projection acting on $L^1(\mathbb{D})$ is of weak-type $(1,1)$. However, for the two-dimensional case, Huo–Wick [11] proved that the Bergman projection is not of weak-type $(1,1)$. From this, we can see that dimensionality may have an effect on the weak-type regularity of the Bergman projection. Besides, according to Chakrabarti–Zeytuncu [3], the Bergman projection is L^q-bounded if and only if $q \in (\frac{4}{3}, 4)$ over the classical Hartogs triangle $\mathbb{H} \subset \mathbb{D}^2$ which is given by

$$\mathbb{H} := \left\{(z_1,z_2) \in \mathbb{D}^2 : |z_1| < |z_2| < 1\right\}.$$

Later, this result is also covered by the work of Edholm–McNeal [9]. Huo–Wick [11] and Christopherson–Koenig [7] have characterized the weak-type regularity of the Bergman projection of the classical Hartogs triangle $\mathbb{H}$ and the rational power-generalized 2-dimensional Hartogs triangles $\mathbb{H}_{\frac{m}{n}}$ $(\mathbb{H}_{\frac{m}{n}} := \{(z_1,z_2) \in \mathbb{D}^2 : |z_1|^m < |z_2|^n < 1\})$, respectively.

For related work on 2-dimensional classical Hartogs triangle $\mathbb{H}$, refer to [20, 21]. A similar result for the harmonic Bergman projection on the punctured unit ball $\mathbb{B} \setminus \{0\}$ in $\mathbb{R}^3$ was proved by Koenig–Wang [13]. It has been proved by Chen [5] that the Bergman projection is bounded on L^q if and only if $q \in (\frac{2n}{n+1}, \frac{2n}{n-1})$ over the n-dimensional ($n \geq 2$) classical Hartogs triangle $\mathbb{H}^n$, where

$$\mathbb{H}^n := \{(z_1, z_2, \dots, z_n) \in \mathbb{D}^n : |z_1| < |z_2| < \cdots < |z_n| < 1\}.$$

This result is also generalized to the n-dimensional ($n \geq 2$) generalized Hartogs triangles by Bender–Chakrabarti–Edholm–Mainkar [1] and Zhang [24]. See also [16, 22, 25–27] for related work on generalized Hartogs triangles. Inspired by their work, we would like to study the weak-type regularity of the Bergman projection over the n-dimensional ($n \geq 2$) classical Hartogs triangle $\mathbb{H}^n$ at the endpoints.

The following two theorems are the main results in this paper, which will be proved in Sects. 2 and 3, respectively.

Theorem 1.1 *The Bergman projection on the n-dimensional* ($n \geq 2$) *classical Hartogs triangle* $\mathbb{H}^n$ *is not of weak-type* $(\frac{2n}{n+1}, \frac{2n}{n+1})$.

Theorem 1.2 *The Bergman projection on the n-dimensional* ($n \geq 2$) *classical Hartogs triangle* $\mathbb{H}^n$ *is of weak-type* $(\frac{2n}{n-1}, \frac{2n}{n-1})$.

We generalize the result of the 2-dimensional case which is developed by Huo–Wick [11, Theorems 4.1 and 4.2]. Our proof of Theorem 1.1 mainly relies on the Bergman projection of the multiparameter function $\overline{z_2}^{a_2}|z_2|^{-b_2p'} \cdots \overline{z_n}^{a_n}|z_n|^{-b_np'}$ for proper p', a_i, b_i, where $i = 2, \dots, n$. And we will prove Theorem 1.2 by showing that $\mathbb{H}^n$ is biholomorphically equivalent to $\mathbb{D} \times (\mathbb{D}^*)^{n-1}$ ($\mathbb{D}^* := \mathbb{D} \setminus \{0\}$) and $\mathbf{P}^+_{\mathbb{D}^n}$ is L^q-bounded for $1 < q < \infty$.

The paper essentially follows the order established in this Introduction.

Throughout this paper, we will use the notation $A \lesssim B$, which is an inequality up to a constant: $A \leq cB$ for some constant c. The relevant constants in all such inequalities do not depend on any relevant variable. If $A \lesssim B$ and $B \lesssim A$ hold simultaneously, then we say $A \approx B$. We denote the Lebesgue measure of a Borel set by the notation $|\cdot|$.

2 Failure of weak-type estimate of the Bergman projection at lower endpoint

In this section, we will prove that the Bergman projection $\mathbf{P}$ on $\mathbb{H}^n$ is not of weak-type $(\frac{2n}{n+1}, \frac{2n}{n+1})$. To get started, we set $q := \frac{2n}{n+1}$ and abbreviate $\mathbf{P}_{\mathbb{H}^n}$ to $\mathbf{P}$. We just need to construct a function $f_\lambda \in L^q(\mathbb{H}^n)$ such that

$$\left|\left\{(z_1, z_2, \dots, z_n) \in \mathbb{H}^n : \left|\mathbf{P}(f_\lambda)(z_1, z_2, \dots, z_n)\right| > \lambda\right\}\right| \geq c_\lambda \frac{\|f_\lambda\|^q_{L^q(\mathbb{H}^n)}}{\lambda^q},$$

where c_λ is a constant related to λ and satisfies $c_\lambda \to \infty$ as $\lambda \to \infty$.

The following lemma gives an orthogonal basis of $A^2(\mathbb{H}^n)$, which plays a major role in this section.

Lemma 2.1 ([24, Lemma 4.1]) *For* $n \geq 2$, *we define*

$$\chi := \left\{\tau = (\tau_1, \dots, \tau_n) \in \mathbb{Z}^n : \tau_1 \geq 0, \sum_{i=1}^{j} \tau_i + j \geq 1, j = 2, \dots, n\right\}.$$

Then $\{z^\tau : \tau \in \chi\}$ *is an orthogonal basis on* $A^2(\mathbb{H}^n)$, *where* $\tau = (\tau_1, \dots, \tau_n)$ *are multiindices and* $z^\tau := z_1^{\tau_1} \cdots z_n^{\tau_n}$.

Now, let us start with the proof of the theorem of this section.

The proof of Theorem 1.1 For $\lambda > 0$, we define

$$f_\lambda(z_1, \dots, z_n) = \overline{z_2}^{a_2} |z_2|^{-b_2 p'} \overline{z_3}^{a_3} |z_3|^{-b_3 p'} \cdots \overline{z_n}^{a_n} |z_n|^{-b_n p'},$$

where $p' = \frac{p}{p-1}$ denotes the conjugate index of p and $p > 1$ is a constant associated to λ, $a_i \in \mathbb{N} \cup \{0\}$ and $b_i \in \mathbb{R}$ for $i = 2, \dots, n$.

Let us calculate $|\mathbf{P}(f_\lambda)(z_1, z_2, \dots, z_n)|$ as follows:

$$\begin{aligned}
&\mathbf{P}(f_\lambda)(z_1, z_2, \dots, z_n) \\
&\quad = \int_{\mathbb{H}^n} \sum_{\substack{\tau_1 \geq 0 \\ \tau_1+\tau_2 \geq -1 \\ \vdots \\ \tau_1+\tau_2+\cdots+\tau_n \geq 1-n}} \frac{\overline{w_2}^{a_2} |w_2|^{-b_2 p'} \cdots \overline{w_n}^{a_n} |w_n|^{-b_n p'} z_1{}^{\tau_1} \overline{w_1}^{\tau_1} z_2{}^{\tau_2} \overline{w_2}^{\tau_2} \cdots z_n{}^{\tau_n} \overline{w_n}^{\tau_n}}{\| w_1{}^{\tau_1} w_2{}^{\tau_2} \cdots w_n{}^{\tau_n} \|^2_{L^2(\mathbb{H}^n)}} \\
&\qquad dV(w_1, w_2, \dots, w_n).
\end{aligned} \tag{2.1}$$

By using polar coordinates, one can easily get that

$$\int_{\mathbb{H}^n} \overline{w_2}^{a_2} |w_2|^{-b_2 p'} \cdots \overline{w_n}^{a_n} |w_n|^{-b_n p'} \overline{w_1}^{\tau_1} \overline{w_2}^{\tau_2} \cdots \overline{w_n}^{\tau_n} \, dV(w_1, w_2, \dots, w_n) \neq 0$$

if and only if

$$\tau_1 = 0 \quad \text{and} \quad \tau_k = -a_k$$

for $k = 2, 3, \dots, n$.

