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Strong convergence of split equality variational inequality, variational inclusion, and multiple sets fixed point problems in Hilbert spaces with application

Charu Batra[1], Renu Chugh[2], Rajeev Kumar[1], Khaled Suwais[3], Sally Almanasra[3] and Nabil Mlaiki[4*]

[*]Correspondence:
nmlaiki@psu.edu.sa
[4]Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

This paper introduces an innovative inertial simultaneous cyclic iterative algorithm designed to address a range of mathematical problems within the realm of split equality variational inequalities. Specifically, the algorithm accommodates finite families of split equality variational inequality problems, infinite families of split equality variational inclusion problems, and multiple-sets split equality fixed point problems involving demicontractive operators in infinite-dimensional Hilbert spaces. The algorithm integrates well-established methods, including the cyclic method, the inertial method, the viscosity approximation method, and the projection method. We establish the strong convergence of this proposed algorithm, demonstrating its applicability in various scenarios and unifying disparate findings from existing literature. Additionally, a numerical example is presented to validate the primary convergence theorem.

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Keywords: Split equality problems; Variational inequality problems; Variational inclusion problems; Multiple-sets fixed point problem; Demicontractive operators

1 Introduction

The variational inequality problem (VIP) was developed by Stampacchia [1] in 1964, and it provides useful mathematical tools for investigating interesting issues like game theory, economic equilibrium mechanics, partial differential equations, and optimization theory; see [2–8]. Due to their active role and significance in nonlinear analysis, VIPs are currently growing in both theory and practice; for examples, see [8, 9]. The variational inequality problem (VIP), one of the significant problems, has drawn the interest of numerous mathematicians throughout the years. One can define VIP as finding $\bar{r}^* \in Q$ such that

$$\langle L(\bar{r}^*), p - \bar{r}^* \rangle \geq 0 \quad \text{for all } p \in Q, \tag{1.1}$$

where L is a nonlinear mapping. The set of solutions of the VIP is represented by $\mathrm{VIP}(Q, L)$. One of the most often used methods for studying VIPs is the projection method, which can be applied to demonstrate the equivalence between fixed points and VIPs.

In 1994, Censor and Elfving [10] introduced the split feasibility problem (SFP) for modeling inverse problems. SFP identifies a point in a closed convex subset of a Hilbert space such that the image of that point under a particular bounded linear operator belongs to a closed convex subset of a different Hilbert space. Currently, SFPs are implemented in a number of fields, including signal processing [11], computer tomography [12], image restoration [13, 14], and intensity modulated radiation treatment (IMRT) [12, 15]. Recently, numerous authors have presented various split-type problems; see [11, 12, 15] for significant developments in this direction or related topics. In 2005, Censor et al. [12] proposed multiple-sets split feasibility problems (MSSFPs), which have applications in the inverse problem of intensity-modulated radiation therapy. In 2009, Censor and Segal [16] proposed split common fixed point problems (SCFPPs) to generalize the split feasibility problem, the convex feasibility problem, and the multiple-sets split feasibility problem. For more related problems, see [17–23].

The split equality problem (SEP) was firstly proposed by Moudafi [24] in 2013. Partial and asymmetric relationships between the variables of two spaces are permitted by SEP. Moudafi [25] studied the split equality fixed point problem (SEFPP), which involves strongly nonexpansive mapping. Additionally, Moudafi and Al-Shemas [26] proved a weak convergence result and proposed the simultaneous iterative method. One can see the above-mentioned iterative approaches need the norm of the bounded linear operators $\|A\|$ and $\|B\|$. Calculating the norm of a bounded linear operator can be difficult in some situations. Lopez et al. [27] provided a valuable solution to this numerical challenge. In 2014, Zhao [28] improved the result of Moudafi [16, 24, 25]. By considering the step size that does not need previous knowledge of the operator norm and obtained the weak convergence result for quasi-nonexpansive mappings. A significant convergence result involving quasi-nonexpansive mappings for solving the split common fixed point problem was demonstrated by Shehu et al. [29] in 2017. The split equality variational inequality problem (SEVIP) has been extensively studied and used to solve many real-world issues, including modeling intensity-modulated radiation therapy treatment planning [30]. Mathematically, SEVIPs are very general since they include common solutions of the variational inequality problem [31], split equality zero point problem [32], split equality feasibility problem [33], and common zeros of mappings [34]. SEVIP is stated as a problem of finding $(\bar{r}^*, \bar{s}^*) \in Q_1 \times Q_2$ such that

$$\left(\bar{r}^*, \bar{s}^*\right) \in \mathrm{VIP}\left(Q_1, L^x\right) \times \mathrm{VIP}\left(Q_2, L^y\right) \quad \text{and} \quad A\bar{r}^* = B\bar{s}^*, \tag{1.2}$$

where A, B are bounded linear operators, Q_1, Q_2 are nonempty closed convex subsets of H_1 and H_2, respectively, and $L^x : H_1 \to H_1$, $L^y : H_2 \to H_2$ are nonlinear operators.

In 2014, Wu et al. [35] proposed the multiple-sets split equality fixed point problem (MSEFPP) stated as a problem of finding $(\bar{r}^*, \bar{s}^*) \in H_1 \times H_2$ such that

$$\bar{r}^* \in \bigcap_{k_1=1}^{N} \mathrm{Fix}(S_{k_1}), \qquad \bar{s}^* \in \bigcap_{k_2=1}^{P} \mathrm{Fix}(U_{k_2}) \quad \text{and} \quad A\bar{r}^* = B\bar{s}^*, \tag{1.3}$$

where $S_{k_1} : H_1 \to H_1$, $U_{k_2} : H_2 \to H_2$ are nonlinear operators with $1 \le k_1 \le N$, $1 \le k_2 \le P$ and A, B are bounded linear operators. MSEFPP includes the multiple-sets split equality problem (MSEP), the split equality fixed point problem (SEFPP), the multiple-sets split feasibility problem (MSFP), the split equality problem (SEP), and many others; see [36]. MSEFPP had a significant impact on the growth of various fields including signal processing and image restoration.

Variational inclusion problems had a significant impact on the growth of many fields including mathematical programming, optimal control, mathematical economics, variational inequalities, game theory, complementarity problems, etc. In 2011, Moudafi [33] studied the split monotone variational inclusion problem (SMVIP). SMVIP is to find $\bar{r}^* \in H$ such that

$$0 \in K(\bar{r}^*) \quad \text{and} \quad 0 \in \bigcap_{i=1}^{\infty} T(A\bar{r}^*), \tag{1.4}$$

where A is a bounded linear operators, $K : H_1 \to H_1$ and $T : H_2 \to H_2$ are set-valued maximal monotone mappings. The general split equality variational inclusion problem (GSEMVIP) was studied by Chang et al. [37] in 2014, generalizing the SMVIP. GSEMVIP is stated as finding $(\bar{r}^*, \bar{s}^*) \in H_1 \times H_2$ such that

$$0 \in \bigcap_{i=1}^{\infty} K_i(\bar{r}^*), \qquad 0 \in \bigcap_{i=1}^{\infty} T_i(\bar{s}^*) \quad \text{and} \quad A\bar{r}^* = B\bar{s}^*. \tag{1.5}$$

The set of solutions to the GSEMVIP is represented by Ω. Chang et al. [37] proved a strong convergence theorem for GSEMVIP. In 2017, Latif and Eslamian [38] introduced a method to find the common solution of three split equality problems, namely variational inequality, equilibrium, and fixed point problems of nonexpansive semigroups. In 2019, Kazmi et al. [39] proposed an iterative method to find the common solution of three split equality problems, namely variational inclusion, generalized general variational-like inequality, and fixed point problems for quasi-nonexpansive mapping.

The inertial extrapolation methods were extensively used as an acceleration technique. The researchers used inertial extrapolation to build numerous iterative algorithms. The critical characteristic of inertial extrapolation is that the following iteration is determined using the results of the previous two iterations. The efficiency of its original (non-inertial) algorithms, like [40, 41], has been significantly enhanced by inertial approaches, as demonstrated by numerical studies. As a result, a lot of research is currently focused on exploiting inertial extrapolation to improve current methods (see [42, 43] and the references therein).

Suppose that H_1, H_2, and H_3 are real Hilbert spaces. Assume that $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators. Suppose that $\{K_i\}_{i=1}^{\infty} : H_1 \to 2^{H_1}$ and $\{T_i\}_{i=1}^{\infty} : H_2 \to 2^{H_2}$ are maximal monotone mappings. Let $S_{k_1} : H_1 \to H_1$ and $U_{k_2} : H_2 \to H_2$ be demicontractive operators such that $I - S_{k_1}$ and $I - U_{k_2}$ are demiclosed at zero for $1 \le k_1 \le N$, $1 \le k_2 \le P$ with constants $0 \le \kappa_{k_1} < 1$ and $0 \le \phi_{k_2} < 1$, respectively, where $\kappa = \max \kappa_{k_1}$ and $\phi = \max \phi_{k_2}$. Define $S_{k_1} = S_{k_1(\bmod N)+1}$ and $U_{k_2} = U_{k_2(\bmod P)+1}$. Let $L_j^x : H_1 \to H_1$, $L_j^y : H_2 \to H_2$ be c_j^x, c_j^y-inverse strongly monotone mappings, respectively, $\bar{c}^x = \min_{j=1,2,\ldots,M}\{c_j^x\}$

and $\bar{c}^{y} = \min_{j=1,2,\ldots,M}\{c_j^y\}$. We represent the required solution set by Γ, where

$$\Gamma = \left\{x^* \in \bigcap_{k_1=1}^{N} \mathrm{Fix}(S_{k_1}) \cap \bigcap_{i=1}^{\infty}(K_i)^{-1}(0) \cap \bigcap_{j=1}^{M} \mathrm{VIP}\big(L_j^x, Q_1\big), \right.$$
$$\left. y^* \in \bigcap_{k_2=1}^{M} \mathrm{Fix}(U_{k_2}) \cap \bigcap_{i=1}^{\infty}(T_i)^{-1}(0) \cap \bigcap_{j=1}^{M} \mathrm{VIP}\big(L_j^y, Q_2\big) \text{ such that } Ax^* = By^* \right\}. \quad (1.6)$$

2 Preliminaries

In this study, we take H to be a real Hilbert space with the inner product $\langle .,. \rangle$ and the corresponding norm $\|.\|$. For weak and strong convergence of $\{x_n\}$ to x^*, we use the notations $x_n \rightharpoonup x^*$ and $x_n \to x^*$. Assume that Q is a nonempty, closed, and convex subset of a Hilbert space H. The collection of all fixed points for mapping $\bar{U}$ is denoted by $\mathrm{Fix}(\bar{U})$.

Definition 2.1 Suppose that $\bar{U} : H \to H$ is a mapping. Then $\bar{U}$ is called

(i) contraction if $\lambda \in [0, 1)$ such that

$$\|\bar{U}u - \bar{U}v\| \le \lambda \|\bar{u} - \bar{v}\| \quad \text{for all } \bar{u}, \bar{v} \in H;$$

(ii) λ-demicontractive if there is $\lambda \in [0, 1)$ such that

$$\|\bar{U}\bar{u} - \bar{v}\|^2 \le \|\bar{u} - \bar{v}\|^2 + \lambda\|\bar{u} - \bar{U}\bar{u}\|^2 \quad \text{for all } \bar{u} \in H \text{ and } \bar{v} \in \mathrm{Fix}(\bar{U});$$

(iii) $\bar{\omega}$- inverse strongly monotone (ism) if there is $\bar{\omega} > 0$ such that

$$\langle \bar{U}\bar{u} - \bar{U}\bar{v}, \bar{u} - \bar{v} \rangle \ge \bar{\omega}\|\bar{U}\bar{u} - \bar{U}\bar{v}\|^2 \quad \text{for all } \bar{u}, \bar{v} \in H.$$

Definition 2.2 Suppose that $\bar{U} : H \to 2^H$ is a mapping. Then $\bar{U}$ is called

(i) monotone if

$$\langle \bar{r} - \bar{t}, \bar{u} - \bar{v} \rangle \ge 0 \quad \text{for all } \bar{r}, \bar{t} \in H, \bar{u} \in \bar{U}\bar{r}, \bar{v} \in \bar{U}\bar{t};$$

(ii) maximal monotone if it is monotone and the graph $G(\bar{U})$ is not properly contained in any other monotone mapping, where

$$G(\bar{U}) = \big\{(\bar{x}, \bar{y}) \in H \times H : \bar{y} \in \bar{U}(\bar{x})\big\}.$$

Lemma 2.3 [44] *Let H be a real Hilbert space, then*

(i) $2\langle \bar{r}, \bar{t} \rangle = \|\bar{r}\|^2 + \|\bar{t}\|^2 - \|\bar{r} - \bar{t}\|^2 = \|\bar{r} + \bar{t}\|^2 - \|\bar{r}\|^2 - \|\bar{t}\|^2$ *for all* $\bar{r}, \bar{t} \in H$;

(ii) $\|\bar{r} + \bar{t}\|^2 \le \|\bar{r}\|^2 + 2\langle \bar{t}, \bar{r} + \bar{t} \rangle$ *for all* $\bar{r}, \bar{t} \in H$;

(iii) $\|\alpha_0\bar{r}_0 + \alpha_1\bar{r}_1 + \alpha_2\bar{r}_2\|^27 =$
$\alpha_0\|\bar{r}_0\|^2 + \alpha_1\|\bar{r}_1\|^2 + \alpha_2\|\bar{r}_2\|^2 - \alpha_0\alpha_1\|\bar{r}_0 - \bar{r}_1\|^2 - \alpha_0\alpha_2\|\bar{r}_0 - \bar{r}_2\|^2 - \alpha_2\alpha_1\|\bar{r}_1 - \bar{r}_2\|^2$ *for* $\bar{r}_i \in H$, $\alpha_i \in [0, 1]$, $i = 0, 1, 2$, *and* $\sum_{i=0}^{2} \alpha_i = 1$.

Lemma 2.4 [45] *The metric projection P_Q satisfies*:

(i) $\bar{z} = P_Q\bar{u}$ *iff* $\langle \bar{u} - \bar{z}, \bar{z} - \bar{y} \rangle \le 0$ *for all* $\bar{y}, \bar{z} \in Q$ *and* $\bar{u} \in H$;

(ii) $\|P_Q\bar{u} - \bar{z}\|^2 \le \|\bar{u} - \bar{z}\|^2 - \|P_Q\bar{u} - \bar{u}\|^2$ *for all* $\bar{u} \in H$ *and* $\bar{z} \in Q$.

