Abuchu et al. *Journal of Inequalities and Applications* (2024) 2024:38
https://doi.org/10.1186/s13660-024-03113-5

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Modified inertial viscosity extrapolation method for solving quasi-monotone variational inequality and fixed point problems in real Hilbert spaces

Jacob A. Abuchu[1,2], Austine E. Ofem[1], Hüseyin Işık[3,5*], Godwin C. Ugwunnadi[4,5] and Ojen K. Narain[1]

*Correspondence:
isikhuseyin76@gmail.com
[3]Department of Engineering Science, Bandırma Onyedi Eylül University, Bandırma 10200, Balıkesir, Turkey
[5]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa 0204, Pretoria, South Africa
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce and study a viscous-type extrapolation algorithm for finding a solution of the variational inequality problem and a fixed point constraint of quasi-nonexpansive mappings under the scope of real Hilbert spaces when the underlying cost operator is quasi-monotone. The method involves inertial viscosity approximation and a constructed self-adjustable step size condition that depends solely on the information of the previous step. We establish a strong convergence result of the proposed method under certain mild conditions on the algorithm parameters. Finally, to demonstrate the gain of our method, some numerical examples are presented in comparison with some related methods in literature.

Mathematics Subject Classification: 47H05; 47J20; 47J25; 65K15

Keywords: Quasi-monotone operator; Variational inequality; Strong convergence; Inertial extrapolation method; Viscosity approximation

1 Introduction

Let $\mathcal{K}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$ endowed with the inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let $\mathcal{L}:\mathcal{H}\to\mathcal{H}$ be a real single-valued nonlinear mapping. Then the variational inequality problem (VIP) is formulated as follows:

$$\text{find } q^*\in\mathcal{K} \text{ such that } \langle\mathcal{L}q^*, z-q^*\rangle\geq 0 \text{ for all } z\in\mathcal{K}. \tag{1.1}$$

VIP (1.1) is a very important tool in optimization theory and other fields of applied mathematical sciences (see [2, 5, 9, 10, 12, 13, 15] and references therein). The notion of variational inequality can be traced back to the Italian mathematician Stampacchia [28]. It was used as a tool for modeling problems in mechanics (the VIP was also independently formulated by Fichera [13]). The theory of variational inequalities is a crucial one in studying a wide range of problems in pure and applied sciences in a simple, natural and unified framework. There is a known relationship between the VIP and the fixed point problem

(FPP). Finding the solution of VIP (1.1) is the same as finding the solution of the FPP:

$$\text{find } q^* \in \mathcal{K} \text{ such that } q^* = P_{\mathcal{K}}\big(q^* - \mu\mathcal{L}q^*\big), \tag{1.2}$$

where $0 < \mu \in \mathbb{R}$ and $P_{\mathcal{K}}$ is a metric projection of $\mathcal{H}$ onto $\mathcal{K}$. There are several algorithms for finding the solution of VIP (1.1). Predominant among these are the projection algorithms. Projection algorithms leverage on projections onto the feasible set $\mathcal{K}$ or onto some related sets in order to iteratively obtain a solution. Simplest among these methods is the gradient projection technique by Goldstein [16]:

$$\begin{cases} q_0 \in \mathcal{K} \subset \mathbb{R}^n, \qquad \xi \in (0, \frac{2\mu}{L^2}), \\ q_{n+1} = P_{\mathcal{K}}(q_n - \xi\mathcal{L}q_n), \quad n \geq 0, \end{cases} \tag{1.3}$$

where $\mathcal{L}$ is μ-strongly monotone and L-Lipschitzian and $P_{\mathcal{K}}$ is a metric projection of $\mathbb{R}^n$ onto $\mathcal{K}$. It has been established that the sequence $\{q_n\}$ constructed by (1.3) converges uniquely to the solution of VIP (1.1). The stringent hypothesis associated with the cost operator in (1.3) limits application and efficiency of this gradient method. To mitigate this, many researchers have attempted to weaken some conditions on the cost operator $\mathcal{L}$. For instance, Korpelevich [19] and Antipin [4] proposed and analyzed the extragradient method (EM) for approximating the solution of VIP (1.1) in finite-dimensional Euclidean spaces when the associated operator $\mathcal{L}$ is monotone and Lipschitz continuous. They proposed the iterative method

$$\begin{cases} q_0 \in \mathcal{H}, \qquad \zeta \in (0, \frac{1}{L}), \\ z_n = P_{\mathcal{K}}(q_n - \zeta\mathcal{L}q_n), \\ q_{n+1} = P_{\mathcal{K}}(q_n - \zeta\mathcal{L}z_n), \quad n \geq 0. \end{cases} \tag{1.4}$$

They proved that the sequence $\{q_n\}$ generated by (1.4) converges weakly to the solution of VIP (1.1) in a finite-dimensional space. The iterative EM has been extensively studied and extended by many researchers. Authors like He et al. [18], Nadezhkina and Takahashi [21], Noor [22], Popov [25], and many others extended EM to infinite-dimensional real Hilbert spaces with better conditions on the cost operator $\mathcal{L}$, for instance when $\mathcal{L}$ is pseudo-monotone or quasi-monotone. One copious defect of the EM is the calculation of orthogonal projections onto the feasible set $\mathcal{K}$ twice per iteration. This can pose a serious deficiency if the feasible set $\mathcal{K}$ is structurally complex. Thus, to overcome this drawback, in 2011, Censor et al. [6] proposed an improved iterative method termed the subgradient extragradient method (SEGM); among its main objectives are reducing the number of projections onto $\mathcal{K}$ per iteration and achieving convergence under certain conditions in infinite-dimensional Hilbert spaces. The authors achieved the first objective by replacing the second projection in (1.4) with projection onto constructible half-space χ_n. The iterative algorithm is as follows:

$$\begin{cases} q_0 \in \mathcal{H}, \qquad \zeta \in (0, \frac{1}{L}), \\ z_n = P_{\mathcal{K}}(q_n - \zeta\mathcal{L}q_n), \\ \chi_n = \{q \in \mathcal{H} : \langle q_n - \zeta\mathcal{L}q_n - z_n, q - z_n\rangle \leq 0\}, \\ q_{n+1} = P_{\chi_n}(q_n - \zeta\mathcal{L}z_n), \quad n \geq 0. \end{cases} \tag{1.5}$$

The authors established that the sequence $\{q_n\}$ generated by (1.5) converges weakly to the solution of VIP (1.1) under some conditions on the algorithm parameters. Due to the gain of SEGM (1.5), the iterative method witnessed several modifications (see [6–8, 33] and references therein) with some conditions imposed on the algorithm parameters which guaranteed strong convergence in infinite-dimensional Hilbert spaces.

Also, to mitigate the drawback associated with the EM, Tseng introduced another variant of the EM known as Tseng's EM [37]. The iterative algorithm is as follows:

$$\begin{cases} z_n = P_{\mathcal{K}}(q_n - \lambda \mathcal{L} q_n), \\ q_{n+1} = z_n - \lambda(\mathcal{L} z_n - \mathcal{L} q_n), \quad n \geq 0, \end{cases} \tag{1.6}$$

where $\lambda \in (0, \frac{1}{L})$ and the underlying operator $\mathcal{L}$ is maximally monotone. Tseng's EM has enjoyed several modifications and extensions by many authors (see [30–32, 36, 39, 41, 42, 44, 45] and references therein).

Recently, researchers have devoted their studies to improving the rate of convergence of iterative methods. One of the techniques of achieving faster convergence is by the introduction of an inertial term into the iterative scheme. This has been shown to be an efficient technique for accelerating the convergence of such iterative methods. The inertial technique emanated from a discrete analog of a second-order dissipative dynamical system which is known for its efficiency in improving the convergence rate of iterative methods. The well-known Polyak heavy ball algorithm [24] in convex optimization, which is an inertial extrapolation process for minimizing a smooth convex function, is the first such method. This popular technique has been used by many authors in different methods for approximating the solution of VIP (1.1) and other related optimization problems (see [1, 2, 8, 20, 27] for details).

Recently, Gang et al. [14] proposed the following modified Tseng extragradient iterative algorithm for solving VIPs in real Hilbert spaces when the underlying operator is pseudomonotone and non-Lipschitzian:

$$\begin{cases} q_0, q_1 \in \mathcal{H}, \\ t_n = q_n + \theta_n(q_n - q_{n-1}), \\ z_n = P_{\mathcal{K}}(t_n - \lambda_n \mathcal{L}(t_n)), \\ r_n = z_n - \lambda_n(\mathcal{L}(t_n) - \mathcal{L}(z_n)), \\ q_{n+1} = \beta_n f(q_n) + (1 - \beta_n) r_n, \quad n \geq 1, \end{cases} \tag{1.7}$$

where the step size λ_n is updated using the Armijo line search rule. The authors show that under some conditions, the sequence $\{q_n\}$ generated by (1.7) converges strongly to the unique solution of (1.1).

We have noticed that many modified Tseng extragradient iterative algorithms mostly entail that the cost operator $\mathcal{L}$ is either monotone and Lipschitzian or pseudo-monotone. Now a pertinent question arises: Can we propose a modified Tseng extragradient algorithm with a more robust cost operator $\mathcal{L}$, say, a modified Tseng extragradient method when the cost operator is quasi-monotone? And can we extend the iterative method to solving the FPP when the underlying operator is quasi-nonexpansive?