It follows from Lemma 2.1 that

$$-a_2 - \cdots - a_k \geq 1 - k \tag{2.2}$$

for $k = 2, \dots, n$.

We can take $a_2 = a_3 = \cdots = a_n = 1$ satisfying (2.2). Hence, one may compute $\|f_\lambda\|^q_{L^q(\mathbb{H}^n)}$ as follows:

$$\begin{aligned}
&\|f_\lambda\|^q_{L^q(\mathbb{H}^n)} \\
&\quad = \int_{\mathbb{H}^n} \left| \overline{z_2}^{a_2} |z_2|^{-b_2 p'} \overline{z_3}^{a_3} |z_3|^{-b_3 p'} \cdots \overline{z_n}^{a_n} |z_n|^{-b_n p'} \right|^{\frac{2n}{n+1}} dV(z_1, z_2, \dots, z_n) \\
&\quad = \int_{|z_n|<1} dV(z_n) \int_{|z_{n-1}|<|z_n|} dV(z_{n-1}) \cdots \int_{|z_1|<|z_2|} \left| \overline{z_2} |z_2|^{-b_2 p'} \overline{z_3} |z_3|^{-b_3 p'} \right. \\
&\qquad \left. \cdots \overline{z_n} |z_n|^{-b_n p'} \right|^{\frac{2n}{n+1}} dV(z_1)
\end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&= \int_{|z_n|<1} dV(z_n) \int_{|z_{n-1}|<|z_n|} \frac{2^{n-3}\pi^{n-2}|z_{n-1}|^{(n-2-B_{n-1}p')\frac{2n}{n+1}+2(n-2)}|z_n|^{(1-b_np')\frac{2n}{n+1}}}{\prod_{k=2}^{n-2}((k-1-B_kp')\frac{2n}{n+1}+2k)} dV(z_{n-1}) \\
&= \int_{|z_n|<1} \frac{2^{n-2}\pi^{n-1}|z_n|^{(n-1-B_np')\frac{2n}{n+1}+2(n-1)}}{\prod_{k=2}^{n-1}((k-1-B_kp')\frac{2n}{n+1}+2k)} dV(z_n) \\
&= \frac{2^{n-1}\pi^n}{\prod_{k=2}^{n}((k-1-B_kp')\frac{2n}{n+1}+2k)},
\end{aligned}
$$

provided that

$$
\bigl(k-1-B_kp'\bigr)\frac{2n}{n+1}+2k>0 \quad \text{for } k=2,\dots,n, \tag{2.3}
$$

where $B_k = \sum_{i=2}^k b_i$ for $k=2,\dots,n$.

We can also simplify $|\mathbf{P}(f_\lambda)(z_1,z_2,\dots,z_n)|$ in (2.1) even further as follows:

$$
\begin{aligned}
&\bigl|\mathbf{P}(f_\lambda)(z_1,z_2,\dots,z_n)\bigr| \\
&\quad = \left|\int_{\mathbb{H}^n} \frac{|w_2|^{-b_2p'}|w_3|^{-b_3p'}\cdots|w_n|^{-b_np'}z_2{}^{-1}\cdots z_n{}^{-1}}{\|w_2{}^{-1}\cdots w_n{}^{-1}\|^2_{L^2(\mathbb{H}^n)}} dV(w_1,w_2,\dots,w_n)\right|.
\end{aligned} \tag{2.4}
$$

Now let us estimate

$$
\bigl\|w_2{}^{-1}\cdots w_n{}^{-1}\bigr\|^2_{L^2(\mathbb{H}^n)}
$$

and

$$
\int_{\mathbb{H}^n} |w_2|^{-b_2p'}|w_3|^{-b_3p'}\cdots|w_n|^{-b_np'} dV(w_1,w_2,\dots,w_n)
$$

separately.

A simple calculation gives

$$
\begin{aligned}
&\int_{\mathbb{H}^n} \bigl|w_2{}^{-1}\cdots w_n{}^{-1}\bigr|^2 dV(w_1,w_2,\dots,w_n) \\
&\quad = \int_{|w_n|<1} dV(w_n)\int_{|w_{n-1}|<|w_n|} dV(w_{n-1})\cdots\int_{|w_1|<|w_2|} \bigl|w_2{}^{-1}\cdots w_n{}^{-1}\bigr|^2 dV(w_1) \\
&\quad = \pi^n.
\end{aligned} \tag{2.5}
$$

Likewise, one could just as easily get

$$\begin{aligned}
&\int_{\mathbb{H}^n} |w_2|^{-b_2p'}|w_3|^{-b_3p'}\cdots|w_n|^{-b_np'}\,dV(w_1,w_2,\dots,w_n)\\
&=\int_{|w_n|<1} dV(w_n)\int_{|w_{n-1}|<|w_n|} dV(w_{n-1})\cdots\int_{|w_1|<|w_2|}|w_2|^{-b_2p'}|w_3|^{-b_3p'}\\
&\quad\cdots|w_n|^{-b_np'}\,dV(w_1)\\
&\ \vdots\\
&=\int_{|w_n|<1} dV(w_n)\int_{|w_{n-1}|<|w_n|}\frac{2^{n-3}\pi^{n-2}|w_{n-1}|^{2(n-2)-B_{n-1}p'}|w_n|^{-b_np'}}{\prod_{k=2}^{n-2}(2k-B_kp')}\,dV(w_{n-1})\\
&=\int_{|w_n|<1}\frac{2^{n-2}\pi^{n-1}|w_n|^{2(n-1)-B_np'}}{\prod_{k=2}^{n-1}(2k-B_kp')}\,dV(w_n)\\
&=\frac{2^{n-1}\pi^n}{\prod_{k=2}^{n}(2k-B_kp')},
\end{aligned}\tag{2.6}$$

provided that

$$2k-B_kp'>0\quad\text{for } k=2,\dots,n,\tag{2.7}$$

where $B_k=\sum_{i=2}^k b_i$ for $k=2,\dots,n$.

Combining (2.4)–(2.6), one obtains

$$\left|\mathbf{P}(f_\lambda)(z_1,z_2,\dots,z_n)\right|=\frac{2^{n-1}}{|z_2||z_3|\cdots|z_n|\prod_{k=2}^n(2k-B_kp')}.\tag{2.8}$$

Then it follows from (2.8) that

$$\begin{aligned}
&\left|\left\{(z_1,z_2,\dots,z_n)\in\mathbb{H}^n:\left|\mathbf{P}(f_\lambda)(z_1,z_2,\dots,z_n)\right|>\lambda\right\}\right|\\
&=\int_{\{(z_1,z_2,\dots,z_n)\in\mathbb{H}^n:\frac{2^{n-1}}{|z_2||z_3|\cdots|z_n|\prod_{k=2}^n(2k-B_kp')}>\lambda\}} dV(z_1,z_2,\dots,z_n)\\
&\geq\int_{\{(z_1,z_2,\dots,z_n)\in\mathbb{H}^n:\frac{2^{n-1}}{\lambda\prod_{k=2}^n(2k-B_kp')}>|z_n|^{n-1}\}} dV(z_1,z_2,\dots,z_n).
\end{aligned}\tag{2.9}$$

Here, the appropriate parameters B_k $(2\leq k\leq n)$ and p' will be chosen to ensure that

$$\frac{2}{(\lambda\prod_{k=2}^n(2k-B_kp'))^{\frac{1}{n-1}}}<1\tag{2.10}$$

holds. Then

$$\begin{aligned}
(2.9)&=\int_{|z_n|<\frac{2}{(\lambda\prod_{k=2}^n(2k-B_kp'))^{\frac{1}{n-1}}}} dV(z_n)\int_{|z_{n-1}|<|z_n|} dV(z_{n-1})\cdots\int_{|z_1|<|z_2|} dV(z_1)\\
&=\frac{2^{2n}\pi^n}{n!(\lambda\prod_{k=2}^n(2k-B_kp'))^{\frac{2n}{n-1}}}
\end{aligned}$$

$$\begin{aligned} &\approx \frac{1}{\prod_{k=2}^{n}((k-1-B_kp')\frac{2n}{n+1}+2k)} \frac{\prod_{k=2}^{n}[(k-1-B_kp')\frac{2n}{n+1}+2k]}{(\lambda\prod_{k=2}^{n}(2k-B_kp'))^{\frac{2n}{n-1}}} \\ &\approx \frac{\|f_\lambda\|_{L^q(\mathbb{H}^n)}^q}{\lambda^q} \frac{\prod_{k=2}^{n}((k-1-B_kp')\frac{2n}{n+1}+2k)}{\lambda^{\frac{4n}{(n-1)(n+1)}}(\prod_{k=2}^{n}(2k-B_kp'))^{\frac{2n}{n-1}}}. \end{aligned} \tag{2.11}$$