Lemma 2.5 [46] *Suppose that $\bar{U} : H \to H$ is a κ-demicontractive operator and $\bar{U}_\lambda := (1 - \lambda)I + \lambda\bar{U}$ for any $\lambda \in (0, 1-\kappa)$, where $\kappa < 1$, then for $\bar{v}^* \in \mathrm{Fix}(\bar{U})$ and $\bar{v} \in H$*

$$\left\| \bar{U}_\lambda(\bar{v}) - \bar{v}^* \right\|^2 \leq \left\| \bar{v} - \bar{v}^* \right\|^2 - \lambda(1 - \kappa - \lambda)\|\bar{v} - \bar{U}\bar{v}\|^2. \tag{2.1}$$

Lemma 2.6 [47] *Suppose that $\{s_n\}$ is a bounded sequence in a Hilbert space H, $\lim_{n\to\infty} \|s_{n+1} - s_n\| = 0$, and $z^* \in \omega_w(s_n)$. Then there is a subsequence $\{s_{m_k}\}$ of $\{s_n\}$ such that $m_k(\mod N) + 1 = i$ and $s_{m_k} \rightharpoonup z^*$, where $i \in \{1, 2, \dots, N\}$.*

Lemma 2.7 [48] *Assume that $\{s_n\}$ is a sequence of real numbers and there is a subsequence $\{n_k\}$ of $\{n\}$ satisfying $s_{n_k} < s_{n_k+1}$ for all $k \in \mathbb{N}$. Then there is a nondecreasing sequence $\{m_i\} \subset \mathbb{N}$ such that $m_i \to \infty$ and satisfies*

$$s_{m_i} \leq s_{m_i+1} \quad \text{and} \quad s_i \leq s_{m_i+1}, \tag{2.2}$$

where $i \in \mathbb{N}$ and $m_i = \max\{k \leq i : s_k < s_{k+1}\}$.

Lemma 2.8 [49] *Assume that $\{s_n\} \subset [0, \infty)$, $\{t_n\} \subset (-\infty, \infty)$, and $\{\omega_n\} \subset [0, 1]$ such that*

$$s_{n+1} \leq (1 - \omega_n)s_n + \omega_n t_n \quad \forall n \in \mathbb{N}, \tag{2.3}$$

where $\sum_{n=0}^{\infty} \omega_n = \infty$ and $\limsup_{n\to\infty} t_n \leq 0$, then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.9 [50] *Assume that $L : H \to 2^H$ is a maximal monotone mapping, $\tau > 0$, and J_τ^L is the resolvent mapping of L defined by $J_\tau^L := (I + \tau L)^{-1}$. Suppose that $L^{-1}(0) \neq \phi$. Then $\langle \bar{r}^* - J_\tau^L \bar{r}^*, J_\tau^L \bar{r}^* - q \rangle \geq 0$ for all $\bar{r}^* \in H$, $\tau > 0$ and $q \in L^{-1}(0)$.*

Lemma 2.10 [51] *Let $L : Q \to H$ be a mapping and $\bar{r}^* \in Q$. Then, for $b > 0$, $\bar{r}^* \in \mathrm{VIP}(Q, L)$ iff $\bar{r}^* = P_Q(I - bL)\bar{r}^*$, where P_Q is the metric projection of H onto Q.*

Lemma 2.11 [52] *Let $L_j : Q \to H$ be a c_j-ism mapping with $\bar{c} = \min_{j=1,2,\dots,M}\{c_j\}$, where $j = 1, 2, \dots, M$ and $\bigcap_{j=1}^{M} \mathrm{VIP}(Q, L_j) \neq \phi$, then $\bigcap_{j=1}^{M} \mathrm{VIP}(Q, L_j) = \mathrm{VIP}(Q, \sum_{j=1}^{M} c_j L_j)$, where $0 < b_j < 1$ for every $j = 1, 2, \dots, M$ and $\sum_{j=1}^{M} c_j = 1$.*

Lemma 2.12 [53] *Let $\bar{U} : Q \to Q$ be a μ-demicontractive mapping and $\mathrm{Fix}(\bar{U}) \neq \phi$, then $\mathrm{Fix}(\bar{U})$ is closed and convex.*

Inspired and motivated by ongoing research in this direction, we study a new inertial simultaneous cyclic iterative algorithm to find the common solution of three split equality problems, namely the finite family of variational inequalities, infinite family of variational inclusion, and multiple-sets fixed point problems for demicontractive operators. The proposed algorithm does not require prior knowledge of the operator norm, and we demonstrate our algorithm's strong convergence under some mild circumstances. We also discuss how our findings can be used to solve the problem of intensity-modulated radiation therapy (IMRT). In addition, we provide a numerical example to illustrate the numerical behavior of the suggested method and to compare it with different methods.

3 Main results

Let H_1, H_2, H_3 be Hilbert spaces, Q_1 and Q_2 be closed convex subsets of H_1 and H_2, respectively. Assume that $\psi_1 : H_1 \to H_1$, $\psi_2 : H_2 \to H_2$ are λ_1, λ_2 contractions mappings with $\lambda = \max\{\lambda_1, \lambda_2\}$ and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two (nonzero) bounded linear operators. Let A^* and B^* denote the adjoint of A and B respectively. Let $L_j^x : H_1 \to H_1$, $L_j^y : H_2 \to H_2$ be c_j^x, c_j^y-inverse strongly monotone mappings, respectively, $\bar{c}^x = \min_{j=1,2,\dots,M}\{c_j^x\}$ and $\bar{c}^y = \min_{j=1,2,\dots,M}\{c_j^y\}$. Suppose that $\{K_i\}_{i=1}^{\infty} : H_1 \to 2^{H_1}$ and $\{T_i\}_{i=1}^{\infty} : H_2 \to 2^{H_2}$ are maximal monotone mappings. Let $S_{k_1} : H_1 \to H_1$ and $U_{k_2} : H_2 \to H_2$ be demicontractive operators such that $I - S_{k_1}$ and $I - U_{k_2}$ are demiclosed at zero for $1 \le k_1 \le N$, $1 \le k_2 \le P$ with constants $0 \le \kappa_{k_1} < 1$ and $0 \le \phi_{k_2} < 1$ respectively, where $\kappa = \max \kappa_{k_1}$ and $\phi = \max \phi_{k_2}$. Define $S_{k_1} = S_{k_1(\mod N)+1}$ and $U_{k_2} = U_{k_2(\mod P)+1}$.

Algorithm 3.1 Consider $0 < a < \nu_n^x \le 2\bar{c}^x$, $0 < b < \nu_n^y \le 2\bar{c}^y$ for some $a, b \in \mathbb{R}$, $b_j^x, b_j^y \in (0,1)$ for $j = 1, 2, \dots, M$, $\{\delta_n\}, \{\mu_n\}, \{\eta_{n,i}\}, \{\sigma_n\} \subset [\bar{d}, \bar{e}] \subset (0,1)$, $a_n \in (\delta, 1 - \phi - \delta)$, $\beta \in (0,1)$, $\sum_{n=1}^{\infty} \tau_n < \infty$ and $\{\kappa_n\} \subset [\kappa, \bar{\kappa}]$, where $0 < \kappa \le \bar{\kappa}$, $\delta > 0$, $\bar{d}$ and $\bar{e} \in \mathbb{R}$. Choose $x_0, x_1 \in Q$ and γ_n such that $0 \le \gamma_n \le \bar{\gamma}_n$, where

$$\bar{\gamma}_n = \begin{cases} \beta & \text{if } x_n = x_{n-1} \text{ and } y_n = y_{n-1} \\ \min\{\frac{\tau_n}{\sqrt{\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|}}, \beta\} & \text{if otherwise.} \end{cases} \tag{3.1}$$

Compute

$$\begin{cases} w_n = x_n + \gamma_n(x_n - x_{n-1}), \\ t_n = y_n + \gamma_n(y_n - y_{n-1}), \\ h_n = \sigma_n \psi_1(x_n) + (1 - \sigma_n) P_{Q_1}(I - \nu_n^x \sum_{j=1}^{M} b_j^x L_j^x) w_n, \\ x_{n+1} = \delta_n h_n + \mu_n((1 - a_n) h_n + a_n S_n(h_n)) \\ \qquad + \sum_{i=1}^{\infty} \eta_{n,i}(J_{\theta_i}^{K_i}(h_n - \rho_n(A^*(Ah_n - Bv_n)))), \\ v_n = \sigma_n \psi_2(y_n) + (1 - \sigma_n) P_{Q_2}(I - \nu_n^y \sum_{j=1}^{M} b_j^y L_j^y) t_n, \\ y_{n+1} = \delta_n v_n + \mu_n((1 - a_n) v_n + a_n U_n(v_n)) \\ \qquad + \sum_{i=1}^{\infty} \eta_{n,i}(J_{\theta_i}^{T_i}(v_n + \rho_n(B^*(Ah_n - Bv_n)))), \end{cases} \tag{3.2}$$

where the stepsize ρ_n is given as

$$\rho_n \in \left(\epsilon, \frac{2\|Ah_n - Bv_n\|^2}{\|A^*(Ah_n - Bv_n)\|^2 + \|B^*(Ah_n - Bv_n)\|^2} - \epsilon\right),$$

$n \in \Delta$ and $\epsilon > 0$ otherwise $\rho_n = \rho$, and the index set $\Delta = \{n; Ah_n - Bv_n \neq 0\}$.

Remark 3.2 Assume that the solution set Γ is nonempty, then $\{\rho_n\}$ in Algorithm 3.1 is well defined; see [54].

Lemma 3.3 *Let the solution set Ω be nonempty, $\{x_n\}$ and $\{y_n\}$ be bounded sequences. Suppose that $\|l_n - J_{\theta_i}^{(K_i,T_i)}(I - \rho_n(H^*H)l_n)\| \to 0$, $\|h_n - w_n\| \to 0$, $\|v_n - t_n\| \to 0$, $\|x_n - w_n\| \to 0$, and $\|y_n - t_n\| \to 0$. Then $l^* = (x^*, y^*)$ is a solution of Problem (1.5) iff $l^* = J_{\theta_i}^{(K_i,T_i)}(I -$*

$\rho_n(H^*H))l^*$ *for every* $\theta_i > 0$ *and* $\rho_n > 0$, *where* $i \geq 1$, $n \in \mathbb{N}$,

$$H = \begin{bmatrix} A & -B \end{bmatrix}, \qquad H^* = \begin{bmatrix} A^* \\ -B^* \end{bmatrix},$$

$$H^*H = \begin{bmatrix} A^*A & -A^*B \\ -B^*A & B^*B \end{bmatrix} \quad \textit{and} \quad J_{\theta_i}^{(K_i,T_i)} = \begin{bmatrix} J_{\theta_i}^{K_i} \\ J_{\theta_i}^{T_i} \end{bmatrix}.$$

Proof Now, we have to show that $l^* = (x^*, y^*) \in \Omega$. For this we have to show for $i \geq 1$, $l^* = J_{\theta_i}^{(K_i,T_i)}(I - \rho_n(H^*H))l^*$. Suppose that $l^* \neq J_{\theta_i}^{(K_i,T_i)}(I - \rho_n(H^*H))l^*$. As $\{x_n\}$ and $\{y_n\}$ are bounded, take subsequences $\{x_{n_q}\}$ of $\{x_n\}$ and $\{y_{n_q}\}$ of $\{y_n\}$ such that $x_{n_q} \rightharpoonup x^* \in H_1$ and $y_{n_q} \rightharpoonup y^* \in H_2$. Additionally, from given conditions, there are subsequences $\{h_{n_q}\}$ $\{w_{n_q}\}$, $\{v_{n_q}\}$, and $\{t_{n_q}\}$ respectively such that $h_{n_q} \rightharpoonup x^*$, $w_{n_q} \rightharpoonup x^*$, $v_{n_q} \rightharpoonup y^*$, and $t_{n_q} \rightharpoonup y^*$. Define $l_n = (h_n, v_n)$. Using the given condition, we have

$$\lim_{q\to\infty} \left\| l_{n_q} - \left(J_{\theta_i}^{(K_i,T_i)}\left(I - \rho_{n_q}\left(H^*H\right)l_{n_q}\right)\right) \right\| = 0. \tag{3.3}$$

Now, using Opial's condition and equation (3.3), we get

$$\begin{aligned}
\liminf_{q\to\infty} \left\| l_{n_q} - l^* \right\| &< \liminf_{q\to\infty} \left\| l_{n_q} - J_{\theta_i}^{(K_i,T_i)}\left(I - \rho_n\left(H^*H\right)\right)l^* \right\| \\
&\leq \liminf_{q\to\infty} \left\| l_{n_q} - J_{\theta_i}^{(K_i,T_i)}\left(I - \rho_n\left(H^*H\right)\right)l_{n_q} \right\| \\
&\quad + \liminf_{q\to\infty} \left\| J_{\theta_i}^{(K_i,T_i)}\left(I - \rho_n\left(H^*H\right)\right)l_{n_q} - J_{\theta_i}^{(K_i,T_i)}\left(I - \rho_n\left(H^*H\right)\right)l^* \right\| \\
&\leq \liminf_{q\to\infty} \left\| l_{n_q} - l^* \right\|,
\end{aligned}$$

which is a contradiction. Hence $l^* = J_{\theta_i}^{(K_i,T_i)}(I - \rho_n(H^*H))l^*$ for $i \geq 1$.

Conversely, assume that $l^* = J_{\theta_i}^{(K_i,T_i)}(I - \rho_n(H^*H))l^*$. Subsequently,

$$\begin{cases} x^* = J_{\theta_i}^{K_i}(x^* - \rho_n(A^*(Ax^* - By^*))) \\ y^* = J_{\theta_i}^{T_i}(y^* + \rho_n(B^*(Ax^* - By^*))). \end{cases} \tag{3.4}$$

As the solution set is nonempty, $K_i^{-1}(0)$ and $T_i^{-1}(0)$ are nonempty. Using Lemma 2.9 and equation (3.4), we have

$$\left\langle x^* - \left(x^* - \rho_n\left(A^*\left(Ax^* - By^*\right)\right)\right), x_1 - x^* \right\rangle \geq 0 \quad \text{for all } x_1 \in K_i^{-1}(0),$$

which implies

$$\left\langle Ax^* - By^*, Ax_1 - Ax^* \right\rangle \geq 0 \quad \text{for all } x_1 \in K_i^{-1}(0). \tag{3.5}$$

Similarly,

$$\left\langle Ax^* - By^*, By^* - By_1 \right\rangle \geq 0 \quad \text{for all } y_1 \in T_i^{-1}(0). \tag{3.6}$$

Adding equations (3.5) and (3.6), we have

$$\langle Ax^* - By^*, Ax_1 - Ax^* + By^* - By_1\rangle \geq 0 \quad \text{for all } x_1 \in K_i^{-1}(0) \text{ and } y_1 \in T_i^{-1}(0). \tag{3.7}$$

On simplifying

$$\|Ax^* - By^*\|^2 \leq \langle Ax^* - By^*, Ax_1 - By_1\rangle \quad \text{for all } x_1 \in K_i^{-1}(0) \text{ and } y_1 \in T_i^{-1}(0).$$

As the solution set is nonempty, let $l' = (x', y') \in \Omega$ for each $i \geq 1$. Therefore, $x' \in K_i^{-1}(0)$ and $y' \in T_i^{-1}(0)$ and $Ax' = By'$. Taking $x_1 = x'$ and $y_1 = y'$ in equation (3.7), we have

$$\|Ax^* - By^*\| = 0,$$

which implies $Ax^* = By^*$, and using $Ax^* = By^*$ in equation (3.4), we have

$$\begin{cases} x^* = J_{\theta_i}^{K_i}(x^*) \\ y^* = J_{\theta_i}^{T_i}(y^*), \end{cases} \tag{3.8}$$

which implies $0 \in K_i(x^*)$ and $0 \in T_i(y^*)$ for all $i \geq 1$. Hence, l^* is the solution of Problem (1.5). □

Theorem 3.4 *Let the sequence* $\{(x_n, y_n)\}$ *be generated by the iterative Algorithm* 3.1 *and a solution set* Γ *be nonempty. Suppose that the following conditions are satisfied*:

(i) $\sum_{j=1}^M b_j^x = \sum_{j=1}^M b_j^y = 1$, $\sum_{n=1}^\infty \nu_n^x < \infty$, $\sum_{n=1}^\infty \nu_n^y < \infty$,

$$0 < a < \nu_n^x \leq 2\bar{c}^x, \qquad 0 < b < \nu_n^y \leq 2\bar{c}^y \quad \text{for some } a, b \in \mathbb{R};$$

(ii) $\tau_n = o(\sigma_n)$, $\lim_{n\to\infty} \frac{\tau_n}{\sigma_n} = 0$, $\lim_{n\to\infty} \sigma_n = 0$, $\sum_{n=1}^\infty \sigma_n = \infty$;

(iii) $\delta_n + \mu_n + \sum_{i=1}^\infty \eta_{n,i} = 1$ *for* $n \geq 0$,

then the sequence generated by the iterative Algorithm 3.1 *converges strongly to* $(\bar{r}^*, \bar{s}^*) \in \Gamma$.