Motivated by the work of Gang et al. [14] and many others in this direction, we give affirmative answers to these questions. We propose a modified Tseng extragradient algorithm for approximating the solution of quasi-monotone VIPs with a fixed point constraint of quasi-nonexpansive mappings in the framework of real Hilbert spaces. Our algorithm is a viscous iterative one that has an inertial extrapolation term incorporated and is embellished with relaxation. In this work, the cost operator $\mathcal{L}$ is assumed to be quasi-monotone instead of the usual condition that $\mathcal{L}$ is monotone, as in [37], or pseudo-monotone, as in [14] and most literature. We obtain strong convergence results for the sequence constructed by this method under some mild assumptions on the algorithm parameters. Finally, we give some numerical examples to demonstrate the applicability and efficiency of our proposed method.

2 Preliminaries

In this section, we will recall some definitions and present results that will help us in our convergence analysis in the subsequent section. Let $\mathcal{H}$ be a real Hilbert space and let $\mathcal{K}$ be a nonempty, closed and convex subset of $\mathcal{H}$. For every $q \in \mathcal{H}$ there exists a unique nearest point in $\mathcal{K}$ denoted by $P_{\mathcal{K}}q$ such that

$$\|q - P_{\mathcal{K}}q\| = \inf\{\|q - z\| : z \in \mathcal{K}\}.$$

$P_{\mathcal{K}}$ is termed the metric projection of $\mathcal{H}$ onto $\mathcal{K}$ and is known to be nonexpansive. Moreover, $P_{\mathcal{K}}$ is associated with the following characterization.

Lemma 2.1 ([18]) *Let $P_{\mathcal{K}} : \mathcal{H} \to \mathcal{K}$ be a metric projection of $\mathcal{H}$ onto $\mathcal{K}$. Then the following hold*:

(i) $\|P_{\mathcal{K}}q - P_{\mathcal{K}}s\|^2 \le \langle P_{\mathcal{K}}q - P_{\mathcal{K}}s, q - s\rangle$, $\forall q, s \in \mathcal{H}$;

(ii) $z = P_{\mathcal{K}}q$ *if and only if* $\langle q - z, s - z\rangle \le 0$, $\forall s \in \mathcal{K}$;

(iii) $\|q - P_{\mathcal{K}}s\|^2 + \|P_{\mathcal{K}}s - s\|^2 \le \|q - s\|^2$, $\forall q \in \mathcal{H}$, $s \in \mathcal{K}$.

Definition 2.1 ([43, 44]) Let $\mathcal{H}$ be real Hilbert space and let $\mathcal{K}$ be a nonempty, closed and convex subset of $\mathcal{H}$. Let $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ be a real single-valued mapping. Then $\mathcal{F}$ is said to be:

(a) α-Lipschitz continuous if there exists $\alpha > 0$ such that

$$\|\mathcal{F}q - \mathcal{F}s\| \le \alpha\|q - s\|, \quad \forall q, s \in \mathcal{H};$$

(b) nonexpansive if

$$\|\mathcal{F}q - \mathcal{F}s\| \le \|q - s\|, \quad \forall q, s \in \mathcal{H};$$

(c) quasi-nonexpansive if the fixed point set of $\mathcal{F}$, $F(\mathcal{F})$, is nonempty and

$$\|\mathcal{F}q - p\| \le \|q - p\|, \quad \forall q \in \mathcal{H}, p \in F(\mathcal{F});$$

(d) β-contraction if there exists $\beta \in [0, 1)$ such that

$$\|\mathcal{F}q - \mathcal{F}s\| \le \beta\|q - s\|, \quad \forall q, s \in \mathcal{H}.$$

Definition 2.2 ([34]) Let $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ be a real single-valued operator and let $\mathcal{K}$ be a nonempty, closed and convex subset of $\mathcal{H}$. Then $\mathcal{G}$ is said to be:

(i) γ-strongly monotone on $\mathcal{K}$ if there exists $\gamma > 0$ such that

$$\langle \mathcal{G}q - \mathcal{G}s, q - s\rangle \geq \gamma \|q - s\|^2, \quad \forall q, s \in \mathcal{K};$$

(ii) monotone on $\mathcal{K}$ if

$$\langle \mathcal{G}q - \mathcal{G}s, q - s\rangle \geq 0, \quad \forall q, s \in \mathcal{K};$$

(iii) pseudo-monotone on $\mathcal{K}$ if

$$\langle \mathcal{G}q, s - q\rangle \geq 0 \quad \Longrightarrow \quad \langle \mathcal{G}s, s - q\rangle \geq 0, \quad \forall q, s \in \mathcal{K};$$

(iv) quasi-monotone on $\mathcal{K}$ if

$$\langle \mathcal{G}q, q - s\rangle > 0 \quad \Longrightarrow \quad \langle \mathcal{G}s, q - s\rangle \geq 0, \quad \forall q, s \in \mathcal{K};$$

(v) sequentially weakly continuous if for each sequence $\{q_n\}$ that converges weakly to a point $q^* \in \mathcal{H}$ the sequence $\{\mathcal{G}q_n\}$ converges weakly to a point $\mathcal{G}q^*$.

Lemma 2.2 ([38]) *Let $\mathcal{H}$ be a real Hilbert space with $\mu, \nu \in \mathcal{H}$ and $\alpha \in \mathbb{R}$. Then the following hold*:

(i) $2\langle \mu, \nu\rangle = \|\mu\|^2 + \|\nu\|^2 - \|\mu - \nu\|^2 = \|\mu + \nu\|^2 - \|\mu\|^2 - \|\nu\|^2$;
(ii) $\|\mu - \nu\|^2 \leq \|\mu\|^2 + 2\langle \nu, \nu - \mu\rangle$;
(iii) $\|\alpha\mu + (1-\alpha)\nu\|^2 = \alpha\|\mu\|^2 + (1-\alpha)\|\nu\|^2 - \alpha(1-\alpha)\|\mu - \nu\|^2$.

Lemma 2.3 ([17, 46]) *Let $\mathcal{K}$ be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$ and let $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ be an L-Lipschitzian and quasi-monotone operator. Let $z \in \mathcal{K}$. If for some $q^* \in \mathcal{K}$ we have $\langle \mathcal{L}(z), q^* - z\rangle \geq 0$, then at least one of the following must hold*:

$$\langle \mathcal{L}(q^*), q^* - z\rangle \geq 0 \quad \text{or} \quad \langle \mathcal{L}(z), q^* - z\rangle \leq 0, \quad \forall q^* \in \mathcal{K}.$$

Lemma 2.4 [26] *Let $\{\phi_n\}$ be a sequence of positive real numbers. Let $\{\delta_n\}$ and $\{\rho_n\}$ be sequences in $(0,1)$ with $\sum_{n=1}^{\infty}\delta_n = \infty$. Suppose that ϕ_n satisfies the inequality*

$$\phi_{n+1} \leq (1-\delta_n)\phi_n + \delta_n\rho_n, \quad \forall n \in \mathbb{N}.$$

If $\limsup_{k\to\infty}\rho_{n_k} \leq 0$ for every subsequence ϕ_{n_k} of ϕ_n satisfying the condition $\liminf_{k\to\infty}(\phi_{n_k+1} - \phi_{n_k}) \geq 0$, then $\lim_{n\to\infty}\phi_n = 0$.

3 Main results

Throughout this work, we shall use $q_n \to q^*$ (respectively $q_n \rightharpoonup q^*$) to denote that the sequence $\{q_n\}$ converges strongly (respectively weakly) to a point q^* as $n \to \infty$. For the purpose of convergence analysis of our method, we shall make the following assumptions.

Assumption 3.1 Suppose that:

(B1) the real Hilbert space $\mathcal{H}$ has a nonempty, closed and convex subset $\mathcal{K}$;

(B2) the operator $\mathcal{L}:\mathcal{H}\to\mathcal{H}$ is quasi-monotone, ℓ_0-Lipschitzian and sequentially weakly continuous;

(B3) $\mathcal{T}:\mathcal{H}\to\mathcal{H}$ is a quasi-nonexpansive mapping which is semiclosed at the origin such that $F(\mathcal{T})\neq\emptyset$, $\mathcal{S}_i:\mathcal{H}\to\mathcal{H}$ is a sequence of nonexpansive mappings and $h:\mathcal{H}\to\mathcal{H}$ is a λ-contraction;

(B4) the solution set of VIP (1.1) $VI(\mathcal{K},\mathcal{L})\neq\emptyset$ and $\Gamma:=VI(\mathcal{K},\mathcal{L})\cap F(\mathcal{T})\neq\emptyset$;

(B5) the control sequences $\{\sigma_n\}$ and $\{\xi_n\}$ are positive real sequences in $(0,1)$ with $\{\sigma_n\}$ satisfying the property $\lim_{n\to\infty}\sigma_n=0$, $\sum_{n=1}^{\infty}\sigma_n=\infty$, with $\xi_n=\circ(\sigma_n)$.

Algorithm 3.2 *Initialization:* Choose $\zeta_0>0$, $\lambda\in[0,1)$, $\theta\in[0,1]$, $q_0,q_1\in\mathcal{H}$.