Substituting $p' = \frac{p}{p-1}$ into $\frac{\prod_{k=2}^{n}((k-1-B_kp')\frac{2n}{n+1}+2k)}{(\prod_{k=2}^{n}(2k-B_kp'))^{\frac{2n}{n-1}}}$ in (2.11), we get

$$\begin{aligned} &\frac{\prod_{k=2}^{n}((k-1-B_kp')\frac{2n}{n+1}+2k)}{(\prod_{k=2}^{n}(2k-B_kp'))^{\frac{2n}{n-1}}} \\ &\quad= \frac{(p-1)^{2n}\prod_{k=2}^{n}((k-1-B_k\frac{p}{p-1})\frac{2n}{n+1}+2k)}{(\prod_{k=2}^{n}(2k(p-1)-B_kp))^{\frac{2n}{n-1}}} \\ &\quad= \frac{(p-1)^{n+1}\prod_{k=2}^{n}(((k-1-B_k)\frac{2n}{n+1}+2k)p-\frac{2n}{n+1}(k-1)-2k)}{(\prod_{k=2}^{n}((2k-B_k)p-2k))^{\frac{2n}{n-1}}}. \end{aligned} \tag{2.12}$$

Here one needs to choose appropriate B_k for $k = 2, \dots, n$ to make sure that the following is true:

$$(k-1-B_k)\frac{2n}{n+1}+2k > 0, \tag{2.13}$$

$$2k - B_k > 0. \tag{2.14}$$

From (2.13) and (2.14), it is easy to see that

$$\begin{aligned} (2.12) &\approx \frac{(p-1)^{n+1}\prod_{k=2}^{n}(p-\frac{\frac{2n}{n+1}(k-1)+2k}{(k-1-B_k)\frac{2n}{n+1}+2k})}{(\prod_{k=2}^{n}(p-\frac{2k}{2k-B_k}))^{\frac{2n}{n-1}}} \\ &= \frac{(p-1)^{n+1}\prod_{k=2}^{n}(p-\frac{2nk+k-n}{-nB_k+2nk+k-n})}{(\prod_{k=2}^{n}(p-\frac{2k}{2k-B_k}))^{\frac{2n}{n-1}}}. \end{aligned} \tag{2.15}$$

We can take $B_k = 2k-1$ for $k = 2, 3, \dots, n$ and $p = 2n + \lambda^{-\delta}$ with $\delta \in (0,1)$ to be chosen shortly. Substituting $B_k = 2k-1$ and $p = 2n + \lambda^{-\delta}$ into the left-hand sides of (2.3), (2.7), (2.10), (2.13), and (2.14), we obtain

$$\begin{aligned} \text{LHS of (2.3)} &= \left(k-1-B_kp'\right)\frac{2n}{n+1}+2k \\ &= \frac{(k-1)(p-1)-(2k-1)p}{p-1}\frac{2n}{n+1}+2k \\ &= 2\frac{n(k-1)(p-1)-(2k-1)pn+k(n+1)(p-1)}{(p-1)(n+1)} \\ &= 2\frac{n-k+k\lambda^{-\delta}}{(p-1)(n+1)} > 0, \end{aligned}$$

$$\text{LHS of (2.7)} = 2k - B_kp' = \frac{2(n-k)+\lambda^{-\delta}}{p-1} > 0,$$

$$\text{LHS of (2.10)} = \frac{2}{(\lambda\prod_{k=2}^{n}(2k-B_kp'))^{\frac{1}{n-1}}}$$

$$= \frac{2}{\left(\frac{\lambda^{1-\delta}}{2n-1+\lambda^{-\delta}}\prod_{k=2}^{n-1}\left(\frac{2(n-k)+\lambda^{-\delta}}{p-1}\right)\right)^{\frac{1}{n-1}}} < 1 \quad \text{as } \lambda \to \infty,$$

$$\text{LHS of (2.13)} = (k-1-B_k)\frac{2n}{n+1} + 2k = \frac{2k}{n+1} > 0,$$

$$\text{LHS of (2.14)} = 2k - B_k = 1 > 0.$$

So (2.3), (2.7), (2.10), (2.13), and (2.14) are satisfied.

Substituting $B_k = 2k-1$, $p = 2n+\lambda^{-\delta}$ into (2.15) and combining (2.9), (2.11), (2.12), and (2.15), one has

$$\begin{aligned} &\left|\left\{(z_1,z_2,\dots,z_n)\in\mathbb{H}^n : \left|\mathbf{P}(f_\lambda)(z_1,z_2,\dots,z_n)\right| > \lambda\right\}\right| \\ &\quad\gtrsim \frac{\|f_\lambda\|^q_{L^q(\mathbb{H}^n)}}{\lambda^q}\frac{(2n-1+\lambda^{-\delta})^{n+1}\lambda^{-\delta}\prod_{k=2}^{n-1}(-1+\frac{n}{k}+\lambda^{-\delta})}{\lambda^{\frac{4n}{(n-1)(n+1)}}\lambda^{\frac{-2n\delta}{n-1}}\left(\prod_{k=2}^{n-1}(2n-2k+\lambda^{-\delta})\right)^{\frac{2n}{n-1}}}. \end{aligned} \tag{2.16}$$

When λ tends to ∞, we can estimate (2.16) as follows:

$$(2.16) \approx \frac{\|f_\lambda\|^q_{L^q(\mathbb{H}^n)}}{\lambda^q}\lambda^{-\delta+\frac{2n}{n-1}\delta-\frac{4n}{(n-1)(n+1)}}.$$

If we choose

$$-\delta + \frac{2n}{n-1}\delta - \frac{4n}{(n-1)(n+1)} > 0,$$

i.e.,

$$\delta > \frac{4n}{(n+1)^2},$$

then $\lambda^{-\delta+\frac{2n}{n-1}\delta-\frac{4n}{(n-1)(n+1)}} \to \infty$ as $\lambda \to \infty$.

Note that $4n < (n+1)^2$ since $n \geq 2$. So one can choose $\delta \in (\frac{4n}{(n+1)^2}, 1)$ such that

$$\left|\left\{(z_1,z_2,\dots,z_n)\in\mathbb{H}^n : \left|\mathbf{P}(f_\lambda)(z_1,z_2,\dots,z_n)\right| > \lambda\right\}\right| \gtrsim \frac{\|f_\lambda\|^q_{L^q(\mathbb{H}^n)}}{\lambda^q}\lambda^{-\delta+\frac{2n}{n-1}\delta-\frac{4n}{(n-1)(n+1)}}$$

and $\lambda^{-\delta+\frac{2n}{n-1}\delta-\frac{4n}{(n-1)(n+1)}} \to \infty$ as $\lambda \to \infty$.

We complete the proof. □

3 Proof of weak-type estimate of the Bergman projection at upper endpoint

In this section, set $q := \frac{2n}{n-1}$ and abbreviate $\mathbf{P}_{\mathbb{H}^n}$ to $\mathbf{P}$. We will show that the Bergman projection is of weak-type $(\frac{2n}{n-1}, \frac{2n}{n-1})$. Let us begin with some preliminaries. The Bergman kernel on $\mathbb{D}^n$ is given by

$$K_{\mathbb{D}^n}(z,w) = \frac{1}{\pi^n\prod_{k=1}^n(1-z_k\overline{w_k})^2},$$

where

$$z = (z_1,z_2,\dots,z_n)\in\mathbb{D}^n \quad \text{and} \quad w = (w_1,w_2,\dots,w_n)\in\mathbb{D}^n.$$