Proof Firstly, we will show that the sequence $\{(x_n, y_n)\}$ is bounded. Take $(\bar{r}^*, \bar{s}^*) \in \Gamma$. Let

$$u_n^1 = h_n - \rho_n\big(A^*(Ah_n - Bv_n)\big) \quad \text{and} \quad u_n^2 = v_n + \rho_n\big(B^*(Ah_n - Bv_n)\big), \tag{3.9}$$

then using Lemma 2.3, we estimate

$$\begin{aligned} \|u_n^1 - \bar{r}^*\|^2 &= \|h_n - \rho_n\big(A^*(Ah_n - Bv_n)\big) - \bar{r}^*\|^2 \\ &= \|h_n - \bar{r}^*\|^2 + \rho_n^2\|A^*(Ah_n - Bv_n)\|^2 - 2\rho_n\langle h_n - \bar{r}^*, A^*(Ah_n - Bv_n)\rangle \\ &= \|h_n - \bar{r}^*\|^2 + \rho_n^2\|A^*(Ah_n - Bv_n)\|^2 - 2\rho_n\langle Ah_n - A\bar{r}^*, Ah_n - Bv_n\rangle \\ &= \|h_n - \bar{r}^*\|^2 + \rho_n^2\|A^*(Ah_n - Bv_n)\|^2 - \rho_n\|Ah_n - A\bar{r}^*\|^2 \\ &\quad - \rho_n\|Ah_n - Bv_n\|^2 + \rho_n\|Bv_n - A\bar{r}^*\|^2 \end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\left\|u_n^2-\bar{s}^*\right\|^2 &= \left\|v_n-\bar{s}^*\right\|^2+\rho_n^2\left\|B^*(Ah_n-Bv_n)\right\|^2-\rho_n\left\|Bv_n-B\bar{s}^*\right\|^2\\ &\quad-\rho_n\|Ah_n-Bv_n\|^2+\rho_n\left\|Ah_n-B\bar{s}^*\right\|^2. \end{aligned}\tag{3.11}$$

As $A\bar{r}^*=B\bar{s}^*$ and adding equations (3.10) and (3.11), we have

$$\begin{aligned}&\left\|u_n^1-\bar{r}^*\right\|^2+\left\|u_n^2-\bar{s}^*\right\|^2\\ &\quad=\left\|h_n-\bar{r}^*\right\|^2+\left\|v_n-\bar{s}^*\right\|^2+\rho_n^2\left(\left\|A^*(Ah_n-Bv_n)\right\|^2+\left\|B^*(Ah_n-Bv_n)\right\|^2\right)\\ &\qquad-2\rho_n\|Ah_n-Bv_n\|^2\\ &\quad=\left\|h_n-\bar{r}^*\right\|^2+\left\|v_n-\bar{s}^*\right\|^2-\rho_n\left[2\|Ah_n-Bv_n\|^2-\rho_n\left(\left\|A^*(Ah_n-Bv_n)\right\|^2\right.\right.\\ &\qquad\left.\left.+\left\|B^*(Ah_n-Bv_n)\right\|^2\right)\right]\\ &\quad\le\left\|h_n-\bar{r}^*\right\|^2+\left\|v_n-\bar{s}^*\right\|^2-\rho_n\epsilon\left(\left\|A^*(Ah_n-Bv_n)\right\|^2+\left\|B^*(Ah_n-Bv_n)\right\|^2\right) &(3.12)\\ &\quad\le\left\|h_n-\bar{r}^*\right\|^2+\left\|v_n-\bar{s}^*\right\|^2. &(3.13)\end{aligned}$$

Let $z_n^1=P_{Q_1}(I-\nu_n^x\sum_{j=1}^M b_j^xL_j^x)w_n$ and $z_n^2=P_{Q_2}(I-\nu_n^y\sum_{j=1}^M b_j^yL_j^y)t_n$.

As $\bar{r}^*\in\mathrm{VIP}(L_j^x,Q_1)$. Also, from the definition of inverse strongly monotone and from condition (i), we have

$$\begin{aligned}\left\|z_n^1-\bar{r}^*\right\|^2&=\left\|P_{Q_1}\left(I-\nu_n^x\sum_{j=1}^M b_j^xL_j^x\right)w_n-P_{Q_1}\left(I-\nu_n^x\sum_{j=1}^M b_j^xL_j^x\right)\bar{r}^*\right\|^2\\ &\le\left\|\left(I-\nu_n^x\sum_{j=1}^M b_j^xL_j^x\right)w_n-\left(I-\nu_n^x\sum_{j=1}^M b_j^xL_j^x\right)\bar{r}^*\right\|^2\\ &=\left\|\left(w_n-\bar{r}^*\right)-\nu_n^x\sum_{j=1}^M b_j^x\left(L_j^xw_n-L_j^x\bar{r}^*\right)\right\|^2\\ &\le\left\|w_n-\bar{r}^*\right\|^2-2\nu_n^x\sum_{j=1}^M b_j^x\left\langle w_n-\bar{r}^*,L_j^xw_n-L_j^x\bar{r}^*\right\rangle\\ &\quad+\sum_{j=1}^M b_j^x\left(\nu_n^x\right)^2\left\|L_j^xw_n-L_j^x\bar{r}^*\right\|^2\\ &\le\left\|w_n-\bar{r}^*\right\|^2-2\nu_n^x\sum_{j=1}^M b_j^xc_j^x\left\|L_j^xw_n-L_j^x\bar{r}^*\right\|^2+\sum_{j=1}^M b_j^x\left(\nu_n^x\right)^2\left\|L_j^xw_n-L_j^x\bar{r}^*\right\|^2\\ &\le\left\|w_n-\bar{r}^*\right\|^2-\nu_n^x\sum_{j=1}^M b_j^x\left(2\bar{c}^x-\nu_n^x\right)\left\|L_j^xw_n-L_j^x\bar{r}^*\right\|^2 &(3.14)\\ &\le\left\|w_n-\bar{r}^*\right\|^2. &(3.15)\end{aligned}$$

Similarly,

$$\|z_n^2 - \bar{s}^*\|^2 = \|t_n - \bar{s}^*\|^2 - \nu_n^y \sum_{j=1}^{M} b_j^y (2\bar{c}^y - \nu_n^y) \|L_j^y t_n - L_j^y \bar{s}^*\|^2 \tag{3.16}$$

$$\leq \|t_n - \bar{s}^*\|^2. \tag{3.17}$$

Adding equations (3.14) and (3.16), we have

$$\|z_n^1 - \bar{r}^*\|^2 + \|z_n^2 - \bar{s}^*\|^2 \leq \|w_n - \bar{r}^*\|^2 + \|t_n - \bar{s}^*\|^2 - \nu_n^x \sum_{j=1}^{M} b_j^x (2\bar{c}^x - \nu_n^x) \|L_j^x w_n - L_j^x \bar{r}^*\|^2$$

$$- \nu_n^y \sum_{j=1}^{M} b_j^y (2\bar{c}^y - \nu_n^y) \|L_j^y t_n - L_j^y \bar{s}^*\|^2 \tag{3.18}$$

$$\leq \|w_n - \bar{r}^*\|^2 + \|t_n - \bar{s}^*\|^2. \tag{3.19}$$

As $\bar{r}^* \in \mathrm{VIP}(L_j^x, Q_1)$, we have

$$\begin{aligned}
\|z_n^1 - \bar{r}^*\|^2 &= \left\| P_{Q_1}\left(I - \nu_n^x \sum_{j=1}^{M} b_j^x L_j^x\right) w_n - P_{Q_1}\left(I - \nu_n^x \sum_{j=1}^{M} b_j^x L_j^x\right) \bar{r}^* \right\|^2 \\
&\leq \left\langle z_n^1 - \bar{r}^*, \left(I - \nu_n^x \sum_{j=1}^{M} b_j^x L_j^x\right) w_n - \left(I - \nu_n^x \sum_{j=1}^{M} b_j^x L_j^x\right) \bar{r}^* \right\rangle \\
&\leq \frac{1}{2}\{\|z_n^1 - \bar{r}^*\|^2 + \left\| \left(I - \nu_n^x \sum_{j=1}^{M} b_j^x L_j^x\right) w_n - \left(I - \nu_n^x \sum_{j=1}^{M} b_j^x L_j^x\right) \bar{r}^* \right\|^2 \\
&\quad - \left\| (z_n^1 - w_n) + \nu_n^x \sum_{j=1}^{M} b_j^x \left(L_j^x(w_n) - L_j^x(\bar{r}^*)\right) \right\|^2.
\end{aligned} \tag{3.20}$$

Using $I - \nu_n^x \sum_{j=1}^{M} b_j^x L_j^x$ is nonexpansive and on rearranging the terms, we have

$$\begin{aligned}
\|z_n^1 - \bar{r}^*\|^2 &\leq \|w_n - \bar{r}^*\|^2 - \|z_n^1 - w_n\|^2 - \left(\nu_n^x \sum_{j=1}^{M} b_j^x\right)^2 \|L_j^x(w_n) - L_j^x(\bar{r}^*)\|^2 \\
&\quad + 2\nu_n^x \sum_{j=1}^{M} b_j^x \langle w_n - z_n^1, L_j^x(w_n) - L_j^x(\bar{r}^*)\rangle \\
&\leq \|w_n - \bar{r}^*\|^2 - \|z_n^1 - w_n\|^2 + 2\nu_n^x \sum_{j=1}^{M} b_j^x \|w_n - z_n^1\| \|L_j^x(w_n) - L_j^x(\bar{r}^*)\|.
\end{aligned} \tag{3.21}$$

Similarly,

$$\|z_n^2 - \bar{s}^*\|^2 \leq \|t_n - \bar{s}^*\|^2 - \|z_n^2 - t_n\|^2 + 2\nu_n^y \sum_{j=1}^{M} b_j^y \|t_n - z_n^2\| \|L_j^y(t_n) - L_j^y(\bar{s}^*)\|. \tag{3.22}$$

Adding equations (3.21) and (3.22), we have

$$\begin{aligned}\|z_n^1-\bar{r}^*\|^2+\|z_n^2-\bar{s}^*\|^2 &\le \|w_n-\bar{r}^*\|^2+\|t_n-\bar{s}^*\|^2-\|z_n^1-w_n\|^2-\|z_n^2-t_n\|^2\\ &\quad+2\nu_n^x\sum_{j=1}^M b_j^x\|w_n-z_n^1\|\|L_j^x(w_n)-L_j^x(\bar{r}^*)\|\\ &\quad+2\nu_n^y\sum_{j=1}^M b_j^y\|t_n-z_n^2\|\|L_j^y(t_n)-L_j^y(\bar{s}^*)\|. \end{aligned}\tag{3.23}$$

Using equation (3.2), we get

$$\begin{aligned}\|w_n-\bar{r}^*\| &= \|x_n+\gamma_n(x_n-x_{n-1})-\bar{r}^*\|\\ &\le \|x_n-\bar{r}^*\|+\gamma_n\|x_n-x_{n-1}\|. \end{aligned}\tag{3.24}$$

Similarly,

$$\|t_n-\bar{s}^*\|\le\|y_n-\bar{s}^*\|+\gamma_n\|y_n-y_{n-1}\|. \tag{3.25}$$

As $\lim_{n\to\infty}\frac{\tau_n}{\sigma_n}=0$, there is $M_1>0$ such that $\tau_n\le\sigma_n M_1$. Using equations (3.24) and (3.25) and using Minkowski's inequality, we get

$$\begin{aligned}&\left(\|w_n-\bar{r}^*\|^2+\|t_n-\bar{s}^*\|^2\right)^{\frac12}\\ &\quad\le\left(\left(\|x_n-\bar{r}^*\|+\gamma_n\|x_n-x_{n-1}\|\right)^2+\left(\|y_n-\bar{s}^*\|+\gamma_n\|y_n-y_{n-1}\|\right)^2\right)^{\frac12}\\ &\quad\le\left(\|x_n-\bar{r}^*\|^2+\|y_n-\bar{s}^*\|^2\right)^{\frac12}+\gamma_n\left(\|x_n-x_{n-1}\|^2+\|y_n-y_{n-1}\|^2\right)^{\frac12}\\ &\quad\le\left(\|x_n-\bar{r}^*\|^2+\|y_n-\bar{s}^*\|^2\right)^{\frac12}+\tau_n\\ &\quad\le\left(\|x_n-\bar{r}^*\|^2+\|y_n-\bar{s}^*\|^2\right)^{\frac12}+\sigma_n M_1. \end{aligned}\tag{3.26}$$

Now, using Lemma 2.5, we have

$$\begin{aligned}\|(S_n)_{a_n}(h_n)-\bar{r}^*\|^2 &= \|((1-a_n)h_n+a_nS_n(h_n))-\bar{r}^*\|^2\\ &\le\|h_n-\bar{r}^*\|^2-a_n(1-a_n-\kappa_{k_1})\|S_n(h_n)-h_n\|^2\\ &\le\|h_n-\bar{r}^*\|^2-a_n(1-a_n-\kappa)\|S_n(h_n)-h_n\|^2 \qquad (3.27)\\ &\le\|h_n-\bar{r}^*\|^2 \qquad (3.28)\end{aligned}$$

and

$$\|(U_n)_{a_n}(v_n)-\bar{s}^*\|^2\le\|v_n-\bar{s}^*\|^2-a_n(1-a_n-\phi)\|U_n(v_n)-v_n\|^2. \tag{3.29}$$