Iterative process steps: Given the iterates q_{n-1} and q_n for each $n\geq 1$, choose θ_n such that $0\leq\theta_n\leq\bar{\theta}_n$, where

$$\bar{\theta}_n=\min\left\{\theta,\frac{\xi_n}{\|q_n-q_{n-1}\|}\right\},\quad \text{if } q_n\neq q_{n-1}; \text{otherwise, set } \bar{\theta}_n=\theta.$$

Step 1: Set $n=1$. We calculate the iterate q_{n+1} as follows:

$$w_n=q_n+\theta_n(\mathcal{S}_iq_n-\mathcal{S}_iq_{n-1}),$$

and we compute

$$z_n=P_{\mathcal{K}}(w_n-\zeta_n\mathcal{L}w_n).$$

If $z_n=w_n$, stop; z_n is the required solution. Else, execute step 2.

Step 2: Compute

$$q_{n+1}=\sigma_nh(q_n)+(1-\sigma_n)\mathcal{T}p_n,$$

where

$$p_n=z_n-\zeta_n(\mathcal{L}z_n-\mathcal{L}w_n),$$

and update $\{\zeta_n\}$ as follows:

$$\zeta_{n+1}=\begin{cases}\min\{\frac{\mu\|w_n-z_n\|}{\|\mathcal{L}w_n-\mathcal{L}z_n\|},\zeta_n\}, & \text{if } \mathcal{L}w_n\neq\mathcal{L}z_n,\\ \zeta_n, & \text{otherwise.}\end{cases}\tag{3.1}$$

Set $n:=n+1$ and return to *Step* 1.

Remark 3.3 We first highlight some novelties of Algorithm 3.2 with respect to others in the literature.

(i) In [14, 44], the authors introduced modified Tseng extragradient-type algorithms for solving VIPs in the framework of real Hilbert spaces. We observed that their iterative methods used a class of pseudo-monotone operators, while in this work we propose a modified inertial Tseng extragradient algorithm for solving VIPs in real Hilbert spaces when the associated operator is quasi-monotone and there is a fixed point constraint of quasi-nonexpansive mappings. It can be observed that the class operators considered in this work are more general and include some important classes of operators; specifically, the class of quasi-monotone operator includes strongly monotone operators, monotone operators, pseudo-monotone operators, and others (see [23] for more details).

(ii) In the proposed algorithm, the self-adjustable step size condition $\zeta_{n+1} \in \min\{\frac{\mu\|w_n - z_n\|}{\|\mathcal{L}w_n - \mathcal{L}z_n\|}, \zeta_n\}$ applied in Algorithm 3.2 is very simple and does not possess any inner loop, unlike the line search technique employed in [14, 33], which uses inner loops and might consume additional computation time for determining the step size. It also does not require prior knowledge of the operator norm $\|\mathcal{L}\|$.

(iii) The proposed method involves inertial and relaxation terms, which are vital in improving the rate of convergence for solving VIP (1.1).

(iv) The proof of our convergence analysis (that is, strong convergence of Theorem 4.6) does not follow the usual "two cases approach" as seen in many papers handling optimization problems (see [3, 11, 27, 29, 44]).

4 Convergence analysis

Lemma 4.1 (See for instance Lemma 3.1 in [40]) *The sequence $\{\zeta_n\}$ generated by Algorithm 3.2 is monotonically nonincreasing and bounded below by* $\min\{\frac{\mu}{\ell_0}, \zeta_0\}$. *Moreover,* $\|\mathcal{L}z_n - \mathcal{L}w_n\| \leq \frac{\mu}{\zeta_{n+1}}\|z_n - w_n\|$, $\forall n \geq 1$.

Proof From the construction of the sequence $\{\zeta_n\}$, it is obvious to see that $\{\zeta_n\}$ is monotone decreasing. Since the operator $\mathcal{L}$ is ℓ_0-Lipschitz continuous, we have for $\mathcal{L}w_n \neq \mathcal{L}z_n$

$$\frac{\mu\|z_n - w_n\|}{\|\mathcal{L}z_n - \mathcal{L}w_n\|} \geq \frac{\mu\|z_n - w_n\|}{\ell_0\|z_n - w_n\|} = \frac{\mu}{\ell_0}.$$

By induction it is clear that $\{\zeta_n\}$ is bounded below with lower bound $\min\{\frac{\mu}{\ell_0}, \zeta_0\}$. Since every nonincreasing monotone sequence that is bounded is convergent, it follows that the limit of $\{\zeta_n\}$ exists and we can denote this by

$$\lim_{n\to\infty} \zeta_n = \zeta > 0. \tag{4.1}$$

□

Lemma 4.2 *Let $\{p_n\}$, $\{z_n\}$ and $\{w_n\}$ be sequences generated by Algorithm 3.2 under Assumption 3.1. Then $\{p_n\}$ satisfies the inequality*

$$\|p_n - q^*\|^2 \leq \|w_n - q^*\|^2 - (1 - \mu^2)\|z_n - w_n\|^2. \tag{4.2}$$

Proof Let $q^* \in \Gamma$. Using the definition of p_n in Step 2 of Algorithm 3.2, Lemma 2.1, Lemma 2.3 and Lemma 4.1, we have

$$\begin{aligned}\|p_n - q^*\|^2 &= \|z_n - \zeta_n(\mathcal{L}z_n - \mathcal{L}w_n) - q^*\|^2 \\ &= \|z_n - q^*\|^2 + \zeta_n^2\|\mathcal{L}z_n - \mathcal{L}w_n\|^2 - 2\zeta_n\langle z_n - q^*, \mathcal{L}z_n - \mathcal{L}w_n\rangle\end{aligned}$$

$$
\begin{aligned}
&= \|z_n - w_n + w_n - q^*\|^2 + \zeta_n^2\|\mathcal{L}z_n - \mathcal{L}w_n\|^2 - 2\zeta_n\langle z_n - q^*, \mathcal{L}z_n - \mathcal{L}w_n\rangle \\
&= \|w_n - q^*\|^2 + \|z_n - w_n\|^2 + 2\langle w_n - q^*, z_n - w_n\rangle + \zeta_n^2\|\mathcal{L}z_n - \mathcal{L}w_n\|^2 \\
&\quad - 2\zeta_n\langle z_n - q^*, \mathcal{L}z_n - \mathcal{L}w_n\rangle \\
&= \|w_n - q^*\|^2 + \|z_n - w_n\|^2 - 2\langle z_n - w_n, z_n - w_n\rangle + 2\langle z_n - q^*, z_n - w_n\rangle \\
&\quad + \zeta_n^2\|\mathcal{L}z_n - \mathcal{L}w_n\|^2 - 2\zeta_n\langle z_n - q^*, \mathcal{L}z_n - \mathcal{L}w_n\rangle \\
&= \|w_n - q^*\|^2 + \|z_n - w_n\|^2 - 2\|z_n - w_n\|^2 + 2\langle z_n - q^*, z_n - w_n\rangle \\
&\quad + \zeta_n^2\|\mathcal{L}z_n - \mathcal{L}w_n\|^2 - 2\zeta_n\langle z_n - q^*, \mathcal{L}z_n - \mathcal{L}w_n\rangle \\
&= \|w_n - q^*\|^2 - \|z_n - w_n\|^2 + 2\langle z_n - q^*, z_n - w_n\rangle + \zeta_n^2\|\mathcal{L}z_n - \mathcal{L}w_n\|^2 \\
&\quad - 2\zeta_n\langle z_n - q^*, \mathcal{L}z_n - \mathcal{L}w_n\rangle \\
&= \|w_n - q^*\|^2 - \|z_n - w_n\|^2 + 2\langle z_n - q^*, z_n - w_n\rangle + \zeta_n^2\|\mathcal{L}z_n - \mathcal{L}w_n\|^2 \\
&\quad - 2\zeta_n\langle \mathcal{L}z_n, z_n - q^*\rangle + 2\zeta_n\langle \mathcal{L}w_n, z_n - q^*\rangle \\
&= \|w_n - q^*\|^2 - \|z_n - w_n\|^2 + 2\langle z_n - w_n + \zeta_n\mathcal{L}w_n, z_n - q^*\rangle \\
&\quad + \zeta_n^2\|\mathcal{L}z_n - \mathcal{L}w_n\|^2 - 2\zeta_n\langle \mathcal{L}z_n, z_n - q^*\rangle.
\end{aligned}
$$

Since $\mathcal{L}$ is quasi-monotone on $\mathcal{K}$ and $z_n \in \mathcal{K}$, by Lemma 2.3 we have

$$\langle \mathcal{L}z_n, z_n - q^*\rangle \geq 0. \tag{4.3}$$

Using the characterization of the metric projection in Lemma 2.1, we obtain

$$\langle w_n - z_n - \zeta_n\mathcal{L}w_n, z_n - q^*\rangle \geq 0. \tag{4.4}$$

With the help of (4.3) and (4.4), we get

$$\|p_n - q^*\|^2 \leq \|w_n - q^*\|^2 - \|z_n - w_n\|^2 + \zeta_n^2\|\mathcal{L}z_n - \mathcal{L}w_n\|^2. \tag{4.5}$$