It is easy to see that the mapping $(z_1,z_2,\dots,z_n)\mapsto(\frac{z_1}{z_2},\frac{z_2}{z_3},\dots,\frac{z_{n-1}}{z_n},z_n)$ is a biholomorphism from $\mathbb{H}^n$ onto $\mathbb{D}\times(\mathbb{D}^*)^{n-1}$. From the biholomorphic transformation formula in [14], we get

$$\begin{aligned}
K_{\mathbb{H}^n}(z,w) &= \det\left(\frac{\partial(\frac{z_1}{z_2},\dots,\frac{z_{n-1}}{z_n},z_n)}{\partial(z_1,\dots,z_n)}\right)K_{\mathbb{D}\times(\mathbb{D}^*)^{n-1}}\left(\frac{z_1}{z_2},\dots,\frac{z_{n-1}}{z_n},z_n;\frac{w_1}{w_2},\dots,\frac{w_{n-1}}{w_n},w_n\right)\\
&\quad\times\overline{\det\left(\frac{\partial(\frac{w_1}{w_2},\dots,\frac{w_{n-1}}{w_n},w_n)}{\partial(w_1,\dots,w_n)}\right)}\\
&=\frac{1}{\prod_{k=2}^n z_k}K_{\mathbb{D}^n}\left(\frac{z_1}{z_2},\dots,\frac{z_{n-1}}{z_n},z_n;\frac{w_1}{w_2},\dots,\frac{w_{n-1}}{w_n},w_n\right)\frac{1}{\prod_{k=2}^n\overline{w_k}}\\
&=\frac{1}{\pi^n(\prod_{k=2}^n z_k)(\prod_{k=2}^n\overline{w_k})(1-z_n\overline{w_n})^2\prod_{k=1}^{n-1}(1-\frac{z_k\overline{w_k}}{z_{k+1}\overline{w_{k+1}}})^2},
\end{aligned}\tag{3.1}$$

where

$$z=(z_1,z_2,\dots,z_n)\in\mathbb{H}^n\quad\text{and}\quad w=(w_1,w_2,\dots,w_n)\in\mathbb{H}^n.$$

The following lemma is a crucial technique of proving Theorem 1.2 and stated as follows.

Lemma 3.1 *The absolute Bergman projection $\mathbf{P}^+_{\mathbb{D}^n}$ is L^q-bounded for $1<q<\infty$.*

Proof Let $f\in L^q(\mathbb{D}^n)$ $(1<q<\infty)$, $z=(z_1,z_2,\dots,z_n)\in\mathbb{D}^n$ and $w=(w_1,w_2,\dots,w_n)\in\mathbb{D}^n$. A simple calculation gives

$$\begin{aligned}
&\mathbf{P}^+_{\mathbb{D}^n}(f)(z)\\
&\quad=\int_{\mathbb{D}^n}\left|K_{\mathbb{D}^n}(z,w)\right|f(w)\,dV(w)\\
&\quad=\int_{\mathbb{D}^n}\frac{f(w_1,w_2,\dots,w_n)}{\pi^n\prod_{i=1}^n|1-z_i\overline{w_i}|^2}\,dV(w_1,w_2,\dots,w_n)\\
&\quad=\int_{\mathbb{D}}\frac{1}{\pi|1-z_n\overline{w_n}|^2}\,dV(w_n)\int_{\mathbb{D}^{n-1}}\frac{f(w_1,w_2,\dots,w_n)}{\pi^{n-1}\prod_{i=1}^{n-1}|1-z_i\overline{w_i}|^2}\,dV(w_1,\dots,w_{n-1}).
\end{aligned}\tag{3.2}$$

Now let us complete the proof in several steps.

Step 1. Set

$$\begin{aligned}
\widetilde{g_n}(w_n)&:=g_n(z_1,\dots,z_{n-1},w_n)\\
&=\int_{\mathbb{D}^{n-1}}\frac{f(w_1,w_2,\dots,w_n)}{\pi^{n-1}\prod_{i=1}^{n-1}|1-z_i\overline{w_i}|^2}\,dV(w_1,\dots,w_{n-1}).
\end{aligned}\tag{3.3}$$

Substituting (3.3) into (3.2), one obtains

$$(3.2)=\int_{\mathbb{D}}\frac{\widetilde{g_n}(w_n)}{\pi|1-z_n\overline{w_n}|^2}\,dV(w_n).$$

From [11, Lemma 2.2], $\mathbf{P}^+_{\mathbb{D}}$ is L^q-bounded for $1<q<\infty$.

Then

$$\begin{aligned}
&\int_{\mathbb{D}^n} \left|\mathbf{P}^+_{\mathbb{D}^n}(f)(z)\right|^q dV(z_1,\dots,z_n) \\
&= \int_{\mathbb{D}^{n-1}} \int_{\mathbb{D}} \left| \int_{\mathbb{D}} \frac{\widetilde{g_n}(w_n)}{\pi |1 - z_n \overline{w_n}|^2} \, dV(w_n) \right|^q dV(z_n)\, dV(z_1,\dots,z_{n-1}) \\
&\lesssim \int_{\mathbb{D}^{n-1}} \int_{\mathbb{D}} \left|\widetilde{g_n}(z_n)\right|^q dV(z_n)\, dV(z_1,\dots,z_{n-1}). \qquad (3.4)
\end{aligned}$$

Step 2.

Set

$$\begin{aligned}
\widetilde{g_{n-1}}(w_{n-1}) &:= g_{n-1}(z_1,\dots,z_{n-2},w_{n-1},z_n) \\
&= \int_{\mathbb{D}^{n-2}} \frac{f(w_1,w_2,\dots,w_{n-1},z_n)}{\pi^{n-2}\prod_{i=1}^{n-2}|1 - z_i \overline{w_i}|^2} \, dV(w_1,\dots,w_{n-2}). \qquad (3.5)
\end{aligned}$$

Substituting (3.5) into (3.3), one has

$$\widetilde{g_n}(z_n) = \int_{\mathbb{D}} \frac{\widetilde{g_{n-1}}(w_{n-1})}{\pi |1 - z_{n-1}\overline{w_{n-1}}|^2} \, dV(w_{n-1}).$$

Since $\mathbf{P}^+_{\mathbb{D}}$ is L^q-bounded for $1 < q < \infty$ by [11, Lemma 2.2], one gets

$$\begin{aligned}
(3.4) &= \int_{\mathbb{D}^{n-1}} \int_{\mathbb{D}} \left| \int_{\mathbb{D}} \frac{\widetilde{g_{n-1}}(w_{n-1})}{\pi |1 - z_{n-1}\overline{w_{n-1}}|^2} \, dV(w_{n-1}) \right|^q dV(z_{n-1})\, dV(z_1,\dots,z_{n-2},z_n) \\
&\lesssim \int_{\mathbb{D}^{n-1}} \int_{\mathbb{D}} \left|\widetilde{g_{n-1}}(z_{n-1})\right|^q dV(z_{n-1})\, dV(z_1,\dots,z_{n-2},z_n).
\end{aligned}$$

Repeat the above process until *Step* $(n-1)$.

Set

$$\begin{aligned}
\widetilde{g_2}(w_2) &:= g_2(z_1,w_2,z_3,\dots,z_n) \\
&= \int_{\mathbb{D}} \frac{f(w_1,w_2,z_3,\dots,z_n)}{\pi |1 - z_1 \overline{w_1}|^2} \, dV(w_1).
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&\int_{\mathbb{D}^n} \left|\mathbf{P}^+_{\mathbb{D}^n}(f)(z)\right|^q dV(z_1,\dots,z_n) \\
&\lesssim \int_{\mathbb{D}^{n-1}} \int_{\mathbb{D}} \left|\widetilde{g_2}(z_2)\right|^q dV(z_2)\, dV(z_1,z_3,\dots,z_n) \\
&= \int_{\mathbb{D}^{n-1}} \int_{\mathbb{D}} \left| \int_{\mathbb{D}} \frac{f(w_1,z_2,z_3,\dots,z_n)}{\pi |1 - z_1 \overline{w_1}|^2} \, dV(w_1) \right|^q dV(z_1)\, dV(z_2,\dots,z_n) \\
&\lesssim \int_{\mathbb{D}^n} \left|f(z_1,z_2,z_3,\dots,z_n)\right|^q dV(z_1,\dots,z_n).
\end{aligned}$$

We complete the proof. □

Now let us prove the main theorem of this section.