Using Lemma 2.3, condition (iii), and equation (3.27), we have

$$\|x_{n+1}-\bar{r}^*\|^2=\left\|\delta_nh_n+\mu_n\left((1-a_n)h_n+a_nS_n(h_n)\right)+\sum_{i=1}^\infty\eta_{n,i}J_{\theta_i}^{K_i}(u_n^1)-\bar{r}^*\right\|^2$$

$$
\begin{aligned}
&\leq \delta_n \|h_n - \bar{r}^*\|^2 + \mu_n\left(\|h_n - \bar{r}^*\|^2 - a_n(1-a_n-\kappa)\|S_n(h_n) - h_n\|^2\right) \\
&\quad + \sum_{i=1}^{\infty} \eta_{n,i} \|J_{\theta_i}^{K_i}(u_n^1) - \bar{r}^*\|^2 \\
&\leq \left(1 - \sum_{i=1}^{\infty} \eta_{n,i}\right)\|h_n - \bar{r}^*\|^2 + \sum_{i=1}^{\infty} \eta_{n,i}\|u_n^1 - \bar{r}^*\|^2 \\
&\quad - \mu_n a_n(1-a_n-\kappa)\|S_n(h_n) - h_n\|^2. \qquad (3.30)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\|y_{n+1} - \bar{s}^*\|^2 &\leq \left(1 - \sum_{i=1}^{\infty} \eta_{n,i}\right)\|v_n - \bar{s}^*\|^2 + \sum_{i=1}^{\infty} \eta_{n,i}\|u_n^2 - \bar{s}^*\|^2 \\
&\quad - \mu_n a_n(1-a_n-\phi)\|U_n(v_n) - v_n\|^2. \qquad (3.31)
\end{aligned}
$$

Adding equations (3.30), (3.31) and using equation (3.13), we have

$$
\begin{aligned}
&\|x_{n+1} - \bar{r}^*\|^2 + \|y_{n+1} - \bar{s}^*\|^2 \\
&\leq \left(1 - \sum_{i=1}^{\infty} \eta_{n,i}\right)\left(\|h_n - \bar{r}^*\|^2 + \|v_n - \bar{s}^*\|^2\right) + \sum_{i=1}^{\infty} \eta_{n,i}\left[\|u_n^1 - \bar{r}^*\|^2 + \|u_n^2 - \bar{s}^*\|^2\right] \\
&\quad - \mu_n a_n(1-a_n-\phi)\|U_n(v_n) - v_n\|^2 - \mu_n a_n(1-a_n-\kappa)\|S_n(h_n) - h_n\|^2 \\
&\leq \|h_n - \bar{r}^*\|^2 + \|v_n - \bar{s}^*\|^2 - \mu_n a_n(1-a_n-\phi)\|U_n(v_n) - v_n\|^2 \\
&\quad - \mu_n a_n(1-a_n-\kappa)\|S_n(h_n) - h_n\|^2 \qquad (3.32) \\
&\leq \|h_n - \bar{r}^*\|^2 + \|v_n - \bar{s}^*\|^2. \qquad (3.33)
\end{aligned}
$$

Using equations (3.2), (3.18), and (3.33), we estimate

$$
\begin{aligned}
&\|x_{n+1} - \bar{r}^*\|^2 + \|y_{n+1} - \bar{s}^*\|^2 \\
&\leq \|h_n - \bar{r}^*\|^2 + \|v_n - \bar{s}^*\|^2 \\
&= \|\sigma_n\psi_1(x_n) + (1-\sigma_n)z_n^1 - \bar{r}^*\|^2 + \|\sigma_n\psi_2(y_n) + (1-\sigma_n)z_n^2 - \bar{s}^*\|^2 \\
&\leq \sigma_n\|\psi_1(x_n) - \bar{r}^*\|^2 + (1-\sigma_n)\|z_n^1 - \bar{r}^*\|^2 + \sigma_n\|\psi_2(y_n) - \bar{s}^*\|^2 \\
&\quad + (1-\sigma_n)\|z_n^2 - \bar{s}^*\|^2 \qquad (3.34) \\
&\leq \sigma_n\left(\|\psi_1(x_n) - \bar{r}^*\|^2 + \|\psi_2(y_n) - \bar{s}^*\|^2\right) + (1-\sigma_n)\Bigg(\|w_n - \bar{r}^*\|^2 + \|t_n - \bar{s}^*\|^2 \\
&\quad - \nu_n^x \sum_{j=1}^{M} b_j^x\left(2\bar{c}^x - \nu_n^x\right)\|L_j^x w_n - L_j^x\bar{r}^*\|^2 - \nu_n^y \sum_{j=1}^{M} b_j^y\left(2\bar{c}^y - \nu_n^y\right)\|L_j^y t_n - L_j^y\bar{s}^*\|^2\Bigg) \qquad (3.35) \\
&\leq \sigma_n\left(\|\psi_1(x_n) - \bar{r}^*\|^2 + \|\psi_2(y_n) - \bar{s}^*\|^2\right) + \|w_n - \bar{r}^*\|^2 + \|t_n - \bar{s}^*\|^2. \qquad (3.36)
\end{aligned}
$$

As $\lambda = \max\{\lambda_1, \lambda_2\}$, using equations (3.2), (3.19), (3.26), and Minkowski's inequality, we have

$$\begin{aligned}
&\big(\|h_n - \bar{r}^*\|^2 + \|\nu_n - \bar{s}^*\|^2\big)^{\frac{1}{2}} \\
&\quad = \big(\|\sigma_n \psi_1(x_n) + (1-\sigma_n) z_n^1 - \bar{r}^*\|^2 + \|\sigma_n \psi_2(y_n) + (1-\sigma_n) z_n^2 - \bar{s}^*\|^2\big)^{\frac{1}{2}} \\
&\quad \le \Big(\big(\sigma_n \|\psi_1(x_n) - \bar{r}^*\| + (1-\sigma_n)\|z_n^1 - \bar{r}^*\|\big)^2 + \big(\sigma_n \|\psi_2(y_n) - \bar{s}^*\| \\
&\qquad + (1-\sigma_n)\|z_n^2 - \bar{s}^*\|\big)^2\Big)^{\frac{1}{2}} \\
&\quad \le \Big(\big(\sigma_n \|\psi_1(x_n) - \bar{r}^*\| + (1-\sigma_n)\|w_n - \bar{r}^*\|\big)^2 + \big(\sigma_n \|\psi_2(y_n) - \bar{s}^*\| \\
&\qquad + (1-\sigma_n)\|t_n - \bar{s}^*\|\big)^2\Big)^{\frac{1}{2}} \\
&\quad \le \sigma_n\big(\|\psi_1(x_n) - \bar{r}^*\|^2 + \|\psi_2(y_n) - \bar{s}^*\|^2\big)^{\frac{1}{2}} + (1-\sigma_n)\big(\|w_n - \bar{r}^*\|^2 + \|t_n - \bar{s}^*\|^2\big)^{\frac{1}{2}} \\
&\quad \le \sigma_n\Big(\big(\|\psi_1(x_n) - \psi_1(\bar{r}^*)\| + \|\psi_1(\bar{r}^*) - \bar{r}^*\|\big)^2 + \big(\|\psi_2(y_n) - \psi_2(\bar{s}^*)\| \\
&\qquad + \|\psi_2(\bar{s}^*) - \bar{s}^*\|\big)^2\Big)^{\frac{1}{2}} \\
&\qquad + (1-\sigma_n)\Big(\big(\|x_n - \bar{r}^*\|^2 + \|y_n - \bar{s}^*\|^2\big)^{\frac{1}{2}} + \sigma_n M_1\Big) \\
&\quad \le \sigma_n\big(\|\psi_1(x_n) - \psi_1(\bar{r}^*)\|^2 + \|\psi_2(y_n) - \psi_2(\bar{s}^*)\|^2\big)^{\frac{1}{2}} + \big(\|\psi_1(\bar{r}^*) - \bar{r}^*\|^2 \\
&\qquad + \|\psi_2(\bar{s}^*) - \bar{s}^*\|^2\big)^{\frac{1}{2}} \\
&\qquad + (1-\sigma_n)\big(\|x_n - \bar{r}^*\|^2 + \|y_n - \bar{s}^*\|^2\big)^{\frac{1}{2}} + \sigma_n M_1 \\
&\quad \le \sigma_n\Big(\big(\lambda_1^2\|x_n - \bar{r}^*\|^2 + \lambda_2^2\|y_n - \bar{s}^*\|^2\big)^{\frac{1}{2}} \\
&\qquad + \big(\|\psi_1(\bar{r}^*) - \bar{r}^*\|^2 + \|\psi_2(\bar{s}^*) - \bar{s}^*\|^2\big)^{\frac{1}{2}}\Big) \\
&\qquad + (1-\sigma_n)\big(\|x_n - \bar{r}^*\|^2 + \|y_n - \bar{s}^*\|^2\big)^{\frac{1}{2}} + \sigma_n M_1 \\
&\quad \le \big(1 - \sigma_n(1-\lambda)\big)\big(\|x_n - \bar{r}^*\|^2 + \|y_n - \bar{s}^*\|^2\big)^{\frac{1}{2}} \\
&\qquad + \sigma_n\big(\|\psi_1(\bar{r}^*) - \bar{r}^*\|^2 + \|\psi_2(\bar{s}^*) - \bar{s}^*\|^2\big)^{\frac{1}{2}} + \sigma_n M_1.
\end{aligned} \tag{3.37}$$

Using equations (3.23), (3.33), and (3.34), we estimate

$$\begin{aligned}
\|x_{n+1} - \bar{r}^*\|^2 + \|y_{n+1} - \bar{s}^*\|^2 &\le \|h_n - \bar{r}^*\|^2 + \|\nu_n - \bar{s}^*\|^2 \\
&\le \sigma_n\big(\|\psi_1(x_n) - \bar{r}^*\|^2 + \|\psi_2(y_n) - \bar{s}^*\|^2\big) \\
&\quad + (1-\sigma_n)\Bigg(\|w_n - \bar{r}^*\|^2 + \|t_n - \bar{s}^*\|^2 - \|z_n^1 - w_n\|^2 \\
&\quad - \|z_n^2 - t_n\|^2 + 2\nu_n^x \sum_{j=1}^{M} b_j^x \|w_n - z_n^1\| \|L_j^x(w_n) - L_j^x(\bar{r}^*)\| \\
&\quad + 2\nu_n^y \sum_{j=1}^{M} b_j^y \|t_n - z_n^2\| \|L_j^y(t_n) - L_j^y(\bar{s}^*)\|\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sigma_n\left(\left\|\psi_1(x_n)-\bar{r}^*\right\|^2+\left\|\psi_2(y_n)-\bar{s}^*\right\|^2\right)+\left\|w_n-\bar{r}^*\right\|^2 \\
&\quad+\left\|t_n-\bar{s}^*\right\|^2-\left\|z_n^1-w_n\right\|^2-\left\|z_n^2-t_n\right\|^2 \\
&\quad+2\nu_n^x\sum_{j=1}^{M}b_j^x\left\|w_n-z_n^1\right\|\left\|L_j^x(w_n)-L_j^x(\bar{r}^*)\right\| \\
&\quad+2\nu_n^y\sum_{j=1}^{M}b_j^y\left\|t_n-z_n^2\right\|\left\|L_j^y(t_n)-L_j^y(\bar{s}^*)\right\|. \qquad (3.38)
\end{aligned}
$$

From Lemma 2.3 and equation (3.28), we have

$$
\begin{aligned}
\left\|x_{n+1}-\bar{r}^*\right\|^2 &= \left\|\delta_n h_n+\mu_n\left((1-a_n)h_n+a_nS_n(h_n)\right)+\sum_{i=1}^{\infty}\eta_{n,i}J_{\theta_i}^{K_i}(u_n^1)-\bar{r}^*\right\|^2 \\
&\leq \delta_n\left\|h_n-\bar{r}^*\right\|^2+\mu_n\left\|(1-a_n)h_n+a_nS_n(h_n)-\bar{r}^*\right\|^2 \\
&\quad+\sum_{i=1}^{\infty}\eta_{n,i}\left\|J_{\theta_i}^{K_i}(u_n^1)-\bar{r}^*\right\|^2-\delta_n\mu_n\left\|h_n-(1-a_n)h_n-a_nS_n(h_n)\right\|^2 \\
&\quad-\delta_n\sum_{i=1}^{\infty}\eta_{n,i}\left\|h_n-J_{\theta_i}^{K_i}(u_n^1)\right\|^2 \\
&\leq \delta_n\left\|h_n-\bar{r}^*\right\|^2+\mu_n\left\|h_n-\bar{r}^*\right\|^2+\sum_{i=1}^{\infty}\eta_{n,i}\left\|J_{\theta_i}^{K_i}(u_n^1)-\bar{r}^*\right\|^2 \\
&\quad-\delta_n\mu_n a_n\left\|h_n-S_n(h_n)\right\|^2-\delta_n\sum_{i=1}^{\infty}\eta_{n,i}\left\|h_n-J_{\theta_i}^{K_i}(u_n^1)\right\|^2 \\
&\leq \left(1-\sum_{i=1}^{\infty}\eta_{n,i}\right)\left\|h_n-\bar{r}^*\right\|^2+\sum_{i=1}^{\infty}\eta_{n,i}\left\|u_n^1-\bar{r}^*\right\|^2 \\
&\quad-\delta_n\mu_n a_n\left\|h_n-S_n(h_n)\right\|^2-\delta_n\sum_{i=1}^{\infty}\eta_{n,i}\left\|h_n-J_{\theta_i}^{K_i}(u_n^1)\right\|^2 \qquad (3.39) \\
&\leq \left(1-\sum_{i=1}^{\infty}\eta_{n,i}\right)\left\|h_n-\bar{r}^*\right\|^2+\sum_{i=1}^{\infty}\eta_{n,i}\left\|u_n^1-\bar{r}^*\right\|^2. \qquad (3.40)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\left\|y_{n+1}-\bar{s}^*\right\|^2 &\leq \left(1-\sum_{i=1}^{\infty}\eta_{n,i}\right)\left\|v_n-\bar{s}^*\right\|^2+\sum_{i=1}^{\infty}\eta_{n,i}\left\|u_n^2-\bar{s}^*\right\|^2 \\
&\quad-\delta_n\mu_n a_n\left\|v_n-U_n(v_n)\right\|^2-\delta_n\sum_{i=1}^{\infty}\eta_{n,i}\left\|v_n-J_{\theta_i}^{T_i}(u_n^2)\right\|^2 \qquad (3.41) \\
&\leq \left(1-\sum_{i=1}^{\infty}\eta_{n,i}\right)\left\|v_n-\bar{s}^*\right\|^2+\sum_{i=1}^{\infty}\eta_{n,i}\left\|u_n^2-\bar{s}^*\right\|^2. \qquad (3.42)
\end{aligned}
$$