Also, from our adaptive step size (3.1),

$$\zeta_n^2\|\mathcal{L}z_n - \mathcal{L}w_n\|^2 \leq \mu^2\|z_n - w_n\|^2. \tag{4.6}$$

Combining (4.5) and (4.6),

$$
\begin{aligned}
\|p_n - q^*\|^2 &\leq \|w_n - z^*\|^2 - \|z_n - w_n\|^2 + \mu^2\|z_n - w_n\|^2 \\
&= \|w_n - q^*\|^2 - (1 - \mu^2)\|z_n - w_n\|^2 \qquad (4.7) \\
&\leq \|w_n - q^*\|^2, \qquad (4.8)
\end{aligned}
$$

which yields the desired result. □

Lemma 4.3 *Let $\{q_n\}$ be a sequence generated by Algorithm 3.2 under Assumption 3.1. Then $\{q_n\}$ is bounded.*

Proof Let $q^* \in \Gamma$. Then from the definition of (w_n) in Algorithm 3.2, we have

$$\begin{aligned}\|w_n - q^*\| &= \|q_n + \theta_n(\mathcal{S}_i q_n - \mathcal{S}_i q_{n-1}) - q^*\| \\ &\le \|q_n - q^*\| + \sigma_n \frac{\theta_n}{\sigma_n}\|\mathcal{S}_i q_n - \mathcal{S}_i q_{n-1}\| \\ &\le \|q_n - q^*\| + \sigma_n\left(\frac{\theta_n}{\sigma_n}\|q_n - q_{n-1}\|\right).\end{aligned} \tag{4.9}$$

Since $\frac{\theta_n}{\sigma_n}\|q_n - q_{n-1}\| \le \frac{\xi_n}{\sigma_n}$, from which by assumption (B5) it follows that $\lim_{n\to\infty}\frac{\theta_n}{\sigma_n}\|q_n - q_{n-1}\| = 0$, the sequence $\{\frac{\theta_n}{\sigma_n}\|q_n - q_{n-1}\|\}$ is bounded. Thus, there exists a real constant $K_1 > 0$ such that $\frac{\theta_n}{\sigma_n}\|q_n - q_{n-1}\| \le K_1$ for all $n \ge 1$. Hence,

$$\|w_n - q^*\| \le \|q_n - q^*\| + \sigma_n K_1. \tag{4.10}$$

Now, using the definition of sequence $\{q_{n+1}\}$ in Algorithm 3.2, Lemma 4.2 and (4.10), we deduce

$$\begin{aligned}\|q_{n+1} - q^*\| &= \|\sigma_n h(q_n) + (1-\sigma_n)\mathcal{T}p_n - q^*\| \\ &= \|\sigma_n[h(q_n) - q^*] + (1-\sigma_n)[\mathcal{T}p_n - q^*]\| \\ &\le \sigma_n\|h(q_n) - q^*\| + (1-\sigma_n)\|\mathcal{T}p_n - q^*\| \\ &\le \sigma_n\|h(q_n) - q^*\| + (1-\sigma_n)\|p_n - q^*\| \\ &= \sigma_n\|h(q_n) - h(q^*) + h(q^*) - q^*\| + (1-\sigma_n)\|p_n - q^*\| \\ &\le \sigma_n\|h(q_n) - h(q^*)\| + \sigma_n\|h(q^*) - q^*\| + (1-\sigma_n)\|p_n - q^*\| \\ &\le \lambda\sigma_n\|q_n - q^*\| + \sigma_n\|h(q^*) - q^*\| + (1-\sigma_n)\|p_n - q^*\| \\ &\le \lambda\sigma_n\|q_n - q^*\| + \sigma_n\|h(q^*) - q^*\| + (1-\sigma_n)\|w_n - q^*\| \\ &\le \lambda\sigma_n\|q_n - q^*\| + \sigma_n\|h(q^*) - q^*\| + (1-\sigma_n)[\|q_n - q^*\| + \sigma_n K_1] \\ &= (1-\sigma_n(1-\lambda))\|q_n - q^*\| + \sigma_n\|h(q^*) - q^*\| + \sigma_n(1-\sigma_n)K_1 \\ &= (1-\sigma_n(1-\lambda))\|q_n - q^*\| + \sigma_n[\|h(q^*) - q^*\| + (1-\sigma_n)K_1] \\ &\le (1-\sigma_n(1-\lambda))\|q_n - q^*\| + \sigma_n(1-\lambda)\left[\frac{\|h(q^*) - q^*\| + K_1}{1-\lambda}\right] \\ &\le \max\{\|q_n - q^*\|, (1-\lambda)^{-1}[\|h(q^*) - q^*\| + K_1]\}.\end{aligned} \tag{4.11}$$

Thus, by induction

$$\|q_n - q^*\| \le \max\{\|q_1 - q^*\|, (1-\lambda)^{-1}[\|h(q^*) - q^*\| + K_1]\}, \quad \forall n \ge 1. \tag{4.12}$$

Since the sequence $\{\|q_n - q^*\|\}$ is bounded, it follows that $\{q_n\}$ is bounded. Hence, $\{w_n\}$, $\{z_n\}$, $\{p_n\}$ and $\{\mathcal{T}p_n\}$ are all bounded. □

Lemma 4.4 *Let $\{z_n\}$ and $\{w_n\}$ be sequences generated by Algorithm 3.2 under Assumption 3.1. Suppose $\{z_{n_\ell}\}$ and $\{w_{n_\ell}\}$ are subsequences of $\{z_n\}$ and $\{w_n\}$, respectively, with $\{z_{n_\ell}\}$ converging weakly to a point $q^* \in \mathcal{H}$ and $\lim_{\ell\to\infty}\|w_{n_\ell} - z_{n_\ell}\| = 0$. Then $q^* \in VI(\mathcal{K}, \mathcal{L})$.*

Proof We know that $z_{n_\ell} = P_\ell(w_{n_\ell} - \zeta_{n_\ell}\mathcal{L}w_{n_\ell})$. Hence, from the characterization of the projection operator (see Lemma 2.1), we get

$$\langle w_{n_\ell} - \zeta_{n_\ell}\mathcal{L}w_{n_\ell} - z_{n_\ell}, q - z_{n_\ell}\rangle \le 0, \quad \forall q \in \mathcal{K}.$$

This implies

$$\langle w_{n_\ell} - z_{n_\ell}, q - z_{n_\ell}\rangle - \zeta_{n_\ell}\langle \mathcal{L}w_{n_\ell}, q - z_{n_\ell}\rangle \le 0,$$
$$\langle z_{n_\ell} - w_{n_\ell}, q - w_{n_\ell}\rangle \le \zeta_{n_\ell}\langle \mathcal{L}w_{n_\ell}, q - z_{n_\ell}\rangle = \zeta_{n_\ell}\langle \mathcal{L}w_{n_\ell}, w_{n_\ell} - z_{n_\ell}\rangle + \zeta_{n_\ell}\langle \mathcal{L}w_{n_\ell}, q - w_{n_\ell}\rangle,$$

and hence

$$\frac{1}{\zeta_{n_\ell}}\langle z_{n_\ell} - w_{n_\ell}, q - z_{n_\ell}\rangle + \langle \mathcal{L}w_{n_\ell}, z_{n_\ell} - w_{n_\ell}\rangle \le \langle \mathcal{L}w_{n_\ell}, q - w_{n_\ell}\rangle. \tag{4.13}$$

Since $\{z_{n_k}\}$ converges weakly to a point $q^* \in \mathcal{H}$ and $\lim_{\ell\to\infty}\|w_{n_\ell} - z_{n_\ell}\| = 0$, it follows that $\{w_{n_\ell}\}$ and $\{\mathcal{L}w_{n_\ell}\}$ are bounded sequences. We have established also from Lemma 4.1 that $\lim_{\ell\to\infty}\zeta_{n_\ell} = \zeta > 0$. So from (4.13), we obtain

$$0 \le \liminf_{\ell\to\infty}\langle \mathcal{L}w_{n_\ell}, q - w_{n_\ell}\rangle \le \limsup_{\ell\to\infty}\langle \mathcal{L}w_{n_\ell}, q - w_{n_\ell}\rangle < \infty, \quad \forall q \in \mathcal{K}. \tag{4.14}$$

We observe that

$$\begin{aligned}\langle \mathcal{L}z_{n_\ell}, q - z_{n_\ell}\rangle &= \langle \mathcal{L}z_{n_\ell}, q - w_{n_\ell} + w_{n_\ell} - z_{n_\ell}\rangle \\ &= \langle \mathcal{L}z_{n_\ell} - \mathcal{L}w_{n_\ell} + \mathcal{L}w_{n_\ell}, q - w_{n_\ell}\rangle + \langle \mathcal{L}z_{n_\ell}, w_{n_\ell} - z_{n_\ell}\rangle \\ &= \langle \mathcal{L}z_{n_\ell} - \mathcal{L}w_{n_\ell}, q - w_{n_\ell}\rangle + \langle \mathcal{L}w_{n_\ell}, q - w_{n_\ell}\rangle + \langle \mathcal{L}z_{n_\ell}, w_{n_\ell} - z_{n_\ell}\rangle. \end{aligned} \tag{4.15}$$

Recall that the operator $\mathcal{L}$ is ℓ_0-Lipschitzian, so

$$\lim_{\ell\to\infty}\|\mathcal{L}w_{n_\ell} - \mathcal{L}z_{n_\ell}\| \le \lim_{\ell\to\infty}\big(\ell_0\|w_{n_k} - z_{n_\ell}\|\big) = 0. \tag{4.16}$$

Combining (4.14), (4.15) and (4.16), we infer

$$0 \le \liminf_{\ell\to\infty}\langle \mathcal{L}z_{n_\ell}, q - z_{n_\ell}\rangle \le \limsup_{\ell\to\infty}\langle \mathcal{L}z_{n_\ell}, q - z_{n_\ell}\rangle < \infty, \quad \forall q \in \mathcal{K}. \tag{4.17}$$

Using (4.17), we shall look at the following two cases.