The proof of Theorem 1.2 Let $f \in L^q(\mathbb{H}^n)$. Then

$$\begin{aligned}\|f\|^q_{L^q(\mathbb{H}^n)} &= \int_{\mathbb{H}^n} \left|f(z_1,z_2,\dots,z_n)\right|^{\frac{2n}{n-1}} dV(z_1,z_2,\dots,z_n) \\ &= \int_{\mathbb{D}^n} \left|f\left(\prod_{k=1}^n z_k, \prod_{k=2}^n z_k,\dots,z_n\right)\right|^{\frac{2n}{n-1}} \left|\prod_{k=2}^n {z_k}^{k-1}\right|^2 dV(z_1,z_2,\dots,z_n) \\ &= \int_{\mathbb{D}^n} \left|f\left(\prod_{k=1}^n z_k, \prod_{k=2}^n z_k,\dots,z_n\right)\prod_{k=2}^n {z_k}^{k-1}\right|^{\frac{2n}{n-1}} \prod_{k=2}^n |z_k|^{\frac{-2k+2}{n-1}} dV(z_1,z_2,\dots,z_n).\end{aligned}$$

Define

$$g(z_1,z_2,\dots,z_n) = f\left(\prod_{k=1}^n z_k, \prod_{k=2}^n z_k,\dots,z_n\right)\prod_{k=2}^n {z_k}^{k-1}.$$

One can easily obtain

$$g \in L^q\left(\mathbb{D}^n, \prod_{k=2}^n |z_k|^{\frac{-2k+2}{n-1}} dV\right)$$

and

$$\|g\|_{L^q(\mathbb{D}^n, \prod_{k=2}^n |z_k|^{\frac{-2k+2}{n-1}} dV)} = \|f\|_{L^q(\mathbb{H}^n)}.$$

By using (3.1) and variable substitutions, one gets

$$\begin{aligned}&\left|\mathbf{P}f(z_1,\dots,z_n)\right| \\ &\quad= \left|\int_{\mathbb{H}^n} f(w_1,\dots,w_n) K_{\mathbb{H}^n}(z_1,\dots,z_n;w_1,\dots,w_n)\, dV(w_1,\dots,w_n)\right| \\ &\quad= \left|\int_{\mathbb{D}^n} f\left(\prod_{k=1}^n w_k,\dots,\prod_{k=n-1}^n w_k, w_n\right) K_{\mathbb{H}^n}\left(z_1,\dots,z_n;\prod_{k=1}^n w_k,\dots,\prod_{k=n-1}^n w_k, w_n\right)\right. \\ &\qquad \left.\times \prod_{k=2}^n |w_k|^{2k-2}\, dV(w_1,\dots,w_n)\right| \\ &\quad= \left|\int_{\mathbb{D}^n} \frac{f(\prod_{k=1}^n w_k,\dots,\prod_{k=n-1}^n w_k, w_n)\prod_{k=2}^n |w_k|^{2k-2}}{\pi^n(\prod_{k=2}^n z_k)(\prod_{k=2}^n\prod_{i=k}^n \overline{w_i})(1-z_n\overline{w_n})^2\prod_{k=1}^{n-1}(1-\frac{z_k\overline{w_k}}{z_{k+1}})^2}\, dV(w_1,\dots,w_n)\right| \\ &\quad= \left|\int_{\mathbb{D}^n} \frac{f(\prod_{k=1}^n w_k,\dots,\prod_{k=n-1}^n w_k, w_n)\prod_{k=2}^n {w_k}^{k-1}}{\pi^n(\prod_{k=2}^n z_k)(1-z_n\overline{w_n})^2\prod_{k=1}^{n-1}(1-\frac{z_k\overline{w_k}}{z_{k+1}})^2}\, dV(w_1,\dots,w_n)\right|. \end{aligned} \tag{3.6}$$

In a similar way, a simple calculation implies

$$\begin{aligned}&\left|\mathbf{P}_{\mathbb{D}^n}(g)\left(\frac{z_1}{z_2},\dots,\frac{z_{n-1}}{z_n},z_n\right)\right| \\ &\quad= \left|\int_{\mathbb{D}^n} g(w_1,\dots,w_n) K_{\mathbb{D}^n}\left(\frac{z_1}{z_2},\dots,\frac{z_{n-1}}{z_n},z_n;w_1,\dots,w_n\right) dV(w_1,\dots,w_n)\right|\end{aligned}$$

$$= \left| \int_{\mathbb{D}^n} f\left(\prod_{k=1}^n w_k, \dots, \prod_{k=n-1}^n w_k, w_n \right) \right.$$
$$\left. \times \frac{\prod_{k=2}^n {w_k}^{k-1}}{\pi^n (1 - z_n \overline{w_n})^2 \prod_{k=1}^{n-1} (1 - \frac{z_k \overline{w_k}}{z_{k+1}})^2} \, dV(w_1, w_2, \dots, w_n) \right|. \tag{3.7}$$

Comparing (3.6) with (3.7), it is easy to see that

$$\left| \mathbf{P} f(z_1, z_2, \dots, z_n) \right| = \frac{|\mathbf{P}_{\mathbb{D}^n}(g)(\frac{z_1}{z_2}, \dots, \frac{z_{n-1}}{z_n}, z_n)|}{\prod_{k=2}^n |z_k|}.$$

Hence we can evaluate $|\{(z_1, z_2, \dots, z_n) \in \mathbb{H}^n : |\mathbf{P}(f)(z_1, z_2, \dots, z_n)| > \lambda\}|$ as follows:

$$\left| \left\{ (z_1, z_2, \dots, z_n) \in \mathbb{H}^n : \left| \mathbf{P}(f)(z_1, z_2, \dots, z_n) \right| > \lambda \right\} \right|$$
$$= \int_{\{(z_1, z_2, \dots, z_n) \in \mathbb{H}^n : |\mathbf{P}(f)(z_1, z_2, \dots, z_n)| > \lambda\}} dV(z_1, z_2, \dots, z_n)$$
$$= \int_{\{(z_1, z_2, \dots, z_n) \in \mathbb{H}^n : \frac{|\mathbf{P}_{\mathbb{D}^n}(g)(\frac{z_1}{z_2}, \dots, \frac{z_{n-1}}{z_n}, z_n)|}{\prod_{k=2}^n |z_k|} > \lambda\}} dV(z_1, z_2, \dots, z_n)$$
$$= \int_{\{(z_1, z_2, \dots, z_n) \in \mathbb{D}^n : \frac{|\mathbf{P}_{\mathbb{D}^n}(g)(z_1, z_2, \dots, z_n)|}{\prod_{k=2}^n |z_k|^{k-1}} > \lambda\}} \prod_{k=2}^n |z_k|^{2k-2} \, dV(z_1, z_2, \dots, z_n)$$
$$= \int_{\{(z_1, z_2, \dots, z_n) \in \mathbb{D}^n : |z_n| \le \frac{1}{2} \quad \text{and} \quad \frac{|\mathbf{P}_{\mathbb{D}^n}(g)(z_1, z_2, \dots, z_n)|}{\prod_{k=2}^n |z_k|^{k-1}} > \lambda\}} \prod_{k=2}^n |z_k|^{2k-2} \, dV(z_1, z_2, \dots, z_n) \tag{3.8}$$
$$+ \int_{\{(z_1, z_2, \dots, z_n) \in \mathbb{D}^n : |z_n| > \frac{1}{2} \quad \text{and} \quad \frac{|\mathbf{P}_{\mathbb{D}^n}(g)(z_1, z_2, \dots, z_n)|}{\prod_{k=2}^n |z_k|^{k-1}} > \lambda\}} \prod_{k=2}^n |z_k|^{2k-2} \, dV(z_1, z_2, \dots, z_n). \tag{3.9}$$

Now we just need to prove (3.8) $\lesssim \frac{\|f\|_{L^q(\mathbb{H}^n)}^q}{\lambda^q}$ and (3.9) $\lesssim \frac{\|f\|_{L^q(\mathbb{H}^n)}^q}{\lambda^q}$. To this end, it is sufficient to show

$$\int_{\{I_1 : \frac{|\mathbf{P}_{\mathbb{D}^n}(g)(z_1, \dots, z_n)|}{\prod_{k=2}^n |z_k|^{k-1}} > \lambda\}} \prod_{k=2}^n |z_k|^{2k-2} \, dV(z_1, \dots, z_n) \lesssim \frac{\|f\|_{L^q(\mathbb{H}^n)}^q}{\lambda^q} \tag{3.10}$$

and

$$\int_{\{I_2 : \frac{|\mathbf{P}_{\mathbb{D}^n}(g)(z_1, \dots, z_n)|}{\prod_{k=2}^n |z_k|^{k-1}} > \lambda\}} \prod_{k=2}^n |z_k|^{2k-2} \, dV(z_1, \dots, z_n) \lesssim \frac{\|f\|_{L^q(\mathbb{H}^n)}^q}{\lambda^q}. \tag{3.11}$$

Here,

$$I_1 := \left\{ (z_1, z_2, \dots, z_n) \in \mathbb{D}^n : |z_{j_1}| \le \frac{1}{2}, \dots, |z_{j_m}| \le \frac{1}{2}, |z_{j_{m+1}}| > \frac{1}{2}, \dots, |z_{j_{n-1}}| > \frac{1}{2}, |z_n| \le \frac{1}{2} \right\},$$

where the set of the numbers $j_1, \dots, j_{n-1}$ is an any fixed rearrangement of $1, 2, \dots, n-1$ and

$$I_2 := \left\{ (z_1, z_2, \dots, z_n) \in \mathbb{D}^n : |z_{t_1}| \le \frac{1}{2}, \dots, |z_{t_s}| \le \frac{1}{2}, |z_{t_{s+1}}| > \frac{1}{2}, \dots, |z_{t_{n-1}}| > \frac{1}{2}, |z_n| > \frac{1}{2} \right\},$$

where the set of the numbers $t_1, \dots, t_{n-1}$ is an any fixed rearrangement of $1, 2, \dots, n-1$.