Using equation (3.13) and adding equations (3.39) and (3.41), we get

$$
\begin{aligned}
\|x_{n+1}-\bar{r}^*\|^2+\|y_{n+1}-\bar{s}^*\|^2 \le{}& \left(1-\sum_{i=1}^{\infty}\eta_{n,i}\right)\|h_n-\bar{r}^*\|^2+\sum_{i=1}^{\infty}\eta_{n,i}\|u_n^1-\bar{r}^*\|^2 \\
&+\left(1-\sum_{i=1}^{\infty}\eta_{n,i}\right)\|v_n-\bar{s}^*\|^2+\sum_{i=1}^{\infty}\eta_{n,i}\|u_n^2-\bar{s}^*\|^2 \\
&-\delta_n\mu_n a_n\left(\|h_n-S_n(h_n)\|^2+\|U_n(v_n)-v_n\|^2\right) \\
&-\delta_n\sum_{i=1}^{\infty}\eta_{n,i}\left(\|h_n-J_{\theta_i}^{K_i}(u_n^1)\|^2+\|J_{\theta_i}^{T_i}(u_n^2)-v_n\|^2\right) \\
\le{}& \left(1-\sum_{i=1}^{\infty}\eta_{n,i}\right)\left(\|h_n-\bar{r}^*\|^2+\|v_n-\bar{s}^*\|^2\right) \\
&+\sum_{i=1}^{\infty}\eta_{n,i}\left(\|h_n-\bar{r}^*\|^2+\|v_n-\bar{s}^*\|^2\right) \\
&-\delta_n\mu_n a_n\left(\|h_n-S_n(h_n)\|^2+\|U_n(v_n)-v_n\|^2\right) \\
&-\delta_n\sum_{i=1}^{\infty}\eta_{n,i}\left(\|h_n-J_{\theta_i}^{K_i}(u_n^1))\|^2+\|J_{\theta_i}^{T_i}(u_n^2)-v_n\|^2\right) \\
\le{}& \|h_n-\bar{r}^*\|^2+\|v_n-\bar{s}^*\|^2 \\
&-\delta_n\mu_n a_n\left(\|h_n-S_n(h_n)\|^2+\|U_n(v_n)-v_n\|^2\right) \\
&-\delta_n\sum_{i=1}^{\infty}\eta_{n,i}\left(\|h_n-J_{\theta_i}^{K_i}(u_n^1)\|^2+\|(J_{\theta_i}^{T_i}(u_n^2)-v_n\|^2\right) \qquad (3.43)\\
\le{}& \|h_n-\bar{r}^*\|^2+\|v_n-\bar{s}^*\|^2. \qquad (3.44)
\end{aligned}
$$

Using equation (3.12) and adding equations (3.40) and (3.42), we get

$$
\begin{aligned}
\|x_{n+1}-\bar{r}^*\|^2+\|y_{n+1}-\bar{s}^*\|^2 \le{}& \left(1-\sum_{i=1}^{\infty}\eta_{n,i}\right)\|h_n-\bar{r}^*\|^2+\sum_{i=1}^{\infty}\eta_{n,i}\|u_n^1-\bar{r}^*\|^2 \\
&+\left(1-\sum_{i=1}^{\infty}\eta_{n,i}\right)\|v_n-\bar{s}^*\|^2+\sum_{i=1}^{\infty}\eta_{n,i}\|u_n^2-\bar{s}^*\|^2 \\
\le{}& \left(1-\sum_{i=1}^{\infty}\eta_{n,i}\right)\left(\|h_n-\bar{r}^*\|^2+\|v_n-\bar{s}^*\|^2\right) \\
&+\sum_{i=1}^{\infty}\eta_{n,i}\Big(\|h_n-\bar{r}^*\|^2+\|v_n-\bar{s}^*\|^2 \\
&-\rho_n\epsilon\left(\|A^*(Ah_n-Bv_n)\|^2+\|B^*(Ah_n-Bv_n)\|^2\right)\Big) \\
\le{}& \|h_n-\bar{r}^*\|^2+\|v_n-\bar{s}^*\|^2 \\
&-\sum_{i=1}^{\infty}\eta_{n,i}\rho_n\epsilon\Big(\|A^*(Ah_n-Bv_n)\|^2
\end{aligned}
$$

$$+ \left\|B^*(Ah_n - Bv_n)\right\|^2\big). \tag{3.45}$$

From equations (3.37) and (3.44), we have

$$\begin{aligned}
&\big(\left\|x_{n+1} - \bar{r}^*\right\|^2 + \left\|y_{n+1} - \bar{s}^*\right\|^2\big)^{\frac{1}{2}} \\
&\quad\le \big(\left\|h_n - \bar{r}^*\right\|^2 + \left\|v_n - \bar{s}^*\right\|^2\big)^{\frac{1}{2}} \\
&\quad\le \big(1 - \sigma_n(1-\lambda)\big)\big(\left\|x_n - \bar{r}^*\right\|^2 + \left\|y_n - \bar{s}^*\right\|^2\big)^{\frac{1}{2}} \\
&\qquad + \sigma_n\big(\left\|\psi_1(\bar{r}^*) - \bar{r}^*\right\|^2 + \left\|\psi_2(\bar{s}^*) - \bar{s}^*\right\|^2\big)^{\frac{1}{2}} + \sigma_n M_1 \\
&\quad\le \max\bigg\{\big(\left\|x_n - \bar{r}^*\right\|^2 + \left\|y_n - \bar{s}^*\right\|^2\big)^{\frac{1}{2}}, \\
&\qquad \frac{1}{1-\lambda}\big[\big(\left\|\psi_1(\bar{r}^*) - \bar{r}^*\right\|^2 + \left\|\psi_2(\bar{s}^*) - \bar{s}^*\right\|^2\big)^{\frac{1}{2}} + M_1\big]\bigg\} \\
&\quad\le \max\bigg\{\big(\left\|x_1 - \bar{r}^*\right\|^2 + \left\|y_1 - \bar{s}^*\right\|^2\big)^{\frac{1}{2}}, \\
&\qquad \frac{1}{1-\lambda}\big[\big(\left\|\psi_1(\bar{r}^*) - \bar{r}^*\right\|^2 + \left\|\psi_2(\bar{s}^*) - \bar{s}^*\right\|^2\big)^{\frac{1}{2}} + M_1\big]\bigg\}.
\end{aligned} \tag{3.46}$$

Hence $\{x_n\}$ and $\{y_n\}$ are bounded. Subsequently, $\{w_n\}$, $\{t_n\}$, $\{h_n\}$, and $\{v_n\}$ are also bounded.

$$\begin{aligned}
\left\|w_n - \bar{r}^*\right\|^2 &= \left\|x_n + \gamma_n(x_n - x_{n-1}) - \bar{r}^*\right\|^2 \\
&\le \left\|x_n - \bar{r}^*\right\|^2 + 2\gamma_n\langle x_n - x_{n-1}, w_n - \bar{r}^*\rangle \\
&\le \left\|x_n - \bar{r}^*\right\|^2 + 2\gamma_n\|x_n - x_{n-1}\|\left\|w_n - \bar{r}^*\right\| \\
&\le \left\|x_n - \bar{r}^*\right\|^2 + 2\tau_n\left\|w_n - \bar{r}^*\right\|.
\end{aligned} \tag{3.47}$$

Similarly,

$$\left\|t_n - \bar{s}^*\right\|^2 \le \left\|y_n - \bar{s}^*\right\|^2 + 2\tau_n\left\|t_n - \bar{s}^*\right\|. \tag{3.48}$$

Adding equations (3.47) and (3.48), we have

$$\left\|w_n - \bar{r}^*\right\|^2 + \left\|t_n - \bar{s}^*\right\|^2 \le \left\|x_n - \bar{r}^*\right\|^2 + \left\|y_n - \bar{s}^*\right\|^2 + 2\tau_n\big[\left\|w_n - \bar{r}^*\right\| + \left\|t_n - \bar{s}^*\right\|\big]. \tag{3.49}$$

From equation (3.2), we have

$$\begin{aligned}
\left\|h_n - \bar{r}^*\right\|^2 &= \left\|\sigma_n\psi_1(x_n) + (1-\sigma_n)z_n^1 - \bar{r}^*\right\|^2 \\
&= \sigma_n^2\left\|\psi_1(x_n) - \bar{r}^*\right\|^2 + (1-\sigma_n)^2\left\|z_n^1 - \bar{r}^*\right\|^2 \\
&\quad + 2\sigma_n(1-\sigma_n)\langle\psi_1(x_n) - \bar{r}^*, z_n^1 - \bar{r}^*\rangle \\
&= \sigma_n^2\left\|\psi_1(x_n) - \bar{r}^*\right\|^2 + (1-\sigma_n)^2\left\|z_n^1 - \bar{r}^*\right\|^2 + 2\sigma_n\langle\psi_1(x_n) - \bar{r}^*, z_n^1 - \bar{r}^*\rangle \\
&\quad - 2\sigma_n^2\langle\psi_1(x_n) - \bar{r}^*, z_n^1 - \bar{r}^*\rangle \\
&\le \sigma_n^2\left\|\psi_1(x_n) - \bar{r}^*\right\|^2 + (1-\sigma_n)^2\left\|z_n^1 - \bar{r}^*\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ 2\sigma_n\langle\psi_1(x_n) - \psi_1(\bar{r}^*), z_n^1 - \bar{r}^*\rangle + 2\sigma_n\langle\psi_1(\bar{r}^*) - \bar{r}^*, z_n^1 - \bar{r}^*\rangle \\
&+ 2\sigma_n^2\|\psi_1(x_n) - \bar{r}^*\|\|z_n^1 - \bar{r}^*\| \\
\leq{}& \sigma_n^2\|\psi_1(x_n) - \bar{r}^*\|^2 + (1-\sigma_n)^2\|z_n^1 - \bar{r}^*\|^2 \\
&+ 2\sigma_n\|\psi_1(x_n) - \psi_1(\bar{r}^*)\|\|z_n^1 - \bar{r}^*\| \\
&+ 2\sigma_n\langle\psi_1(\bar{r}^*) - \bar{r}^*, z_n^1 - \bar{r}^*\rangle + 2\sigma_n^2\|\psi_1(x_n) - \bar{r}^*\|\|z_n^1 - \bar{r}^*\| \\
\leq{}& \sigma_n^2\|\psi_1(x_n) - \bar{r}^*\|^2 + (1-\sigma_n)^2\|z_n^1 - \bar{r}^*\|^2 + 2\sigma_n\lambda_1\|x_n - \bar{r}^*\|\|z_n^1 - \bar{r}^*\| \\
&+ 2\sigma_n\langle\psi_1(\bar{r}^*) - \bar{r}^*, z_n^1 - \bar{r}^*\rangle + 2\sigma_n^2\|\psi_1(x_n) - \bar{r}^*\|\|z_n^1 - \bar{r}^*\| \\
\leq{}& \sigma_n^2\|\psi_1(x_n) - \bar{r}^*\|^2 + (1-\sigma_n)^2\|z_n^1 - \bar{r}^*\|^2 + \sigma_n\lambda_1(\|x_n - \bar{r}^*\|^2 \\
&+ \|z_n^1 - \bar{r}^*\|^2) + 2\sigma_n\langle\psi_1(\bar{r}^*) - \bar{r}^*, z_n^1 - \bar{r}^*\rangle \\
&+ 2\sigma_n^2\|\psi_1(x_n) - \bar{r}^*\|\|z_n^1 - \bar{r}^*\|.
\end{aligned} \tag{3.50}
$$

Similarly,

$$
\begin{aligned}
\|v_n - \bar{s}^*\|^2 \leq{}& \sigma_n^2\|\psi_2(y_n) - \bar{s}^*\|^2 + (1-\sigma_n)^2\|z_n^2 - \bar{s}^*\|^2 + \sigma_n\lambda_2(\|y_n - \bar{s}^*\|^2 \\
&+ \|z_n^2 - \bar{s}^*\|^2) + 2\sigma_n\langle\psi_2(\bar{s}^*) - \bar{s}^*, z_n^2 - \bar{s}^*\rangle \\
&+ 2\sigma_n^2\|\psi_2(y_n) - \bar{s}^*\|\|z_n^2 - \bar{s}^*\|.
\end{aligned} \tag{3.51}
$$

Adding equations (3.50) and (3.51) and using equations (3.18) and (3.49), we get

$$
\begin{aligned}
&\|h_n - \bar{r}^*\|^2 + \|v_n - \bar{s}^*\|^2 \\
&\leq \sigma_n^2(\|\psi_1(x_n) - \bar{r}^*\|^2 + \|\psi_2(y_n) - \bar{s}^*\|^2) \\
&\quad + (1-\sigma_n)^2(\|z_n^1 - \bar{r}^*\|^2 + \|z_n^2 - \bar{s}^*\|^2) + \sigma_n\lambda(\|x_n - \bar{r}^*\|^2 \\
&\quad + \|y_n - \bar{s}^*\|^2) + \sigma_n\lambda(\|z_n^1 - \bar{r}^*\|^2 + \|z_n^2 - \bar{s}^*\|^2) \\
&\quad + 2\sigma_n(\langle\psi_1(\bar{r}^*) - \bar{r}^*, z_n^1 - \bar{r}^*\rangle + \langle\psi_2(\bar{s}^*) - \bar{s}^*, z_n^2 - \bar{s}^*\rangle) \\
&\quad + 2\sigma_n^2(\|\psi_1(x_n) - \bar{r}^*\|.\|z_n^1 - \bar{r}^*\| + \|\psi_2(y_n) - \bar{s}^*\|.\|z_n^2 - \bar{s}^*\|) \\
&\leq (1 - 2(1-\lambda)\sigma_n)(\|x_n - \bar{r}^*\|^2 + \|y_n - \bar{s}^*\|^2) + \sigma_n^2 M_2 \\
&\quad + 2\sigma_n(\langle\psi_1(\bar{r}^*) - \bar{r}^*, z_n^1 - \bar{r}^*\rangle + \langle\psi_2(\bar{s}^*) - \bar{s}^*, z_n^2 - \bar{s}^*\rangle) \\
&\quad + 2\tau_n((1-\sigma_n)^2 + \sigma_n\lambda)(\|w_n - \bar{r}^*\| + \|t_n - \bar{s}^*\|),
\end{aligned} \tag{3.52}
$$

where

$$
\begin{aligned}
M_2 = \sup_{n\in\mathbb{N}}\{&\|x_n - \bar{r}^*\|^2 + \|y_n - \bar{s}^*\|^2 + \|\psi_1(x_n) - \bar{r}^*\|^2 \\
&+ \|\psi_2(y_n) - \bar{s}^*\|^2 + 2\|\psi_1(x_n) - \bar{r}^*\|.\|z_n^1 - \bar{r}^*\| &(3.53)\\
&+ 2\|\psi_2(y_n) - \bar{s}^*\|.\|z_n^2 - \bar{s}^*\|\}. &(3.54)
\end{aligned}
$$

Hence,

$$\begin{aligned}\|x_{n+1}-\bar{r}^*\|^2+\|y_{n+1}-\bar{s}^*\|^2 &\leq \bigl(1-2(1-\lambda)\sigma_n\bigr)\bigl(\|x_n-\bar{r}^*\|^2+\|y_n-\bar{s}^*\|^2\bigr)\\
&\quad+\frac{2\sigma_n(1-\lambda)}{2(1-\lambda)}\Bigl(2\bigl(\langle\psi_1(\bar{r}^*)-\bar{r}^*,z_n^1-\bar{r}^*\rangle\\
&\quad+\langle\psi_2(\bar{s}^*)-\bar{s}^*,z_n^2-\bar{s}^*\rangle\bigr)+4\frac{\tau_n}{\sigma_n}\|w_n-\bar{r}^*\|+\|t_n-\bar{s}^*\|\\
&\quad+\sigma_nM_2\Bigr),\end{aligned} \tag{3.55}$$

which gives

$$\bar{p}_{n+1}\leq(1-d_n)\bar{p}_n+d_n\bar{s}_n, \tag{3.56}$$

where $\bar{p}_n=\|x_n-\bar{r}^*\|^2+\|y_n-\bar{s}^*\|^2$, $d_n=2\sigma_n(1-\lambda)$ and

$$\begin{aligned}\bar{s}_n&=\frac{1}{2(1-\lambda)}\Bigl[\bigl(\langle\psi_1(\bar{r}^*)-\bar{r}^*,z_n^1-\bar{r}^*\rangle+\langle\psi_2(\bar{s}^*)-\bar{s}^*,z_n^2-\bar{s}^*\rangle\bigr)\\
&\quad+4\frac{\tau_n}{\sigma_n}\bigl(\|w_n-\bar{r}^*\|+\|t_n-\bar{s}^*\|\bigr)+\sigma_nM_2\Bigr].\end{aligned}$$