Case A: Suppose that $\limsup_{k\to\infty}\langle \mathcal{L}z_{n_\ell}, q - z_{n_\ell}\rangle > 0, \forall q \in \mathcal{K}$. Then there exists a subsequence $(z_{n_{\ell_m}})$ of sequence (z_{n_ℓ}) such that $\lim_{m\to\infty}\langle \mathcal{L}z_{n_{\ell_m}}, q - z_{n_{\ell_m}}\rangle > 0$. This implies that one can find $m_0 \ge 1$ such that

$$\langle \mathcal{L}z_{n_{\ell_m}}, q - z_{n_{\ell_m}}\rangle > 0, \quad \forall m \ge m_0.$$

By the quasi-monotonicity of the mapping $\mathcal{L}$, it follows that

$$\langle \mathcal{L}q, q - z_{n_{\ell_m}}\rangle \ge 0, \quad \forall q \in \mathcal{K}, m \ge m_0. \tag{4.18}$$

If we pass to the limit as $m \to \infty$ in (4.18), we get

$$\lim_{m\to\infty} \langle \mathcal{L}q, q - z_{n_{\ell_m}} \rangle = \langle \mathcal{L}q, q - q^* \rangle \geq 0, \quad \forall q \in \mathcal{K}.$$

Hence, $q^* \in VI(\mathcal{K}, \mathcal{L})$.

Case B: Suppose in (4.17),

$$\limsup_{k\to\infty} \langle \mathcal{L}z_{n_\ell}, q - z_{n_\ell} \rangle = 0. \tag{4.19}$$

We construct a nonincreasing positive sequence $\{\eta_\ell\}$ defined by

$$\eta_\ell := \left| \langle \mathcal{L}z_{n_\ell}, q - z_{n_\ell} \rangle \right| + \frac{1}{\ell + 1}. \tag{4.20}$$

Clearly $\eta_\ell \to 0$ as $\ell \to \infty$, so combining (4.19) and (4.20), we get

$$\langle \mathcal{L}z_{n_\ell}, q - z_{n_\ell} \rangle + \eta_\ell > 0. \tag{4.21}$$

Since $\{z_{n_\ell}\} \subset \mathcal{K}$, this implies that $\{\mathcal{L}z_{n_\ell}\}$ is strictly nonzero. We let $\lim_{\ell\to\infty} \|\mathcal{L}z_{n_\ell}\| = K_2 > 0$.

We can infer that

$$\|\mathcal{L}z_{n_\ell}\| > \frac{K_2}{2}. \tag{4.22}$$

Also, we let $\{\psi_{n_\ell}\}$ be a sequence given by $\psi_{n_\ell} = \frac{\mathcal{L}z_{n_\ell}}{\|\mathcal{L}z_{n_\ell}\|^2}$. It follows that

$$\langle \mathcal{L}z_{n_\ell}, \psi_{n_\ell} \rangle = 1. \tag{4.23}$$

By combining (4.21) and (4.23), we obtain

$$\langle \mathcal{L}z_{n_\ell}, q - z_{n_\ell} \rangle + \eta_\ell \langle \mathcal{L}z_{n_\ell}, \varepsilon_{n_\ell} \rangle > 0, \tag{4.24}$$

so

$$\langle \mathcal{L}z_{n_\ell}, q + \eta_k \psi_{n_\ell} - z_{n_\ell} \rangle > 0.$$

Since $\mathcal{L}$ is quasi-monotone on $\mathcal{H}$, we get

$$\left\langle \mathcal{L}(q + \eta_\ell \psi_{n_\ell}), q + \eta_\ell \psi_{n_\ell} - z_{n_\ell} \right\rangle \geq 0. \tag{4.25}$$

But we note that

$$\begin{aligned} \langle \mathcal{L}q, q + \eta_\ell \psi_{n_\ell} - z_{n_\ell} \rangle &= \left\langle \mathcal{L}q - \mathcal{L}(q + \eta_\ell \psi_{n_\ell}) + \mathcal{L}(q + \eta_\ell \psi_{n_\ell}), q + \eta_\ell \psi_{n_\ell} - z_{n_\ell} \right\rangle \\ &= \left\langle \mathcal{L}q - \mathcal{L}(q + \eta_\ell \psi_{n_\ell}), q + \eta_\ell \psi_{n_\ell} - z_{n_\ell} \right\rangle \\ &\quad + \left\langle \mathcal{L}(q + \eta_\ell \psi_{n_\ell}), q + \eta_\ell \psi_{n_\ell} - z_{n_\ell} \right\rangle. \end{aligned} \tag{4.26}$$

Combining (4.25) and (4.26) and applying the Cauchy–Schwartz inequality, we obtain

$$\begin{aligned}\langle \mathcal{L}q, q + \eta_\ell \psi_{n_\ell} - z_{n_\ell}\rangle &\geq \langle \mathcal{L}q - \mathcal{L}(q + \eta_k \psi_{n_\ell}), q + \eta_\ell \psi_{n_\ell} - z_{n_\ell}\rangle \\ &\geq - \|\mathcal{L}q - \mathcal{L}(q + \eta_\ell \psi_{n_\ell})\| \|q + \eta_\ell \psi_{n_\ell} - y_{n_\ell}\|. \end{aligned} \tag{4.27}$$

Since $\mathcal{L}$ is ℓ_0-Lipschitz continuous, we have

$$\langle \mathcal{L}q, q + \eta_\ell \psi_{n_k} - z_{n_\ell}\rangle + \ell_0 \|\eta_\ell \psi_{n_\ell}\| \|q + \eta_\ell \psi_{n_\ell} - z_{n_\ell}\| \geq 0. \tag{4.28}$$

Combining (4.22) and (4.28) and taking into consideration the definition of the sequence $\{\varepsilon_{n_\ell}\}$, we have

$$\langle \mathcal{L}q, q + \eta_\ell \psi_{n_\ell} - z_{n_\ell}\rangle + \frac{2\ell_0}{K_2} \eta_\ell \|q + \eta_\ell \psi_{n_\ell} - z_{n_\ell}\| \geq 0. \tag{4.29}$$

Using (4.29), since $\eta_\ell \to 0$ and $z_{n_\ell} \to q^*$ as $\ell \to \infty$, we get

$$\begin{aligned}&\lim_{\ell\to\infty} \left(\langle \mathcal{L}q, q + \eta_k \psi_{n_\ell} - z_{n_\ell}\rangle + \frac{2\ell_0}{K_2} \eta_\ell \|z + \eta_\ell \psi_{n_\ell} - z_{n_\ell}\| \right) \\ &\quad = \langle \mathcal{L}q, q - q^*\rangle \geq 0, \quad \forall q \in \mathcal{K}. \end{aligned} \tag{4.30}$$

Thus, $q^* \in VI(\mathcal{K}, \mathcal{L})$, completing the proof. □

Lemma 4.5 *Let $\{q_n\}$ be a sequence generated by Algorithm 3.2 under Assumption 3.1. Then for all $n \geq 1$, $\{q_n\}$ satisfies the inequality*

$$\Lambda_{n+1} \leq (1 - \varrho_n)\Lambda_n + \varrho_n \partial_n, \tag{4.31}$$

where $\Lambda_{n+1} := \|q_{n+1} - q^*\|^2$, $\Lambda_n := \|q_n - q^*\|^2$, $\varrho_n := (1-\lambda)\sigma_n$ and $\partial_n := \frac{K_3K_4 + \langle h(q^*) - q^*, q_{n+1} - q^*\rangle}{1-\lambda}$ for some real constants K_3, K_4.