Let us begin with (3.10). For $|z_n| \le \frac{1}{2}$, one has

$$\begin{aligned}\left|K_{\mathbb{D}^n}(z_1,z_2,\ldots,z_n;w_1,w_2,\ldots,w_n)\right| &= \frac{1}{\pi^n \prod_{k=1}^n |1-z_k\overline{w_k}|^2}\\ &\approx \frac{1}{\pi^{n-1}\prod_{k=1}^{n-1}|1-z_k\overline{w_k}|^2}\end{aligned}$$

and

$$\begin{aligned}\left|\mathbf{P}_{\mathbb{D}^n}(g)(z_1,z_2,\ldots,z_n)\right| &= \left|\int_{\mathbb{D}^n}\frac{g(w_1,w_2,\ldots,w_n)}{\pi^n\prod_{k=1}^n(1-z_k\overline{w_k})^2}\,dV(w_1,w_2,\ldots,w_n)\right|\\ &\le \int_{\mathbb{D}^n}\frac{|g(w_1,w_2,\ldots,w_n)|}{\pi^n\prod_{k=1}^n|1-z_k\overline{w_k}|^2}\,dV(w_1,w_2,\ldots,w_n)\\ &\lesssim \int_{\mathbb{D}^{n-1}}\frac{\int_{\mathbb{D}}|g(w_1,w_2,\ldots,w_n)|\,dV(w_n)}{\pi^{n-1}\prod_{k=1}^{n-1}|1-z_k\overline{w_k}|^2}\,dV(w_1,w_2,\ldots,w_{n-1})\\ &= []\mathbf{P}^+_{\mathbb{D}^{n-1}}\left(\int_{\mathbb{D}}\left|g(w_1,w_2,\ldots,w_n)\right|dV(w_n)\right)(z_1,z_2,\ldots,z_{n-1})\\ &= \mathbf{P}^+_{\mathbb{D}^{n-1}}(G)(z_1,z_2,\ldots,z_{n-1}),\end{aligned}$$

where $G(w_1,w_2,\ldots,w_{n-1}) = \int_{\mathbb{D}}|g(w_1,w_2,\ldots,w_n)|\,dV(w_n)$. Then there exists a constant C such that

$$\begin{aligned}&\text{LHS of (3.10)}\\ &\le \int_{\{I_1:\frac{|\mathbf{P}^+_{\mathbb{D}^{n-1}}(G)(z_1,z_2,\ldots,z_{n-1})|}{\prod_{k=2}^n|z_k|^{k-1}}>C\lambda\}}\prod_{k=2}^n|z_k|^{2k-2}\,dV(z_1,z_2,\ldots,z_n)\\ &= \int_{I_1'}dV(z_1,z_2,\ldots,z_{n-1})\int_{\{|z_n|\le\frac12\text{ and }(\frac{|\mathbf{P}^+_{\mathbb{D}^{n-1}}(G)(z_1,z_2,\ldots,z_{n-1})|}{C\lambda\prod_{k=2}^{n-1}|z_k|^{k-1}})^{\frac{1}{n-1}}>|z_n|\}}\prod_{k=2}^n|z_k|^{2k-2}\,dV(z_n)\\ &\lesssim \int_{I_1'}dV(z_1,z_2,\ldots,z_{n-1})\int_0^{(\frac{|\mathbf{P}^+_{\mathbb{D}^{n-1}}(G)(z_1,z_2,\ldots,z_{n-1})|}{C\lambda\prod_{k=2}^{n-1}|z_k|^{k-1}})^{\frac{1}{n-1}}} r^{2n-1}\prod_{k=2}^{n-1}|z_k|^{2k-2}\,dr\\ &\lesssim \int_{I_1'}\frac{|\mathbf{P}^+_{\mathbb{D}^{n-1}}(G)(z_1,z_2,\ldots,z_{n-1})|^{\frac{2n}{n-1}}}{(\lambda\prod_{k=2}^{n-1}|z_k|^{k-1})^{\frac{2n}{n-1}}}\prod_{k=2}^{n-1}|z_k|^{2k-2}\,dV(z_1,z_2,\ldots,z_{n-1}),\end{aligned}\tag{3.12}$$

where

$$I_1' := \left\{(z_1,z_2,\ldots,z_{n-1})\in\mathbb{D}^{n-1}: |z_{j_1}|\le\frac12,\ldots,|z_{j_m}|\le\frac12, |z_{j_{m+1}}|>\frac12,\ldots,|z_{j_{n-1}}|>\frac12\right\}.$$

For $|z_{j_1}|\le\frac12, |z_{j_2}|\le\frac12,\ldots,|z_{j_m}|\le\frac12, |z_{j_{m+1}}|>\frac12, |z_{j_{m+2}}|>\frac12,\ldots,|z_{j_{n-1}}|>\frac12$, it is easy to see that

$$\begin{aligned}\left|K_{\mathbb{D}^{n-1}}(z_1,z_2,\ldots,z_{n-1};w_1,w_2,\ldots,w_{n-1})\right| &= \frac{1}{\pi^{n-1}\prod_{k=1}^{n-1}|1-z_k\overline{w_k}|^2}\\ &\approx \frac{1}{\pi^{n-m-1}\prod_{k=m+1}^{n-1}|1-z_{j_k}\overline{w_{j_k}}|^2}.\end{aligned}$$

In order to estimate (3.12), we need to simplify $|\mathbf{P}^+_{\mathbb{D}^{n-1}}(G)(z_1,z_2,\dots,z_{n-1})|$ as follows:

$$\begin{aligned}
&\left|\mathbf{P}^+_{\mathbb{D}^{n-1}}(G)(z_1,z_2,\dots,z_{n-1})\right| \\
&\quad= \int_{\mathbb{D}^{n-1}} \frac{\int_{\mathbb{D}} |g(w_1,w_2,\dots,w_n)|\,dV(w_n)}{\pi^{n-1}\prod_{k=1}^{n-1}|1-z_k\overline{w_k}|^2}\,dV(w_1,w_2,\dots,w_{n-1}) \\
&\quad\approx \int_{\mathbb{D}^{n-m-1}} \frac{\int_{\mathbb{D}^{m+1}} |g(w_1,w_2,\dots,w_n)|\,dV(w_{j_1},w_{j_2},\dots,w_{j_m},w_n)}{\pi^{n-m-1}\prod_{k=m+1}^{n-1}|1-z_{j_k}\overline{w_{j_k}}|^2} \\
&\quad\quad dV(w_{j_{m+1}},w_{j_{m+1}},\dots,w_{j_{n-1}}) \\
&\quad= \mathbf{P}^+_{\mathbb{D}^{n-m-1}}\left(\int_{\mathbb{D}^{m+1}}\left|g(w_1,w_2,\dots,w_n)\right|dV(w_{j_1},w_{j_2},\dots,w_{j_m},w_n)\right)(z_{j_{m+1}},z_{j_{m+2}},\dots,z_{j_{n-1}}) \\
&\quad= \mathbf{P}^+_{\mathbb{D}^{n-m-1}}(G_1)(z_{j_{m+1}},z_{j_{m+2}},\dots,z_{j_{n-1}}),
\end{aligned} \tag{3.13}$$

where $G_1(w_{j_{m+1}},w_{j_{m+2}},\dots,w_{j_{n-1}}) = \int_{\mathbb{D}^{m+1}}|g(w_1,w_2,\dots,w_n)|\,dV(w_{j_1},w_{j_2},\dots,w_{j_m},w_n)$.