Case 1. Assume that there exists $N\in\mathbb{N}$ such that $\{\bar{p}_n\}$ is decreasing for $n\geq N$. As $\{\bar{p}_n\}$ is bounded and monotonic and subsequently convergent, as $\lim_{n\to\infty}\sigma_n=0$ and $\tau_n=o(\sigma_n)$, using equations (3.35), (3.43), (3.49), we have

$$\begin{aligned}\|x_{n+1}-\bar{r}^*\|^2+\|y_{n+1}-\bar{s}^*\|^2&\leq\sigma_n\bigl(\|\psi_1(x_n)-\bar{r}^*\|^2+\|\psi_2(y_n)-\bar{s}^*\|^2\bigr)+\|x_n-\bar{r}^*\|^2\\
&\quad+\|y_n-\bar{s}^*\|^2+2\tau_n\bigl[\|w_n-\bar{r}^*\|+\|t_n-\bar{s}^*\|\bigr]\\
&\quad-\delta_n\mu_na_n\bigl(\|h_n-S_n(h_n)\|^2+\|v_n-U_n(v_n)\|^2\bigr)\\
&\quad-\delta_n\sum_{i=1}^{\infty}\eta_{n,i}\bigl(\|h_n-J_{\theta_i}^{K_i}(u_n^1)\|^2+\|J_{\theta_i}^{T_i}(u_n^2)-v_n\|^2\bigr),\end{aligned} \tag{3.57}$$

which implies

$$\begin{aligned}\delta_n\sum_{i=1}^{\infty}\eta_{n,i}\bigl(\|h_n-J_{\theta_i}^{K_i}(u_n^1)\|^2+\|J_{\theta_i}^{T_i}(u_n^2)-v_n\|^2\bigr)&\leq\sigma_n\bigl(\|\psi_1(x_n)-\bar{r}^*\|^2+\|\psi_2(y_n)-\bar{s}^*\|^2\bigr)\\
&\quad+\|x_n-\bar{r}^*\|^2+\|y_n-\bar{s}^*\|^2\\
&\quad-\bigl(\|x_{n+1}-\bar{r}^*\|^2+\|y_{n+1}-\bar{s}^*\|^2\bigr)\\
&\quad+2\tau_n[\|w_n-\bar{r}^*\|+\|t_n-\bar{s}^*\|.\end{aligned} \tag{3.58}$$

Taking limit $n\to\infty$ in equation (3.58) and using condition (ii), we get

$$\lim_{n\to\infty}\|h_n-J_{\theta_i}^{K_i}(u_n^1)\|=\lim_{n\to\infty}\|J_{\theta_i}^{T_i}(u_n^2)-v_n\|=0. \tag{3.59}$$

From equations (3.32), (3.35), and (3.49), we get

$$\begin{aligned}\|x_{n+1}-\bar{r}^*\|^2+\|y_{n+1}-\bar{s}^*\|^2 &\leq \sigma_n\left(\|\psi_1(x_n)-\bar{r}^*\|^2+\|\psi_2(y_n)-\bar{s}^*\|^2\right)\\ &\quad+\|x_n-\bar{r}^*\|^2+\|y_n-\bar{s}^*\|^2+2\tau_n\left[\|w_n-\bar{r}^*\|\right.\\ &\quad\left.+\|t_n-\bar{s}^*\|\right]-\mu_n a_n(1-a_n-\phi)\|U_n(v_n)-v_n\|^2\\ &\quad-\mu_n a_n(1-a_n-\kappa)\|S_n(h_n)-h_n\|^2,\end{aligned}$$

which implies

$$\begin{aligned}&\mu_n a_n(1-a_n-\phi_{k_2})\|U_n(v_n)-v_n\|^2 &(3.60)\\ &\quad+\mu_n a_n(1-a_n-\kappa_{n_{k_1}})\|S_n(h_n)-h_n\|^2\\ &\quad\leq \sigma_(\|\psi_1(x_n)-\bar{r}^*\|^2+\|\psi_2(y_n)-\bar{s}^*\|^2)\\ &\quad+\|x_n-\bar{r}^*\|^2+\|y_n-\bar{s}^*\|^2-\left(\|x_{n+1}-\bar{r}^*\|^2\right.\\ &\quad\left.+\|y_{n+1}-\bar{s}^*\|^2\right)+2\tau_n\left[\|w_n-\bar{r}^*\|+\|t_n-\bar{s}^*\|\right]. &(3.61)\end{aligned}$$

Taking limit $n\to\infty$ in equation (3.60) and using condition (ii), we get

$$\lim_{n\to\infty}\|U_n(v_n)-v_n\|=\lim_{n\to\infty}\|S_n(h_n)-h_n\|=0. \tag{3.62}$$

From equations (3.35), (3.45), and (3.49), we have

$$\begin{aligned}&\sum_{i=1}^{\infty}\eta_{n,i}\rho_n\epsilon\left(\|A^*(Ah_n-Bv_n)\|^2+\|B^*(Ah_n-Bv_n)\|^2\right)\\ &\quad\leq\sigma_n\left(\|\psi_1(x_n)-\bar{r}^*\|^2+\|\psi_2(y_n)-\bar{s}^*\|^2\right)+\|x_n-\bar{r}^*\|^2\\ &\quad+\|y_n-\bar{s}^*\|^2-\left(\|x_{n+1}-\bar{r}^*\|^2+\|y_{n+1}-\bar{s}^*\|^2\right)\\ &\quad+2\tau_n\left[\|w_n-\bar{r}^*\|+\|t_n-\bar{s}^*\|\right]. &(3.63)\end{aligned}$$

Taking limit $n\to\infty$ in equation (3.63) and using condition (ii), we get

$$\lim_{n\to\infty}\|A^*(Ah_n-Bv_n)\|=\lim_{n\to\infty}\|B^*(Ah_n-Bv_n)\|=0. \tag{3.64}$$

As A and B are bounded linear operators and hence

$$\lim_{n\to\infty}\|Ah_n-Bv_n\|=0. \tag{3.65}$$

From equations (3.34), (3.35), and (3.49), we get

$$\begin{aligned}&v_n^x\sum_{j=1}^{M}b_j^x\left(2\bar{c}^x-v_n^x\right)\|L_j^x w_n-L_j^x\bar{r}^*\|^2+v_n^y\sum_{j=1}^{M}b_j^y\left(2\bar{c}^y-v_n^y\right)\|L_j^y t_n-L_j^y\bar{s}^*\|^2\\ &\quad\leq\sigma_n\left(\|\psi_1(x_n)-\bar{r}^*\|^2+\|\psi_2(y_n)-\bar{s}^*\|^2\right)\end{aligned}$$

$$\begin{aligned}&+\|x_n-\bar{r}^*\|^2+\|y_n-\bar{s}^*\|^2-\left(\|x_{n+1}-\bar{r}^*\|^2\right.\\&\left.+\|y_{n+1}-\bar{s}^*\|^2\right)+2\tau_n\left[\|w_n-\bar{r}^*\|+\|t_n-\bar{s}^*\|\right].\end{aligned}\tag{3.66}$$

Taking limit $n\to\infty$ in equation (3.66) and using conditions $(i)-(ii)$, we get

$$\lim_{n\to\infty}\|L_j^x w_n-L_j^x\bar{r}^*\|=\lim_{n\to\infty}\|L_j^y t_n-L_j^y\bar{s}^*\|=0,\quad \text{where } 1\le j\le M.\tag{3.67}$$

From equations (3.35), (3.38), and (3.49), we get

$$\begin{aligned}&\|z_n^1-w_n\|^2+\|z_n^2-t_n\|^2\\&\quad\le\sigma_n\left(\|\psi_1(x_n)-\bar{r}^*\|^2+\|\psi_2(y_n)-\bar{s}^*\|^2\right)+\|x_n-\bar{r}^*\|^2\\&\qquad+\|y_n-\bar{s}^*\|^2-\left(\|x_{n+1}-\bar{r}^*\|^2+\|y_{n+1}-\bar{s}^*\|^2\right)+2\tau_n\left[\|w_n-\bar{r}^*\|\right.\\&\qquad\left.+\|t_n-\bar{s}^*\|\right]+2\nu_n^x\sum_{j=1}^M b_j^x\|w_n-z_n^1\|\left\|L_j^x(w_n)-L_j^x(\bar{r}^*)\right\|\\&\qquad+2\nu_n^y\sum_{j=1}^M b_j^y\|t_n-z_n^2\|\left\|L_j^y(t_n)-L_j^y(\bar{s}^*)\right\|.\end{aligned}\tag{3.68}$$

Taking limit $n\to\infty$ in equation (3.68) and using equation (3.67) and using condition (ii), we get

$$\lim_{n\to\infty}\|z_n^1-w_n\|=\lim_{n\to\infty}\|z_n^2-t_n\|=0.\tag{3.69}$$

From equation (3.2), we have

$$\|h_n-w_n\|\le\sigma_n\|\psi_1(x_n)-w_n\|+(1-\sigma_n)\|z_n^1-w_n\|.\tag{3.70}$$

Similarly,

$$\|v_n-t_n\|\le\sigma_n\|\psi_2(y_n)-t_n\|+(1-\sigma_n)\|z_n^2-t_n\|.\tag{3.71}$$

Taking limit $n\to\infty$ in equations (3.70) and (3.71), using equation (3.69) and $\lim_{n\to\infty}\sigma_n=0$, we get

$$\lim_{n\to\infty}\|h_n-w_n\|=\lim_{n\to\infty}\|v_n-t_n\|=0.\tag{3.72}$$

From equation (3.2), we have

$$\begin{aligned}\|x_{n+1}-h_n\|&\le\delta_n\|h_n-h_n\|+\mu_n\left((1-a_n)\|h_n-h_n\|+a_n\|S_n(h_n)-h_n\|\right)\\&\quad+\sum_{i=1}^\infty\eta_{n,i}\|J_{\theta_i}^{K_i}(u_n^1)-h_n\|\\&\le\mu_n a_n\|S_n(h_n)-h_n\|+\sum_{i=1}^\infty\eta_{n,i}\|J_{\theta_i}^{K_i}(u_n^1)-h_n\|.\end{aligned}\tag{3.73}$$

Similarly,

$$\|y_{n+1} - v_n\| \le \mu_n a_n \big\| U_n(v_n) - v_n \big\| + \sum_{i=1}^{\infty} \eta_{n,i} \big\| J_{\theta_i}^{K_i}\big(u_n^2\big) - v_n \big\|. \tag{3.74}$$

Taking limit $n \to \infty$ in equation (3.73) and (3.74), then using equations (3.59) and (3.62), we get

$$\lim_{n\to\infty} \|x_{n+1} - h_n\| = \lim_{n\to\infty} \|y_{n+1} - v_n\| = 0. \tag{3.75}$$

Also,

$$\|w_n - x_n\| = \gamma_n \|x_n - x_{n-1}\| \le \tau_n. \tag{3.76}$$

Similarly,

$$\|t_n - y_n\| \le \tau_n. \tag{3.77}$$

Since $\tau_n = o(\sigma_n)$ and $\lim_{n\to\infty} \sigma_n = 0$, we have

$$\lim_{n\to\infty} \|w_n - x_n\| = \lim_{n\to\infty} \|t_n - y_n\| = 0. \tag{3.78}$$

Using equations (3.72), (3.75), and (3.78), we get

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = \lim_{n\to\infty} \|y_n - y_{n+1}\| = 0. \tag{3.79}$$

Moreover,

$$\begin{aligned}\|Ax_n - By_n\| &\le \|Ax_n - Ah_n\| + \|Ah_n - Bv_n\| + \|Bv_n - By_n\| \\ &\le \|A\|\|x_n - h_n\| + \|Ah_n - Bv_n\| + \|B\|\|v_n - y_n\|.\end{aligned} \tag{3.80}$$

By equations (3.65), (3.72), and (3.78), we get

$$\lim_{n\to\infty} \|Ax_n - By_n\| = 0. \tag{3.81}$$

As A and B are bounded linear operators, therefore, $Ax_{n_q} \rightharpoonup Ax^*$ and $By_{n_q} \rightharpoonup By^*$. Clearly, it follows from weak semicontinuity of norm that

$$\big\|Ax^* - By^*\big\| \le \liminf_{q\to\infty} \|Ax_{n_q} - By_{n_q}\| = 0. \tag{3.82}$$