Proof Indeed, using the definition of the sequence $\{q_{n+1}\}$ in Algorithm 3.2, Lemma 2.2 and Lemma 4.2, we have

$$\begin{aligned}\|q_{n+1} - q^*\|^2 &= \|\sigma_n h(q_n) + (1-\sigma_n)\mathcal{T}p_n - q^*\|^2 \\ &= \|\sigma_n h(q_n) - \sigma_n h(q^*) + \sigma_n h(q^*) + (1-\sigma_n)\mathcal{T}p_n - q^*\|^2 \\ &= \|\sigma_n(h(q_n) - h(q^*)) + (1-\sigma_n)(\mathcal{T}p_n - q^*) + \sigma_n(h(q^*) - q^*)\|^2 \\ &\leq \|\sigma_n(h(q_n) - h(q^*)) + (1-\sigma_n)(\mathcal{T}p_n - q^*)\|^2 + 2\sigma_n\langle h(q^*) - q^*, q_{n+1} - q^*\rangle \\ &\leq \sigma_n\|h(q_n) - h(q^*)\|^2 + (1-\sigma_n)\|\mathcal{T}p_n - q^*\|^2 \\ &\quad - \sigma_n(1-\sigma_n)\|(h(q_n) - \mathcal{T}p_n) - (h(q^*) - q^*)\|^2 \\ &\quad + 2\sigma_n\langle h(q^*) - q^*, q_{n+1} - q^*\rangle \\ &\leq \sigma_n\lambda^2\|q_n - q^*\|^2 + (1-\sigma_n)\|p_n - q^*\|^2 \\ &\quad - \sigma_n(1-\sigma_n)\|(h(q_n) - \mathcal{T}p_n) - (h(q^*) - q^*)\|^2 \end{aligned}$$

$$
\begin{aligned}
&\quad + 2\sigma_n\langle h(q^*) - q^*, q_{n+1} - q^*\rangle \\
&\leq \sigma_n \|h(q_n) - h(q^*)\|^2 + (1-\sigma_n)\|p_n - q^*\|^2 - \sigma_n(1-\sigma_n)\|h(q_n) - \mathcal{T}p_n\|^2 \\
&\quad + 2\sigma_n\langle h(q^*) - q^*, q_{n+1} - q^*\rangle \\
&\leq \sigma_n\lambda\|q_n - q^*\|^2 + (1-\sigma_n)\left[\|w_n - q^*\|^2 - (1-\mu^2)\|z_n - w_n\|^2\right] \\
&\quad - \sigma_n(1-\sigma_n)\|h(q_n) - \mathcal{T}p_n\|^2 + 2\sigma_n\langle h(q^*) - q^*, q_{n+1} - q^*\rangle \\
&= \sigma_n\lambda\|q_n - q^*\|^2 + (1-\sigma_n)\|w_n - q^*\|^2 - (1-\sigma_n)(1-\mu^2)\|z_n - w_n\|^2 \\
&\quad - \sigma_n(1-\sigma_n)\|h(q_n) - \mathcal{T}p_n\|^2 + 2\sigma_n\langle h(q^*) - q^*, q_{n+1} - q^*\rangle \\
&\leq \sigma_n\lambda\|q_n - q^*\|^2 + (1-\sigma_n)\|w_n - q^*\|^2 - (1-\sigma_n)(1-\mu^2)\|z_n - w_n\|^2 \\
&\quad - \sigma_n(1-\sigma_n)\|h(q_n) - \mathcal{T}p_n\|^2 + 2\sigma_n\langle h(q^*) - q^*, q_{n+1} - q^*\rangle. \qquad (4.32)
\end{aligned}
$$

But we observe that

$$
\begin{aligned}
\|w_n - q^*\|^2 &= \|q_n + \theta_n(\mathcal{S}_i q_n - \mathcal{S}_i q_{n-1}) - q^*\|^2 \\
&= \|(q_n - q^*) + \theta_n(\mathcal{S}_i q_n - \mathcal{S}_i q_{n-1})\|^2 \\
&= \|q_n - q^*\|^2 + 2\theta_n\langle q_n - q^*, \mathcal{S}_i q_n - \mathcal{S}_i q_{n-1}\rangle + \theta_n^2\|\mathcal{S}_i q_n - \mathcal{S}_i q_{n-1}\|^2 \\
&\leq \|q_n - q^*\|^2 + 2\theta_n\|q_n - q^*\|\|\mathcal{S}_i q_n - \mathcal{S}_i q_{n-1}\| + \theta_n^2\|\mathcal{S}_i q_n - \mathcal{S}_i q_{n-1}\|^2 \\
&\leq \|q_n - q^*\|^2 + 2\theta_n\|q_n - q^*\|\|q_n - q_{n-1}\| + \theta_n^2\|q_n - q_{n-1}\|^2 \\
&= \|q_n - q^*\|^2 + \theta_n\|q_n - q_{n-1}\|\left(2\|q_n - q^*\| + \theta_n\|q_n - q_{n-1}\|\right) \\
&= \|q_n - q^*\|^2 + \theta_n\|q_n - q_{n-1}\|K_3, \qquad (4.33)
\end{aligned}
$$

where $K_3 := \sup(2\|q_n - q^*\| + \theta_n\|q_n - q_{n-1}\|) < \infty$.

By combining (4.32) and (4.33),

$$
\begin{aligned}
\|q_{n+1} - q^*\|^2 &\leq \sigma_n\lambda\|q_n - q^*\|^2 + (1-\sigma_n)\left[\|q_n - q^*\|^2 + \theta_n\|q_n - q_{n-1}\|K_3\right] \\
&\quad - (1-\sigma_n)(1-\mu^2)\|z_n - w_n\|^2 - \sigma_n(1-\sigma_n)\|h(q_n) - \mathcal{T}p_n\|^2 \\
&\quad + 2\sigma_n\langle h(q^*) - q^*, q_{n+1} - q^*\rangle \\
&\leq \sigma_n\lambda\|q_n - q^*\|^2 + (1-\sigma_n)\|q_n - q^*\|^2 + \theta_n\|q_n - q_{n-1}\|K_3 \\
&\quad - (1-\sigma_n)(1-\mu^2)\|z_n - w_n\|^2 + 2\sigma_n\langle h(q^*) - q^*, q_{n+1} - q^*\rangle \\
&\leq \left(1 - (1-\lambda)\sigma_n\right)\|q_n - q^*\|^2 + \sigma_n\left[\frac{\theta_n}{\sigma_n}\|q_n - q_{n-1}\|K_3\right] \\
&\quad - (1-\sigma_n)(1-\mu^2)\|z_n - w_n\|^2 - \sigma_n(1-\sigma_n)\|h(q_n) - \mathcal{T}p_n\|^2 \\
&\quad + 2\sigma_n\langle h(q^*) - q^*, q_{n+1} - q^*\rangle \qquad (4.34) \\
&\leq \left(1 - (1-\lambda)\sigma_n\right)\|q_n - q^*\|^2 + \sigma_n K_3 K_4 + 2\sigma_n\langle h(q^*) - q^*, q_{n+1} - q^*\rangle \\
&= \left(1 - (1-\lambda)\sigma_n\right)\|q_n - q^*\|^2
\end{aligned}
$$

$$+ (1-\lambda)\sigma_n \left[\frac{K_3 K_4 + 2\langle h(q^*) - q^*, q_{n+1} - q^* \rangle}{1-\lambda} \right], \tag{4.35}$$

which gives the desired result. □

Theorem 4.6 *Let $\{q_n\}$ be a sequence generated by Algorithm 3.2 under Assumption 3.1. Then $\{q_n\}$ converges strongly to $q^* \in \Gamma := VI(\mathcal{K}, \mathcal{L}) \cap F(\mathcal{T}) \iff \lim_{n\to\infty} \|q_{n+1} - q_n\| = 0$.*

Proof Let $q^* \in \Gamma$. To show that $q_n \to q^* \in \Gamma$, we shall apply Lemma 2.4 and (4.35) and show that

$$\limsup_{\ell\to\infty} \left[\frac{\theta_{n_\ell}}{(1-\lambda)\sigma_{n_\ell}} \|q_{n_\ell} - q_{n_\ell - 1}\| K_3 + \frac{2}{1-\lambda} \langle h(q^*) - q^*, q_{n_\ell + 1} - q^* \rangle \right] \le 0$$

for every subsequence $\{\|q_{n_\ell} - q^*\|\}$ of the sequence $\{\|q_n - q^*\|\}$ satisfying the condition

$$\liminf_{\ell\to\infty} \left(\|q_{n_\ell+1} - q^*\| - \|q_{n_\ell} - q^*\| \right) \ge 0. \tag{4.36}$$

Consider $\{\|q_{n_\ell} - q^*\|\}$ to be the subsequence of $\{\|q_n - q^*\|\}$ satisfying condition (4.36). Then

$$\begin{aligned}
&\liminf_{\ell\to\infty} \left(\|q_{n_\ell+1} - q^*\|^2 - \|q_{n_\ell} - q^*\|^2 \right) \\
&\quad = \liminf_{\ell\to\infty} \left[\left(\|q_{n_\ell+1} - q^*\| - \|q_{n_\ell} - q^*\| \right) \left(\|q_{n_\ell+1} - q^*\| + \|q_{n_\ell} - q^*\| \right) \right] \\
&\quad \ge 0.
\end{aligned} \tag{4.37}$$