Substituting (3.13) into (3.12), one obtains

$$\begin{aligned}
&\int_{I_1'} \frac{|\mathbf{P}^+_{\mathbb{D}^{n-1}}(G)(z_1,z_2,\dots,z_{n-1})|^{\frac{2n}{n-1}}}{(\lambda\prod_{k=2}^{n-1}|z_k|^{k-1})^{\frac{2n}{n-1}}}\prod_{k=2}^{n-1}|z_k|^{2k-2}\,dV(z_1,\dots,z_{n-1}) \\
&\quad\approx \int_{I_1'} \frac{|\mathbf{P}^+_{\mathbb{D}^{n-m-1}}(G_1)(z_{j_{m+1}},\dots,z_{j_{n-1}})|^{\frac{2n}{n-1}}}{(\lambda\prod_{k=2}^{n-1}|z_k|^{k-1})^{\frac{2n}{n-1}}}\prod_{k=2}^{n-1}|z_k|^{2k-2}\,dV(z_1,\dots,z_{n-1}) \\
&\quad\approx \int_{I_1'} \frac{|\mathbf{P}^+_{\mathbb{D}^{n-m-1}}(G_1)(z_{j_{m+1}},\dots,z_{j_{n-1}})|^{\frac{2n}{n-1}}}{(\lambda\prod_{k=1,j_k\neq 1}^{m}|z_{j_k}|^{j_k-1})^{\frac{2n}{n-1}}}\prod_{k=1,j_k\neq 1}^{m}|z_{j_k}|^{2j_k-2}\,dV(z_1,\dots,z_{n-1}) \\
&\quad\lesssim \int_{\mathbb{D}^{n-m-1}} \frac{{G_1'}^q}{\lambda^q}\int_{\mathbb{D}^m}\prod_{k=1,j_k\neq 1}^{m}|z_{j_k}|^{\frac{2(n-j_k)}{n-1}-2}\,dV(z_{j_1},\dots,z_{j_m})\,dV(z_{j_{m+1}},\dots,z_{j_{n-1}}),
\end{aligned} \tag{3.14}$$

where $G_1' := |\mathbf{P}^+_{\mathbb{D}^{n-m-1}}(G_1)(z_{j_{m+1}},\dots,z_{j_{n-1}})|$.

Note that $\frac{2(n-j_k)}{n-1}-2 > -2$ since $j_k \le n-1$, so $\int_{\mathbb{D}^m}\prod_{k=1,j_k\neq 1}^{m}|z_{j_k}|^{\frac{2(n-j_k)}{n-1}-2}\,dV(z_{j_1},\dots,z_{j_m}) < \infty$. Hence,

$$\begin{aligned}
(3.14) &\approx \int_{\mathbb{D}^{n-m-1}} \frac{{G_1'}^q}{\lambda^q}\,dV(z_{j_{m+1}},\dots,z_{j_{n-1}}) \\
&= \frac{\|\mathbf{P}^+_{\mathbb{D}^{n-m-1}}(G_1)\|^q_{L^q(\mathbb{D}^{n-m-1})}}{\lambda^q}.
\end{aligned} \tag{3.15}$$

By Hölder's inequality, one gets

$$\begin{aligned}
&\int_{\mathbb{D}^{n-m-1}} G_1(w_{j_{m+1}},\dots,w_{j_{n-1}})^q\,dV(w_{j_{m+1}},\dots,w_{j_{n-1}}) \\
&\quad\lesssim \int_{\mathbb{D}^{n-m-1}}\int_{\mathbb{D}^{m+1}}\left|g(w_1,w_2,\dots,w_n)\right|^q dV(w_{j_1},w_{j_2},\dots,w_{j_m},w_n) \\
&\quad\quad dV(w_{j_{m+1}},w_{j_{m+2}},\dots,w_{j_{n-1}}) \\
&\quad= \|g\|^q_{L^q(\mathbb{D}^n)} \le \|g\|^q_{L^q(\mathbb{D}^n,\prod_{k=2}^n|z_k|^{\frac{-2k+2}{n-1}}dV)} = \|f\|^q_{L^q(\mathbb{H}^n)}.
\end{aligned} \tag{3.16}$$

Hence,

$$G_1 \in L^q\big(\mathbb{D}^{n-m-1}\big). \tag{3.17}$$

Combining (3.12), (3.13), (3.14), (3.15), (3.16), (3.17), and Lemma 3.1, we have

$$\begin{aligned}\text{LHS of (3.10)} &\lesssim \frac{\|\mathbf{P}^+_{\mathbb{D}^{n-m-1}}(G_1)\|^q_{L^q(\mathbb{D}^{n-m-1})}}{\lambda^q}\\ &\lesssim \frac{\|G_1\|^q_{L^q(\mathbb{D}^{n-m-1})}}{\lambda^q}\\ &\lesssim \frac{\|f\|^q_{L^q(\mathbb{H}^n)}}{\lambda^q}.\end{aligned}$$

This gives (3.10).

Now only (3.11) remains to be dealt with. Let

$$|z_{t_1}| \le \frac{1}{2}, \dots, |z_{t_s}| \le \frac{1}{2}, \qquad |z_{t_{s+1}}| > \frac{1}{2}, \dots, |z_{t_{n-1}}| > \frac{1}{2}, \qquad |z_n| > \frac{1}{2}.$$

Similarly, the set of the numbers $t_1, \dots, t_{n-1}$ is an any fixed rearrangement of $1, 2, \dots, n-1$. Then

$$\begin{aligned}\big|K_{\mathbb{D}^n}(z_1, z_2, \dots, z_n; w_1, w_2, \dots, w_n)\big| &= \frac{1}{\pi^n \prod_{k=1}^n |1 - z_k\overline{w_k}|^2}\\ &\approx \frac{1}{\pi^{n-s}|1 - z_n\overline{w_n}|^2 \prod_{k=s+1}^{n-1} |1 - z_{t_k}\overline{w_{t_k}}|^2}.\end{aligned} \tag{3.18}$$

It follows from (3.18) that

$$\begin{aligned}&\big|\mathbf{P}_{\mathbb{D}^n}(g)(z_1, z_2, \dots, z_n)\big|\\ &\quad= \left|\int_{\mathbb{D}^n} \frac{g(w_1, w_2, \dots, w_n)}{\pi^n \prod_{k=1}^n |1 - z_k\overline{w_k}|^2}\, dV(w_1, w_2, \dots, w_n)\right|\\ &\quad\lesssim \int_{\mathbb{D}^n} \frac{|g(w_1, w_2, \dots, w_n)|}{\pi^{n-s}|1 - z_n\overline{w_n}|^2 \prod_{k=s+1}^{n-1} |1 - z_{t_k}\overline{w_{t_k}}|^2}\, dV(w_1, w_2, \dots, w_n)\\ &\quad\approx \int_{\mathbb{D}^{n-s}} \frac{\int_{\mathbb{D}^s} |g(w_1, w_2, \dots, w_n)|\, dV(w_{t_1}, w_{t_2}, \dots, w_{t_s})}{\pi^{n-s}|1 - z_n\overline{w_n}|^2 \prod_{k=s+1}^{n-1} |1 - z_{t_k}\overline{w_{t_k}}|^2}\, dV(w_{t_{s+1}}, w_{t_{s+2}}, \dots, w_{t_{n-1}}, w_n)\\ &\quad= \mathbf{P}^+_{\mathbb{D}^{n-s}}\left(\int_{\mathbb{D}^s} \big|g(w_1, w_2, \dots, w_n)\big|\, dV(w_{t_1}, w_{t_2}, \dots, w_{t_s})\right)(z_{t_{s+1}}, z_{t_{s+2}}, \dots, z_{t_{n-1}}, z_n)\\ &\quad= \mathbf{P}^+_{\mathbb{D}^{n-s}}(G_2)(z_{t_{s+1}}, z_{t_{s+2}}, \dots, z_{t_{n-1}}, z_n),\end{aligned} \tag{3.19}$$

where $G_2(w_{t_{s+1}}, w_{t_{s+2}}, \dots, w_{t_{n-1}}, w_n) = \int_{\mathbb{D}^s} |g(w_1, w_2, \dots, w_n)|\, dV(w_{t_1}, w_{t_2}, \dots, w_{t_s})$.