This implies $Ax^* = By^*$. Subsequently, $(x^*, y^*) \in \Gamma$. Fix an index $k_1 \in \{1, 2, \dots, N\}$. As the pool of index is finite and $\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0$, from Lemma 2.6, one can find a subsequence $\{x_{m_q}\}$ of $\{x_n\}$ such that $x_{m_q} \rightharpoonup x^*$ and $m_q(\mod N) + 1 = k_1$. Using equations (3.72) and (3.78), we get $h_{m_q} \rightharpoonup x^*$. Additionally,

$$\lim_{q\to\infty} \big\|S_{k_1}(h_{m_q}) - h_{m_q}\big\| = \lim_{q\to\infty} \big\|S_{m_q}(h_{m_q}) - h_{m_q}\big\| = 0 \tag{3.83}$$

and $I - S_{k_1}$ is demiclosed at zero for each $k_1 \in \{1,2,\dots,N\}$, which implies $x^* \in \bigcap_{k_1=1}^{N} \mathrm{Fix}(S_{k_1})$. Similarly,

$$\lim_{q\to\infty} \| U_{k_2}(v_{m_q}) - v_{m_q} \| = \lim_{q\to\infty} \| U_{m_q}(v_{m_q}) - v_{m_q} \| = 0, \tag{3.84}$$

and $I - U_{k_2}$ is demiclosed at zero for each $k_2 \in \{1,2,\dots,P\}$, which implies $y^* \in \bigcap_{k_2=1}^{P} \mathrm{Fix}(U_{k_2})$. As $\{x_n\}$ is bounded, take a subsequence $\{x_{n_q}\}$ of $\{x_n\}$ such that $x_{n_q} \rightharpoonup x^* \in H_1$. Additionally, from the boundedness of $\{h_n\}$, $\{w_n\}$ and from equations (3.72) and (3.78), there are subsequences $\{h_{n_q}\}$, $\{w_{n_q}\}$ respectively such that $h_{n_q} \rightharpoonup x^*$ and $w_{n_q} \rightharpoonup x^*$. Suppose that $x^* \notin \bigcap_{j=1}^{M} \mathrm{VIP}(Q_1, L_j^x)$. Then, from Lemmas (2.10) and (2.11), we have $x^* \notin \mathrm{Fix}(P_{Q_1}(I - \nu_x \sum_{j=1}^{M} b_j^x L_j^x))$. Using Opial's condition and equation (3.69), we estimate

$$\begin{aligned}
\liminf_{q\to\infty} \| w_{n_q} - x^* \| &< \liminf_{q\to\infty} \left\| w_{n_q} - P_{Q_1}\left(I - \nu_x \sum_{j=1}^{M} b_j^x L_j^x\right) x^* \right\| \\
&\leq \liminf_{q\to\infty} \left\| w_{n_q} - P_{Q_1}\left(I - \nu_x \sum_{j=1}^{M} b_j^x L_j^x\right) w_{n_q} \right\| \\
&\quad + \liminf_{q\to\infty} \left\| P_{Q_1}\left(I - \nu_x \sum_{j=1}^{M} b_j^x L_j^x\right) w_{n_q} - P_{Q_1}\left(I - \nu_x \sum_{j=1}^{M} b_j^x L_j^x\right) x^* \right\| \\
&\leq \liminf_{q\to\infty} \| w_{n_q} - x^* \|,
\end{aligned} \tag{3.85}$$

which is a contradiction. Thus $x^* \in \mathrm{Fix}(P_{Q_1}(I - \nu_x \sum_{j=1}^{M} b_j^x L_j^x))$. Similarly, $\{y_n\}$ is bounded, take a subsequence $\{y_{n_q}\}$ of $\{y_n\}$ such that $y_{n_q} \rightharpoonup y^* \in H_2$. Additionally, from the boundedness of $\{v_n\}$, $\{t_n\}$ and from equations (3.72) and (3.78), there are subsequences $\{v_{n_q}\}$ $\{t_{n_q}\}$ respectively such that $v_{n_q} \rightharpoonup y^*$ and $t_{n_q} \rightharpoonup y^*$. Suppose that $y^* \notin \bigcap_{j=1}^{M} \mathrm{VIP}(Q_2, L_j^y)$. Then, from Lemmas (2.10) and (2.11), we have $y^* \notin \mathrm{Fix}(P_{Q_2}(I - \nu_y \sum_{j=1}^{M} b_j^y L_j^y))$. Using Opial's condition and equation (3.69), we estimate

$$\begin{aligned}
\liminf_{q\to\infty} \| t_{n_q} - y^* \| &< \liminf_{q\to\infty} \left\| t_{n_q} - P_{Q_2}\left(I - \nu_y \sum_{j=1}^{M} b_j^y L_j^y\right) y^* \right\| \\
&\leq \liminf_{q\to\infty} \left\| t_{n_q} - P_{Q_2}\left(I - \nu_y \sum_{j=1}^{M} b_j^y L_j^y\right) t_{n_q} \right\| \\
&\quad + \liminf_{q\to\infty} \left\| P_{Q_2}\left(I - \nu_y \sum_{j=1}^{M} b_j^y L_j^y\right) t_{n_q} - P_{Q_2}\left(I - \nu_y \sum_{j=1}^{M} b_j^y L_j^y\right) y^* \right\| \\
&\leq \liminf_{q\to\infty} \| t_{n_q} - y^* \|,
\end{aligned} \tag{3.86}$$

which is a contradiction. Thus $y^* \in \mathrm{Fix}(P_{Q_2}(I - \nu_y \sum_{j=1}^{M} b_j^y L_j^y))$. Using equation (3.59), we get

$$\lim_{q\to\infty} \| l_{n_q} - \left(J_{\theta_i}^{(K_i,T_i)}\left(I - \rho_{n_q}(H^* H) l_{n_q}\right)\right) \| = 0. \tag{3.87}$$

Now, using Lemma 3.3, $l^* \in \Omega$. As $w_{n_q} \rightharpoonup x^*$ and $t_{n_q} \rightharpoonup y^*$, from equation (3.69), we estimate $z^1_{n_q} \rightharpoonup x^*$ and $z^2_{n_q} \rightharpoonup y^*$. Using Lemma 2.4, we get

$$\begin{aligned}
&\limsup_{n\to\infty}\langle z^1_n - \bar{r}^*, \psi_1(\bar{r}^*) - \bar{r}^*\rangle + \langle z^2_n - \bar{s}^*, \psi_2(\bar{s}^*) - \bar{s}^*\rangle \\
&\quad = \limsup_{q\to\infty}\langle z^1_{n_q} - \bar{r}^*, \psi_1(\bar{r}^*) - \bar{r}^*\rangle + \langle z^2_{n_q} - \bar{s}^*, \psi_2(\bar{s}^*) - \bar{s}^*\rangle \\
&\quad = \limsup_{q\to\infty}\langle x^* - \bar{r}^*, \psi_1(\bar{r}^*) - \bar{r}^*\rangle + \langle y^* - \bar{s}^*, \psi_2(\bar{s}^*) - \bar{s}^*\rangle \\
&\quad \le 0. \qquad (3.88)
\end{aligned}$$

As $\lim_{n\to\infty}\sigma_n = 0$ and $\lim_{n\to\infty}\frac{\tau_n}{\sigma_n} = 0$, using equation (3.88), we get $\limsup_{n\to\infty}\bar{s}_n \le 0$, and using Lemma 2.8, we have

$$\lim_{n\to\infty}\bar{p}_n = \lim_{n\to\infty}\|x_n - \bar{r}^*\|^2 + \|y_n - \bar{s}^*\|^2 = 0, \qquad (3.89)$$

which implies $\lim_{n\to\infty}\|x_n - \bar{r}^*\|^2 = \lim_{n\to\infty}\|y_n - \bar{s}^*\|^2 = 0$. Thus $x_n \to \bar{r}^*$ and $y_n \to \bar{s}^*$.

Case 2. Suppose that there is a subsequence $\{\bar{p}_{n_q}\}$ of $\{\bar{p}_n\}$ such that

$$\bar{p}_{n_q+1} \ge \bar{p}_{n_q} \quad \text{for all } q \in \mathbb{N}.$$

Hence, by Lemma 2.7, there is a nondecreasing sequence of natural numbers $\{n_l\} \subset \mathbb{N}$ such that $n_l \to \infty$ as $l \to \infty$, we have

$$\bar{p}_{n_l+1} \ge \bar{p}_{n_l}$$

and

$$\bar{p}_{n_l+1} \ge \bar{p}_l. \qquad (3.90)$$

This with equation (3.63) gives

$$\begin{aligned}
&\sum_{i=1}^{\infty}\eta_{n_l,i}\rho_{n_l}\epsilon\left(\|A^*(Ah_{n_l} - Bv_{n_l})\|^2 + \|B^*(Ah_{n_l} - Bv_{n_l})\|^2\right) \qquad (3.91)\\
&\quad \le \sigma_{n_l}\left(\|\psi_1(x_{n_l}) - \bar{r}^*\|^2 + \|\psi_2(y_{n_l}) - \bar{s}^*\|^2\right) \\
&\qquad + \bar{p}_{n_l} - \bar{p}_{n_l+1} + 2\epsilon_{n_l}\left[\|w_{n_l} - \bar{r}^*\| + \|t_{n_l} - \bar{s}^*\|\right]. \qquad (3.92)
\end{aligned}$$

Taking limit $l \to \infty$, using $\lim_{l\to\infty}\rho_{n_l} = 0$ and $\epsilon_{n_l} = o(\rho_{n_l})$, we get

$$\|A^*(Ah_{n_l} - Bv_{n_l})\| = \|B^*(Ah_{n_l} - Bv_{n_l})\| = 0. \qquad (3.93)$$

Similarly,

$$\nu^x_{n_l}\sum_{j=1}^{M}b^x_j\left(2\bar{c}^x - \nu^x_{n_l}\right)\|L^x_j w_{n_l} - L^x_j\bar{r}^*\|^2 + \nu^y_{n_l}\sum_{j=1}^{M}b^y_j\left(2\bar{c}^y - \nu^y_{n_l}\right)\|L^y_j t_{n_l} - L^y_j\bar{s}^*\|^2$$

$$\begin{aligned}
&\leq \sigma_{n_l}\left(\left\|\psi_1(x_{n_l}) - \bar{r}^*\right\|^2 + \left\|\psi_2(y_{n_l}) - \bar{s}^*\right\|^2\right) \\
&\quad + \bar{p}_{n_l} - \bar{p}_{n_l+1} + 2\epsilon_{n_l}\left[\left\|w_{n_l} - \bar{r}^*\right\| \right. \\
&\quad \left. + \left\|t_{n_l} - \bar{s}^*\right\|\right].
\end{aligned} \tag{3.94}$$

Taking limit $l \to \infty$, using $\lim_{l\to\infty} \rho_{n_l} = 0$, condition (i), and $\epsilon_{n_l} = o(\rho_{n_l})$, we get

$$\lim_{l\to\infty} \left\|L_j^x w_{n_l} - L_j^x \bar{r}^*\right\| = \lim_{l\to\infty} \left\|L_j^y t_{n_l} - L_j^y \bar{s}^*\right\| = 0. \tag{3.95}$$

Similarly, we can prove

$$\lim_{l\to\infty} \left\|z_{n_l}^1 - w_{n_l}\right\| = \lim_{l\to\infty} \left\|z_{n_l}^2 - t_{n_l}\right\| = \lim_{l\to\infty} \|w_{n_l} - x_{n_l}\| = 0 \quad \text{and} \tag{3.96}$$

$$\lim_{l\to\infty} \|t_{n_l} - y_{n_l}\| = \lim_{l\to\infty} \|x_{n_l} - x_{n_l+1}\| = \lim_{l\to\infty} \|y_{n_l} - y_{n_l+1}\| = 0. \tag{3.97}$$

Using the same justification as in Case 1, we can prove

$$\lim_{l\to\infty} \|Ax_{n_l} - By_{n_l}\| = \lim_{l\to\infty} \left\|A^*(Ah_{n_l} - Bv_{n_l})\right\| = 0 \tag{3.98}$$

$$\text{and} \quad \lim_{l\to\infty} \left\|U_{n_l}(v_{n_l}) - v_{n_l}\right\| = \lim_{l\to\infty} \left\|S_{n_l}(h_{n_l}) - h_{n_l}\right\| = \lim_{l\to\infty} \|h_{n_l} - w_{n_l}\| = 0 \tag{3.99}$$

$$\text{and} \quad = \lim_{l\to\infty} \left\|B^*(Ah_{n_l} - Bv_{n_l})\right\| = \lim_{l\to\infty} \|v_{n_l} - t_{n_l}\| = 0. \tag{3.100}$$

Again, following the same steps as in Case 1, we obtain

$$\limsup_{l\to\infty} \left\langle z_{n_l}^1 - \bar{r}^*, \psi_1(\bar{r}^*) - \bar{r}^*\right\rangle + \left\langle z_{n_l}^2 - \bar{s}^*, \psi_2(\bar{s}^*) - \bar{s}^*\right\rangle \leq 0. \tag{3.101}$$

Additionally, from equation (3.56), we obtain

$$\bar{p}_{n_l+1} \leq (1 - d_{n_l})\bar{p}_{n_l} + d_{n_l}\bar{s}_{n_l}, \tag{3.102}$$

which implies

$$\begin{aligned}
d_{n_l}\bar{p}_{n_l} &\leq \bar{p}_{n_l} - \bar{p}_{n_l+1} + d_{n_l}\bar{s}_{n_l} \\
&\leq d_{n_l}\bar{s}_{n_l}.
\end{aligned} \tag{3.103}$$

As $d_{n_l} > 0$, we have $\bar{p}_{n_l} \leq \bar{s}_{n_l}$. Subsequently,

$$\begin{aligned}
\left\|x_{n_l} - \bar{r}^*\right\|^2 + \left\|y_{n_l} - \bar{s}^*\right\|^2 \leq \frac{1}{2(1-\lambda)}\Bigg[&\left(\left\langle\psi_1(\bar{r}^*) - \bar{r}^*, z_{n_l}^1 - \bar{r}^*\right\rangle\right. \\
&\left. + \left\langle\psi_2(\bar{s}^*) - \bar{s}^*, z_{n_l}^2 - \bar{s}^*\right\rangle\right) \\
&+ 4\frac{\epsilon_{n_l}}{\sigma_{n_l}}\left(\left\|w_{n_l} - \bar{r}^*\right\| + \left\|t_{n_l} - \bar{s}^*\right\|\right) + \sigma_{n_l}M_2\Bigg].
\end{aligned} \tag{3.104}$$

As $\lim_{l\to\infty} \sigma_{n_l} = 0$, $\lim_{l\to\infty} \frac{\epsilon_{n_l}}{\sigma_{n_l}} = 0$ and from equation (3.101), we obtain $\bar{p}_{n_l} \to 0$ as $l \to \infty$. Also using equation (3.102), we get $\bar{p}_{n_l+1} \to 0$ as $l \to \infty$. Subsequently, from equation

(3.90), we get $\bar{p}_{n_l+1} \geq \bar{p}_l$. This implies $\lim_{l\to\infty} \bar{p}_l = 0$ i.e. $\|x_l - x^*\|^2 + \|y_l - y^*\|^2 \to 0$ as $l \to \infty$, which implies $x_l \to \bar{r}^*$ and $y_l \to \bar{s}^*$. □

Remark 3.5

(i) Izuchukwu et al. [55] established a strong convergence theorem for finding a common solution to two types of split equality problems, namely fixed point problem and finite families of variational inequalities problems. In this paper we proved a strong convergence theorem for finding a common solution to three types of split equality problems, namely infinite families of variational inclusion problems, finite families of variational inequalities, and a multiple-sets fixed point problem. Hence, our result is more desirable than Izuchukwu et al. [55] results.

(ii) Theorem 3.4 generalizes the findings of Kazmi et al. [56] from the common solution of a multiple-sets split equality fixed point problem and a split equality variational inequality problem to a infinite family of split equality variational inclusion problems, a multiple-sets split equality fixed point problem, and a finite family of split equality variational inequality problems.

(iii) Theorem 3.4 improves and generalizes the result of Guo et al. [57] from the common solution of a split equality variational inclusion problem and a finite family of fixed point problems to an infinite family of split equality variational inclusion problems, a finite family of split equality variational inequality problems, and a multiple-sets split equality fixed point problem. Further, our result generalize the Guo et al. [57] result from nonexpansive mapping to more generalized demicontractive mappings.

4 Applications

Intensity-modulated radiation therapy Intensity-modulated radiation therapy (IMRT) is a cutting-edge radiotherapy technology that treats cancer while minimizing the amount of normal tissue exposed to radiation in the treatment area. IMRT has recently garnered much attention, as evidenced by [15, 58]. We often utilize an optimization technique to minimize the objective function of radiation beam weights, and radiations with varying intensities are transmitted into the body of patients while building IMRT.

The primary goal of IMRT is to deliver a sufficient dosage to the areas that require radiation therapy, known as planned target volumes (PTV), while limiting the amount given to other areas, known as organs at risk (OAR). To achieve a clinically appropriate dose distribution, we primarily evaluate the radiation dosage absorbed by irradiated tissue based on the distribution of beamlet intensities and the radiation intensity given by all beamlets when planning IMRT. Furthermore, we consider the irradiated object's physical and biological features. While the restrictions on the deliverable radiation intensities of the beamlets are represented in the intensity space, the space whose elements are the deliverable radiation intensities of the beamlets, and the limits on the dose received by each voxel of the body are represented in the dose space. The intensity space and the dosage space are Euclidean spaces for vectors.