From (4.34) and (4.37),

$$\begin{aligned}
&\limsup_{\ell\to\infty} \left[(1-\sigma_{n_\ell})(1-\mu^2)\|z_{n_\ell} - w_{n_\ell}\|^2 + \sigma_n(1-\sigma_{n_\ell}) \|h(q_{n_\ell}) - \mathcal{T} p_{n_\ell}\|^2 \right] \\
&\quad \le \limsup_{\ell\to\infty} \left[\left(1 - (1-\lambda)\sigma_{n_\ell}\right) \|q_{n_\ell} - q^*\|^2 - \|q_{n_\ell+1} - q^*\|^2 \right] \\
&\qquad + \limsup_{\ell\to\infty} \left[(1-\lambda)\sigma_{n_\ell} \left(\frac{\theta_{n_\ell}}{(1-\lambda)\sigma_{n_\ell}} \|q_{n_\ell} - q_{n_\ell-1}\| K_3 \right.\right. \\
&\qquad \left.\left. + \frac{2}{1-\lambda} \langle h(q^*) - q^*, q_{n_\ell+1} - q^* \rangle \right) \right] \\
&\quad \le \limsup_{\ell\to\infty} \left[\left(1 - (1-\lambda)\sigma_{n_\ell}\right) \|q_{n_\ell} - q^*\|^2 - \|q_{n_\ell+1} - q^*\|^2 \right] \\
&\qquad + \limsup_{\ell\to\infty} \left[(1-\lambda)\sigma_{n_\ell} \left(\frac{\theta_{n_\ell}}{(1-\lambda)\sigma_{n_\ell}} \|q_{n_\ell} - q_{n_\ell-1}\| K_3 \right.\right. \\
&\qquad \left.\left. + \frac{2}{1-\lambda} \|h(q^*) - q^*\| \|q_{n_\ell+1} - q^*\| \right) \right] \\
&\quad = -\liminf_{\ell\to\infty} \left[\|q_{n_\ell+1} - q^*\|^2 - \|q_{n_\ell} - q^*\|^2 \right] \le 0.
\end{aligned}$$

This implies

$$\lim_{\ell\to\infty} \left[(1-\sigma_{n_\ell})(1-\mu^2)\|z_{n_\ell} - w_{n_\ell}\|^2 + \sigma_n(1-\sigma_{n_\ell}) \|h(q_{n_\ell}) - \mathcal{T} p_{n_\ell}\|^2 \right] = 0. \tag{4.38}$$

Thus,

$$\lim_{\ell\to\infty} \|z_{n_\ell} - w_{n_\ell}\| = 0, \tag{4.39}$$

so

$$\lim_{\ell\to\infty} \left\|h(q_{n_\ell}) - \mathcal{T}p_{n_\ell}\right\| = 0. \tag{4.40}$$

We observe from the definition of the sequence $\{w_{n_\ell}\}$ that

$$\|w_{n_\ell} - q_{n_\ell}\| = \sigma_{n_\ell}\left[\frac{\theta_{n_\ell}}{\sigma_{n_\ell}}\|\mathcal{S}_i q_{n_\ell} - \mathcal{S}_i q_{n_\ell -1}\|\right] \to 0 \quad \text{as } \ell \to \infty. \tag{4.41}$$

Since $\mathcal{L}$ is uniformly continuous and ℓ_0-Lipschitzian, using (4.39), we get

$$\|p_{n_\ell} - z_{n_\ell}\| \le \zeta_{n_\ell}\ell_0\|z_{n_\ell} - w_{n_\ell}\| \to 0 \quad \text{as } \ell \to \infty. \tag{4.42}$$

Also, using (4.39) and (4.42),

$$\|p_{n_\ell} - w_{n_\ell}\| \le \|p_{n_\ell} - z_{n_\ell}\| + \|z_{n_\ell} - w_{n_\ell}\| \to 0 \quad \text{as } \ell \to \infty. \tag{4.43}$$

Combining (4.41) and (4.43),

$$\|p_{n_\ell} - q_{n_\ell}\| \le \|p_{n_\ell} - w_{n_\ell}\| + \|w_{n_\ell} - q_{n_\ell}\| \to 0 \quad \text{as } \ell \to \infty. \tag{4.44}$$

Furthermore, using the definition of $\{q_{n_\ell+1}\}$ in Algorithm 3.2, we deduce

$$q_{n_\ell+1} - p_{n_\ell} = \sigma_{n_\ell}\left(h(q_{n_\ell}) - \mathcal{T}p_{n_\ell}\right) + \mathcal{T}p_{n_\ell} - p_{n_\ell},$$

which immediately implies that

$$\begin{aligned}\|p_{n_\ell} - \mathcal{T}p_{n_\ell}\| &= \|q_{n_\ell+1} - p_{n_\ell}\| + \sigma_{n_\ell}\left\|h(q_{n_\ell}) - \mathcal{T}p_{n_\ell}\right\| \\ &\le \|q_{n_\ell+1} - q_{n_\ell}\| + \|q_{n_\ell} - p_{n_\ell}\| + \sigma_{n_\ell}\left\|h(q_{n_\ell}) - \mathcal{T}p_{n_\ell}\right\|.\end{aligned}$$

Since $\sigma_{n_\ell} \to 0$, $q_{n_\ell+1} - q_{n_\ell} \to 0$ as $\ell \to \infty$, using (4.40) and (4.44), we get

$$\lim_{\ell\to\infty} \|p_{n_\ell} - \mathcal{T}p_{n_\ell}\| = 0. \tag{4.45}$$

Since the sequence $\{q_{n_\ell}\}$ is bounded, there exists a subsequence $\{q_{n_{\ell_k}}\}$ of $\{q_{n_\ell}\}$ such that $\{q_{n_{\ell_k}}\}$ converges weakly to $z^* \in \mathcal{H}$ as $k \to \infty$. Using (4.41), we have $w_{n_{\ell_k}} \rightharpoonup z^* \in \mathcal{H}$ as $k \to \infty$. Also, since $\{p_{n_\ell}\}$ is bounded, there exists a subsequence $\{p_{n_{\ell_k}}\}$ which converges weakly to $z^* \in \mathcal{H}$. Since $\mathcal{T}$ is semiclosed at the origin, by (4.45), we infer that $z^* \in F(\mathcal{T})$. Thus, by (4.45) and Lemma 4.4, $z^* \in \Gamma$.

Furthermore, since $q_{n_{\ell_k}} \rightharpoonup z^*$, it follows that

$$\limsup_{\ell\to\infty}\left\langle h\left(q^*\right) - q^*, q_{n_\ell} - q^*\right\rangle = \lim_{k\to\infty}\left\langle h\left(q^*\right) - q^*, q_{n_{\ell_k}} - q^*\right\rangle$$

$$= \langle h(q^*) - q^*, z^* - q^* \rangle. \tag{4.46}$$

However, since z^* is a unique solution in Γ, this implies that

$$\limsup_{\ell\to\infty} \langle h(q^*) - q^*, q_{n_\ell} - q^* \rangle = \langle h(q^*) - q^*, z^* - q^* \rangle \le 0. \tag{4.47}$$

Furthermore,

$$\begin{aligned} \limsup_{\ell\to\infty} \langle h(q^*) - q^*, q_{n_\ell+1} - q^* \rangle &= \limsup_{\ell\to\infty} \langle h(q^*) - q^*, q_{n_\ell} - q^* \rangle \\ &= \langle h(q^*) - q^*, z^* - q^* \rangle \le 0. \end{aligned} \tag{4.48}$$

Using Assumption 3.1 (B5) and (4.48),

$$\begin{aligned} \limsup_{\ell\to\infty} \partial_{n_\ell} &= \limsup_{\ell\to\infty} \left[\frac{\theta_{n_\ell}}{(1-\lambda)\sigma_{n_\ell}} \|q_{n_\ell} - q_{n_\ell - 1}\| K_3 + \frac{2}{1-\lambda} \langle h(q^*) - q^*, q_{n_\ell+1} - q^* \rangle \right] \\ &\le 0. \end{aligned} \tag{4.49}$$

It follows from Lemma 2.4 that $\lim_{n\to\infty} \|q_n - q^*\| = 0$, completing the proof. □

5 Numerical experiments

In this section, we will give some special numerical examples which show the behavior of our proposed iterative method in comparison to Algorithm (3.4) of Gang et al. [14], Algorithm (3.6) of Thong et al. [35] and Algorithm (3.2) of Zhao et al. [44].

Example 5.1 Let $\mathcal{H} = \ell_2(\mathbb{R})$ be a real Hilbert space of square summable sequences of real numbers equipped with the property

$$\|q_1\|^2 + \|q_2\|^2 + \|q_3\|^2 + \cdots + \|q_n\|^2 + \cdots < +\infty.$$

Let $\mathcal{L} : \mathcal{K} \to \mathcal{K}$ be a mapping defined by

$$\mathcal{L}(q) = (\sigma - \|q\|)q, \quad \forall q \in \mathcal{H},$$

where $\mathcal{K} := \{q \in \mathcal{H} : \|q\| \le \gamma\}$, $\sigma, \gamma \in \mathbb{R}$, such that $\gamma, \sigma > 0$. It is can be easily seen that $\mathcal{L}$ is weakly sequentially continuous on $\mathcal{H}$ and $VI(\mathcal{K}, \mathcal{L}) = \{0\}$. Now for $q, s \in \mathcal{K}$, we get

$$\begin{aligned} \|\mathcal{L}q - \mathcal{L}s\| &= \|(\sigma - \|q\|)q - (\sigma - \|s\|)s\| \\ &= \|\sigma q - \|q\|q - \sigma s + \|s\|s\| \\ &\le \sigma \|q - s\| + \|q\| \|q - s\| + \|s\| \big| \|q\| - \|s\| \big| \\ &\le \sigma \|q - s\| + \gamma \|q - s\| + \gamma \|q - s\| \\ &= \sigma \|q - s\| + 2\gamma \|q - s\| \\ &\le (\sigma + 2\gamma) \|q - s\|. \end{aligned}$$