Hölder's inequality now leads to

$$\begin{aligned}&\int_{\mathbb{D}^{n-s}} G_2(w_{t_{s+1}}, \dots, w_{t_{n-1}}, w_n)^{\frac{2n}{n-1}}\, dV(w_{t_{s+1}}, \dots, w_{t_{n-1}}, w_n)\\ &\quad\lesssim \int_{\mathbb{D}^{n-s}} \int_{\mathbb{D}^s} \big|g(w_1, w_2, \dots, w_n)\big|^{\frac{2n}{n-1}}\, dV(w_{t_1}, w_{t_2}, \dots, w_{t_s})\, dV(w_{t_{s+1}}, w_{t_{s+2}}, \dots, w_{t_{n-1}}, w_n)\end{aligned}$$

$$= \|g\|^q_{L^q(\mathbb{D}^n)} \le \|g\|^q_{L^q(\mathbb{D}^n, \prod_{k=2}^n |z_k|^{\frac{-2k+2}{n-1}} dV)} = \|f\|^q_{L^q(\mathbb{H}^n)}. \tag{3.20}$$

So $G_2 \in L^q(\mathbb{D}^{n-s})$ and let $G_2' := \mathbf{P}^+_{\mathbb{D}^{n-s}}(G_2)(z_{t_{s+1}}, z_{t_{s+2}}, \ldots, z_{t_{n-1}}, z_n)$.

Together with (3.19), one has

$$\begin{aligned}
\text{LHS of (3.11)} &\le \int_{I_2} \left(\frac{|\mathbf{P}_{\mathbb{D}^n}(g)(z_1, z_2, \ldots, z_n)|}{\lambda \prod_{k=2}^n |z_k|^{k-1}} \right)^{\frac{2n}{n-1}} \prod_{k=2}^n |z_k|^{2k-2}\, dV(z_1, z_2, \ldots, z_n) \\
&\lesssim \int_{I_2} \frac{{G_2'}^{\frac{2n}{n-1}}}{(\lambda \prod_{k=2}^n |z_k|^{k-1})^{\frac{2n}{n-1}}} \prod_{k=2}^n |z_k|^{2k-2}\, dV(z_1, z_2, \ldots, z_n) \\
&\approx \int_{I_2} \frac{{G_2'}^{\frac{2n}{n-1}}}{(\lambda \prod_{k=1, t_k \ne 1}^s |z_{t_k}|^{t_k-1})^{\frac{2n}{n-1}}} \prod_{k=1, t_k \ne 1}^s |z_{t_k}|^{2t_k-2}\, dV(z_1, z_2, \ldots, z_n) \\
&= \int_{I_2} \frac{{G_2'}^{\frac{2n}{n-1}}}{\lambda^{\frac{2n}{n-1}}} \prod_{k=1, t_k \ne 1}^s |z_{t_k}|^{\frac{2(n-t_k)}{n-1}-2}\, dV(z_1, z_2, \ldots, z_n).
\end{aligned} \tag{3.21}$$

Note that $\frac{2(n-t_k)}{n-1} - 2 > -2$ since $t_k \le n-1$, so

$$\int_{\mathbb{D}^s} \prod_{k=1, t_k \ne 1}^s |z_{t_k}|^{\frac{2(n-t_k)}{n-1}-2}\, dV(z_{t_1}, \ldots, z_{t_s}) < \infty. \tag{3.22}$$

Then

$$\begin{aligned}
(3.21) &\le \int_{\mathbb{D}^n} \frac{{G_2'}^{\frac{2n}{n-1}}}{\lambda^{\frac{2n}{n-1}}} \prod_{k=1, t_k \ne 1}^s |z_{t_k}|^{\frac{2(n-t_k)}{n-1}-2}\, dV(z_1, z_2, \ldots, z_n) \\
&= \int_{\mathbb{D}^{n-s}} \frac{{G_2'}^{\frac{2n}{n-1}}}{\lambda^{\frac{2n}{n-1}}}\, dV(z_{t_{s+1}}, z_{t_{s+2}}, \ldots, z_{t_{n-1}}, z_n) \\
&\quad \times \int_{\mathbb{D}^s} \prod_{k=1, t_k \ne 1}^s |z_{t_k}|^{\frac{2(n-t_k)}{n-1}-2}\, dV(z_{t_1}, z_{t_2}, \ldots, z_{t_s}) \\
&\approx \int_{\mathbb{D}^{n-s}} \frac{{G_2'}^{\frac{2n}{n-1}}}{\lambda^{\frac{2n}{n-1}}}\, dV(z_{t_{s+1}}, z_{t_{s+2}}, \ldots, z_{t_{n-1}}, z_n).
\end{aligned} \tag{3.23}$$

From Lemma 3.1 and (3.20), it follows that

$$\begin{aligned}
(3.23) &\lesssim \frac{\|G_2\|^{\frac{2n}{n-1}}_{L^q(\mathbb{D}^{n-s})}}{\lambda^{\frac{2n}{n-1}}} \\
&\lesssim \frac{\|f\|^q_{L^q(\mathbb{H}^n)}}{\lambda^q}.
\end{aligned}$$

We complete the proof of the weak-type $(\frac{2n}{n-1}, \frac{2n}{n-1})$. □

Remark 3.2 It is necessary to divide the proof of Theorem 1.2 into two parts, (3.8) and (3.9). The ways to prove (3.8) $\lesssim \frac{\|f\|^q_{L^q(\mathbb{H}^n)}}{\lambda^q}$ and (3.9) $\lesssim \frac{\|f\|^q_{L^q(\mathbb{H}^n)}}{\lambda^q}$ are different and not interchangeable.

(i) If the method of proving (3.9) $\lesssim \frac{\|f\|^q_{L^q(\mathbb{H}^n)}}{\lambda^q}$ is applied to the proof of (3.8) $\lesssim \frac{\|f\|^q_{L^q(\mathbb{H}^n)}}{\lambda^q}$, there will be errors.

When $|z_n| \leq \frac{1}{2}$, let us consider

$$\prod_{k=1,t_k\neq 1}^{s} |z_{t_k}|^{\frac{2(n-t_k)}{n-1}-2}$$

in (3.21). One obtains

$$\prod_{k=1,t_k\neq 1}^{s} |z_{t_k}|^{\frac{2(n-t_k)}{n-1}-2} = |z_n|^{-2} \prod_{k=1,t_k\neq 1}^{s-1} |z_{t_k}|^{\frac{2(n-t_k)}{n-1}-2}.$$

Then

$$\begin{aligned}&\int_{\mathbb{D}^s} \prod_{k=1,t_k\neq 1}^{s} |z_{t_k}|^{\frac{2(n-t_k)}{n-1}-2}\, dV(z_{t_1},\dots,z_{t_s})\\ &\quad= \int_{\mathbb{D}^{s-1}} \prod_{k=1,t_k\neq 1}^{s-1} |z_{t_k}|^{\frac{2(n-t_k)}{n-1}-2}\, dV(z_{t_1},\dots,z_{t_{s-1}}) \int_{\mathbb{D}} |z_n|^{-2}\, dV(z_n).\end{aligned}$$

It is easy to see that

$$\int_{\mathbb{D}^{s-1}} \prod_{k=1,t_k\neq 1}^{s-1} |z_{t_k}|^{\frac{2(n-t_k)}{n-1}-2}\, dV(z_{t_1},\dots,z_{t_{s-1}}) < \infty$$

and

$$\int_{\mathbb{D}} |z_n|^{-2}\, dV(z_n) = \infty.$$

Hence, (3.22) will not hold.

(ii) After a simple calculation, we also find that the method of proving (3.8) $\lesssim \frac{\|f\|^q_{L^q(\mathbb{H}^n)}}{\lambda^q}$ cannot be applied to the proof of (3.9) $\lesssim \frac{\|f\|^q_{L^q(\mathbb{H}^n)}}{\lambda^q}$.

Acknowledgements
The authors are grateful to the referee for comments and suggestions.

Funding
No funding.

Data availability
No data were used to support this study.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Yi Li and Mengjiao Wang wrote the main manuscript text.

Received: 27 July 2023 Accepted: 11 March 2024 Published online: 21 March 2024

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