We employ volumetric modulated arc therapy (VMAT) and study two external sources/treatment equipment, the Varian and the Elekta. Here, we compare one dose space and two intensity spaces. We split the entire patient volume into J-voxels $j = 1, 2, \dots, J$, beam into K-beamlets, $k = 1, 2, \dots, K$. Assume that $a_{jk} \geq 0$ is the dose absorbed at jth as a result of unit intensity from kth beamlet, $x_k \geq 0$ is the intensity of k beamlet, and c_j denotes the dose absorbed by jth voxel given as

$$c_j = \sum_{k=1}^{K} a_{jk} x_k. \tag{4.1}$$

We represent A as a dose influence matrix with nonnegative entries a_{jk}. In the dose space, we will set a lower bound on the dose delivered to PTV and an upper bound on the dose delivered to OAR. Suppose that A_s denotes anatomical structures that are $Q+R$ in number, where A_s are PTVs for $s = 1, 2, \dots, Q$ and A_s are OARs for $s = Q+1, Q+2, \dots, R$. Let l_s and u_s be lower and upper bound doses for jth voxel, and some upper bound constraints are given as

$$J_{\max,s} = \{t \in R^J : t_j \leq u_s, j \in A_s\}, \tag{4.2}$$

the lower bound constraint

$$J_{\min,s} = \{t \in R^J : l_s \leq t_j, j \in A_s\}, \tag{4.3}$$

the equivalent uniform dose (EUD) constraint $U_s : R^J \to R$ given as

$$U_s(t) = \left[\frac{1}{N_s} \sum_{j \in A_s} (t_j)^{\alpha_s}\right]^{\frac{1}{\alpha_s}}. \tag{4.4}$$

Assume that A and B are influenced matrices, X_+ and Y_+ are nonnegative constraints, X_{k_1}, $k_1 = 1, 2, \dots, N$, and $Y_{k_2}, k_2 = 1, 2, \dots, P$, are the beamlet intensity vectors that satisfy the nth constraint for Varian (V) and Elekta (E) machine respectively. Consider the multiple-sets split equality problem of finding

$$\text{Find } \bar{r} \in X_+ \cap \left(\bigcap_{k_1=1}^{N} X_{k_1}\right), \qquad \bar{s} \in Y_+ \cap \left(\bigcap_{k_2=1}^{M} Y_{k_2}\right) \quad \text{such that } A\bar{r} = B\bar{s}. \tag{4.5}$$

Thus, the main objective of IMRT is to find an intensity $\bar{r}$ that is closest to the entire radiation intensity space in the Varian machine and an intensity $\bar{s}$ that is closest to the entire radiation intensity space in the Elekta machine, these both machines transmit an equal dose to the tumor. As we know that $\bar{r^*} \in \mathrm{Fix}(P_Q)$ if and only if $x \in Q$, where P_Q is the metric projection defined on a closed convex subset Q of Hilbert space H.

Take $S_{k_1} = P_{X_+ \cap X_j}$ $(k_1 = 1, 2, \dots, N)$ and $U_{k_2} = P_{Y_+ \cap Y_k}$ $(k_2 = 1, 2, \dots, P)$ in Algorithm 3.1. For the multiple-sets split equality problem, we obtain the following cyclic iterative algorithm.

Algorithm 4.1 Consider $0 < a < \nu_n^x \leq 2\bar{c}^x$, $0 < b < \nu_n^y \leq 2\bar{c}^y$ for some $a, b \in \mathbb{R}$, $b_j^x, b_j^y \in (0,1)$ for $j = 1, 2, \dots, M$, $\{\delta_n\}, \{\mu_n\}, \{\eta_{n,i}\}, \{\sigma_n\} \subset [\bar{d}, \bar{e}] \subset (0,1)$, $a_n \in (\delta, 1-\phi-\delta)$, $\beta \in (0,1)$, $\sum_{n=1}^{\infty} \tau_n < \infty$, and $\{\kappa_n\} \subset [\kappa, \bar{\kappa}]$, where $0 < \kappa \leq \bar{\kappa}$, $\delta > 0$, $\bar{d}$, and $\bar{e} \in \mathbb{R}$. Choose $x_0, x_1 \in Q$

and γ_n such that $0 \le \gamma_n \le \bar{\gamma_n}$, where

$$\bar{\gamma_n} = \begin{cases} \beta & \text{if } x_n = x_{n-1} \text{ and } y_n = y_{n-1} \\ \min\{\frac{\tau_n}{\sqrt{\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|}}, \beta\} & \text{if otherwise.} \end{cases} \tag{4.6}$$

Compute

$$\begin{cases} w_n = x_n + \gamma_n(x_n - x_{n-1}), \\ t_n = y_n + \gamma_n(y_n - y_{n-1}), \\ h_n = \sigma_n \psi_1(x_n) + (1-\sigma_n) P_{Q_1}(I - \nu_n^x \sum_{j=1}^M b_j^x L_j^x) w_n, \\ x_{n+1} = \delta_n h_n + \mu_n((1-a_n)h_n + a_n P_{X_+ \cap X_n}(h_n)) \\ \qquad + \sum_{i=1}^\infty \eta_{n,i}(J_{\theta_i}^{K_i}(h_n - \rho_n(A^*(Ah_n - Bv_n)))), \\ v_n = \sigma_n \psi_2(y_n) + (1-\sigma_n) P_{Q_2}(I - \nu_n^y \sum_{j=1}^M b_j^y L_j^y) t_n, \\ y_{n+1} = \delta_n v_n + \mu_n((1-a_n)v_n + a_n P_{Y_+ \cap Y_n}(v_n)) \\ \qquad + \sum_{i=1}^\infty \eta_{n,i}(J_{\theta_i}^{T_i}(v_n + \rho_n(B^*(Ah_n - Bv_n)))), \end{cases} \tag{4.7}$$

where the stepsize ρ_n is given as

$$\rho_n \in \left(\epsilon, \frac{2\|Ah_n - Bv_n\|^2}{\|A^*(Ah_n - Bv_n)\|^2 + \|B^*(Ah_n - Bv_n)\|^2} - \epsilon\right),$$

$n \in \Delta$ and $\epsilon > 0$ otherwise $\rho_n = \rho$, and the index set $\Delta = \{n; Ah_n - Bv_n \neq 0\}$.

As we know, metric projection is firmly nonexpansive, and thus 0-demicontractive. As a result, using Algorithm 4.1 and the proof described in Theorem 3.4, we can obtain the strong convergence result for approximating the solution of the multiple-sets split equality problem (4.1).

It is not always possible to find an intensity that meets all of the constraints. In this case, we find a solution that is as close to all of the constraints as possible. To do so, we consider the following proximity function:

$$f(r,s) = \frac{a_n}{2}\sum_{k_1=1}^{N} \|P_{X_+ \cap (X_{k_1})} r - r\|^2 + \frac{a_n}{2}\sum_{k_2=1}^{P} \|P_{Y_+ \cap (Y_{k_2})} s - s\|^2 + \frac{1}{2}\|Ar - Bs\|^2. \tag{4.8}$$

The multiple-sets split equality problem (4.1), on the other hand, can be written as the minimization problem

$$\min_{r \in \mathbb{R}^N, s \in \mathbb{R}^P} f(r,s). \tag{4.9}$$

5 Numerical example

In this section, we give a numerical example to compare the convergence of the algorithms given in [32, 37] to Algorithms 3.1.

Example 5.1 Let $H_1 = H_2 = H_3 = \mathbb{R}^4$ be the set of real numbers. Assume that

$$Q_1 = Q_2 = \{(\bar{t_1}, \bar{t_2}, \bar{t_3}, \bar{t_4})^T \in \mathbb{R}^4 : \bar{t_1} + \bar{t_2} - 3\bar{t_3} + \bar{t_4} \ge 0\}.$$

Table 1 Comparison of Alg 3.1 with Eslamian Alg [32] and Chang Alg [37]

Algorithm	Iteration number	Time (seconds)
Alg 3.1	22	0.0596
Eslamian Alg	49	0.9678
Chang Alg	187	1.5156

Table 2 Numerical analysis of Alg 3.1 for different cases

Cases	Iteration number	Execution time in seconds
1	17	0.0596
2	21	0.1051
3	37	0.9043

Table 3 Numerical analysis of Alg 3.1 for different values of ρ_1

ρ_1	Iteration number	Execution time in seconds
0.2	21	0.0596
0.4	23	0.1051
0.6	27	0.9043
0.8	47	1.0099

Let $A, B : \mathbb{R}^4 \to \mathbb{R}^4$ be two bounded linear operators created from a normal distribution with unit variance and mean zero. Consider $\psi_1 : \mathbb{R}^4 \to \mathbb{R}^4$ and $\psi_2 : \mathbb{R}^4 \to \mathbb{R}^4$ as two contraction mappings such that $\psi_1(x) = \frac{x}{24}$ and $\psi_2(x) = \frac{x}{24}$ for all $x \in \mathbb{R}^4$. Further, we take two demicontractive mappings S_{k_1} and $U_{k_1} : \mathbb{R}^4 \to \mathbb{R}^4$ such that $S_{k_1}(x) = \frac{-3k_1}{k_1+1}x$, $U_{k_2}(x) = \frac{-2k_2}{k_2+1}$, where $k_1 = k_2 = 1$, $x = (x_1, x_2, x_3, x_4)^T$. Take $i = 1$, and $K_1 : H_1 \to 2^{H_1}$ and $T_1 : H_2 \to 2^{H_2}$ are defined as $K_1(x) = A^*A(x)$ and $T_1(x) = B^*B(x)$. Take $J = 20$. Let L_j^x and L_j^y be defined by $L_j^y(\bar{t}) = \frac{\bar{t}+2}{5j}$ and $L_j^x(\bar{t}) = \frac{\bar{t}-1}{3j}$ with $b_j^x = \frac{2}{3^j} + \frac{1}{N3^N}$, $b_j^y = \frac{4}{5^j} + \frac{1}{N5^N}$ and $\nu_j^x = \frac{1}{7j^3}$, $\nu_j^y = \frac{1}{5j^3}$. Obviously, the solution set Γ is nonempty as $0 \in \Gamma$. Let $\delta_n = \frac{1}{n+1}$, $\mu_n = \frac{1+n}{3n}$, $\eta_{n,1} = \frac{2n^2-2n-1}{3n(n+1)}$, $\sigma_n = \frac{1}{2n+1}$, $a_n = \frac{1}{n+1}$, and the step size $\tau_n = \frac{1}{n^5}$. Choose $\beta = 0.5$ and initial values $x = (0.1, 0.1, 0.1, 0.1)^T$, $y = (0.2, 0.2, 0.2, 0.2)^T$. We set $E_n = \|x_n - x_{n-1}\| + \|y_n - y_{n-1}\| < 10^{-4}$ as a stopping criterion and plot the graphs of errors E_n over the number of iterations n. Furthermore, we provide numerical data for various values of η_n and the following values of x_0 and x_1.

Case 1: $x_0 = (0.01, 0.01, 0.01, 0.01)$, $x_1 = (0.02, 0.02, 0.02, 0.02)$;

Case 2: $x_0 = (0.1, 0.1, 0.1, 0.1)$, $x_1 = (0.2, 0.2, 0.2, 0.2)$;

Case 3: $x_0 = (1, 1, 1, 1)$, $x_1 = (2, 2, 2, 2)$.

Tables 1–3 and Figs. 1–3 present the numerical results.

6 Conclusion

In this paper, we have proposed a new inertial simultaneous cyclic iterative algorithm with a method for a finite family of split equality variational inequality problems, an infinite family of split equality variational inclusion problems, and a multiple-sets split equality fixed point problem for demicontractive operators in infinite dimensional Hilbert spaces. The proposed algorithm includes several well-known methods such as the cyclic method, the inertial method, the viscosity approximation method, and the projection method. We prove strong convergence of the proposed algorithm. This result extends and unifies various known results in the literature. Finally, we give a numerical example to justify the main convergence theorem.

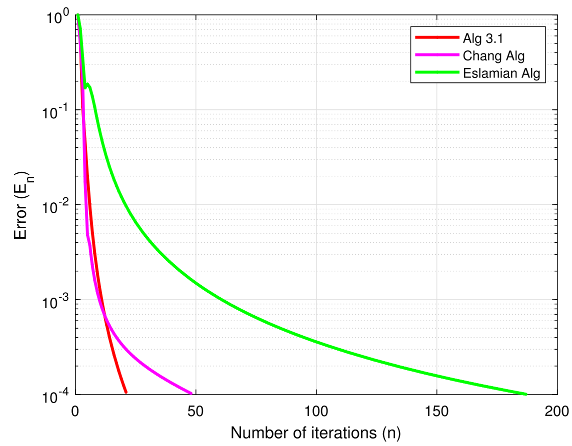

Figure 1 Comparison of Alg 3.1 with Eslamian Alg [32] and Chang Alg [37]

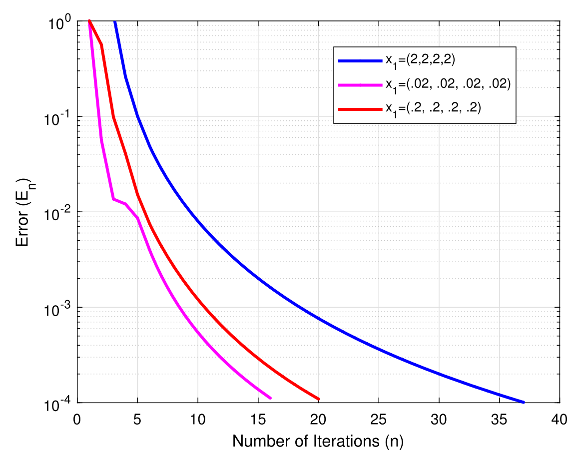

Figure 2 Numerical analysis of Alg 3.1 for different cases

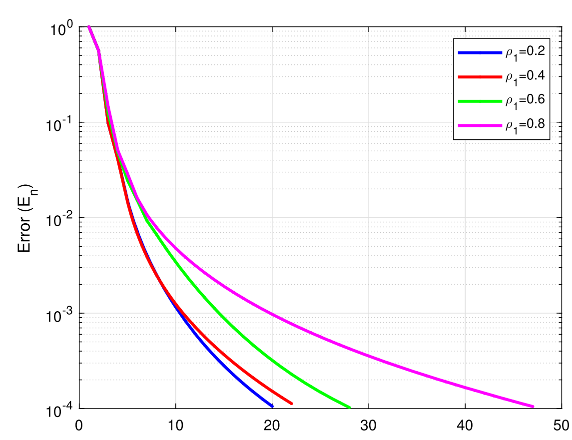

Figure 3 Numerical analysis of Alg 3.1 for different values of ρ_1

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Data availability
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
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Author details
[1]Department of Mathematics, Maharshi Dayanand University, Rohtak 124001, India. [2]Department of Mathematics, Gurugram University, Gurugram 122003, India. [3]Faculty of Computer Studies, Arab Open University, Riyadh 11681, Saudi Arabia. [4]Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia.

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