This implies $\mathcal{L}$ is ℓ_0-Lipschitzian with $\ell_0 = (\sigma + 2\gamma)$. It is also easily verifiable that the operator $\mathcal{L}$ is quasi-monotone for all $0 < \frac{\sigma}{2} < \gamma < \sigma$. To see this, suppose $\langle \mathcal{L}q, s-q\rangle > 0$ for $q, s \in \mathcal{K}$ such that

$$\big(\sigma - \|q\|\big)\langle q, s-q\rangle > 0.$$

Then from the definition of the feasible set $\mathcal{K}$, it follows that

$$\begin{aligned}\langle \mathcal{L}s, s-q\rangle\big(\sigma - \|s\|\big)\langle q, s-q\rangle &\geq \big(\sigma - \|s\|\big)\langle s, s-q\rangle - \big(\sigma - \|s\|\big)\langle q, s-q\rangle \\ &\geq \big(\sigma - \|s\|\big)\big[\langle s, s-q\rangle - \langle q, s-q\rangle\big] \\ &\geq \big(\sigma - \|s\|\big)\langle s-q, s-q\rangle \\ &= \big(\sigma - \|s\|\big)\|s-q\|^2 \geq 0.\end{aligned}$$

The operator $\mathcal{L}$ is quasi-monotone on $\mathcal{K}$. We compute the metric projection on $\mathcal{K}$ as follows:

$$P_{\mathcal{K}}(q) = \begin{cases} q, & \text{if } \|q\| \leq \gamma, \\ \frac{\gamma q}{\|q\|}, & \text{otherwise.}\end{cases}$$

We define a map $\mathcal{T} : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ by $\mathcal{T}q = \frac{q \sin q}{2}$. As seen in [44], $\mathcal{T}$ is quasi-nonexpansive and semiclosed at 0 with $F(\mathcal{T}) = \{0\}$. Finally, let $h : \mathcal{H} \to \mathcal{H}$ be a mapping defined by $h(q) = \frac{1}{2}q$. It can be seen easily that h is $\frac{1}{2}$-contractive. We take $F(q) = \frac{1}{2}q$ [44]. Suppose for this example $\zeta_0 = \frac{1}{2}$, $\mu = 0.5$, $\gamma = 2.0$, $\theta = \frac{2}{5}$, $\zeta_{n+1} = \frac{100}{(n+1)^{1.3}}$, $\sigma_n = \frac{1}{100n^2+1}$ for Algorithm 3.2. Also, if we consider $TOL = \|q_n - q_{n-1}\| \leq 10^{-5}$ as the stopping criterion and choose the following different initial points:

Case A: $q_0 = (1, \frac{1}{2}, \frac{1}{4}, \dots)$, $q_1 = (-2, 1, -\frac{1}{2}, \dots)$,
Case B: $q_0 = (1, \frac{1}{2}, \frac{1}{4}, \dots)$, $q_1 = (-3, 1, -\frac{1}{3}, \dots)$,
Case C: $q_0 = (5, 1, \frac{1}{5}, \dots)$, $q_1 = (-5, 1, -\frac{1}{5}, \dots)$,
Case D: $q_0 = (2, 1, \frac{1}{2}, \dots)$, $q_1 = (-3, 1, -\frac{1}{3}, \dots)$,

then we obtain the results in Table 1.

Example 5.2 ([33]) Let $\mathcal{H} = L_2([0,1])$ be an infinite-dimensional linear space of square integrable functions associated with the inner product $\langle q, s\rangle = \int_0^1 q(t)s(t)\,dt \forall q, s \in L_2([0,1])$, $t \in [0,1]$, and induced norm $\|q\|_{L_2} = (\int_0^1 |q(t)|^2|\,dt)^{\frac{1}{2}}, \forall q \in L_2([0,1])$. Consider a closed unit

Table 1 Numerical results for Example 5.1

Case		Zhao et al.	Thong et al.	Gang et al.	Abuchu et al.
Case A	Sec.	0.0672	0.0446	0.0455	0.0439
	No. of Iter.	67	67	58	38
Case B	Sec.	0.0688	0.0485	0.0227	0.0034
	No. of Iter.	27	26	23	19
Case C	Sec.	0.0511	0.0393	0.0145	0.0145
	No. of Iter.	82	84	69	35
Case D	Sec.	0.0104	0.0088	0.0069	0.0054
	No. of Iter.	95	98	81	41

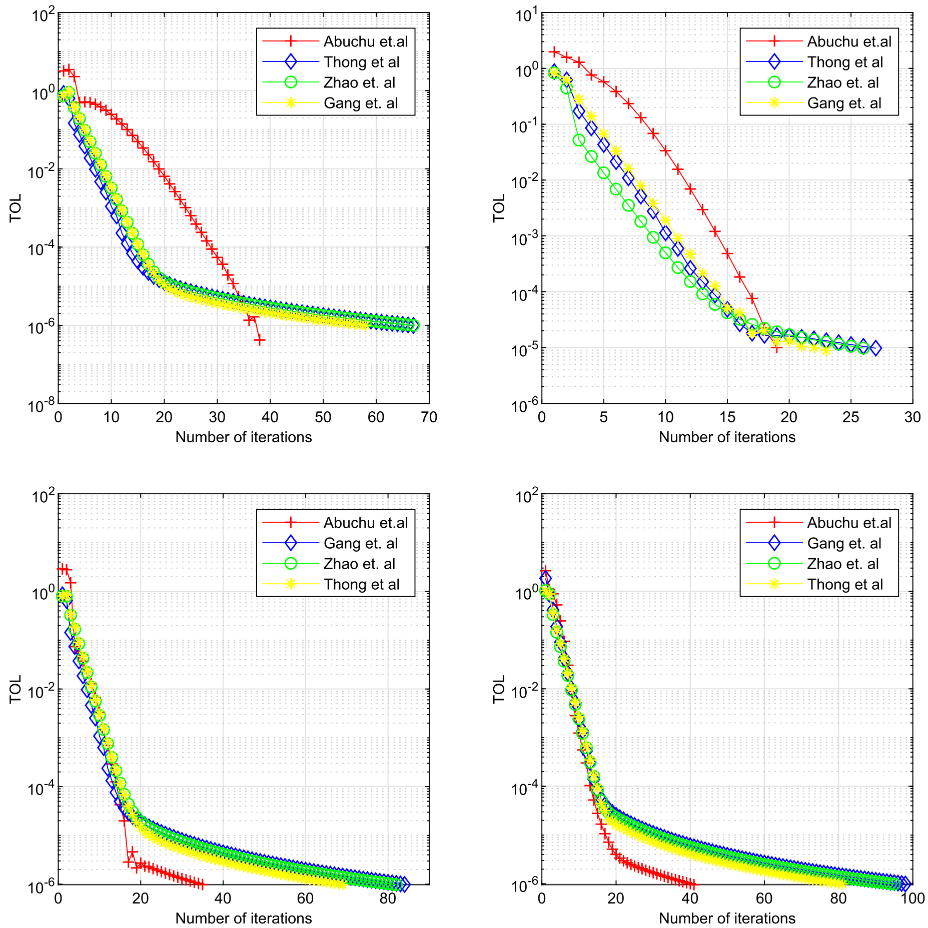

Figure 1 Plot of Errors against number of iterations for Example 5.1; Top Left: Case A; Top Right: Case B; Bottom Left: Case C; Bottom Right: Case D

ball $\mathcal{K} := \{q \in L_2([0,1]) : \|q\| \leq 1\}$. Define a projection from $L_2([0,1])$ onto $\mathcal{K}$ by

$$P_{\mathcal{K}}(q) = \begin{cases} \frac{q}{\|q\|_{L^2}}, & \text{if } \|q\|_{L^2} > 1, \\ q, & \text{if } \|q\|_{L^2} \leq 1. \end{cases}$$

Let $\mathcal{L} : L_2([0,1]) \to L_2([0,1])$ be an operator defined by $(\mathcal{L}q)(t) = \max\{0, \frac{q(t)}{2}\}$, $\forall q \in L_2([0,1])$, t$\in [0,1]$. It is easily verifiable that $\mathcal{L}$ is pseudo-monotone (so, quasi-monotone), and $VI(\mathcal{L},\mathcal{K}) = \{0\}$. Also, let $\mathcal{T} : L_2([0,1]) \to L_2([0,1])$ be a mapping given by $(\mathcal{T}q)(t) = \int_0^1 tq(s)\,ds$, $t \in [0,1]$. The operator $\mathcal{T}$ defined here is nonexpansive and hence quasi-nonexpansive (see [33]). It can also be easily seen that $0 \in F(\mathcal{T})$. Thus, $\Gamma := VI(\mathcal{L},\mathcal{K}) \cap F(\mathcal{T}) \neq \emptyset$. Let $h : L_2([0,1]) \to L_2([0,1])$ be a mapping defined by $(h(q))(t) = \frac{1}{2}q(t)$, $t \in [0,1]$. Then h is a contraction. If we take $TOL = \|q_n - q_{n-1}\| \leq 10^{-5}$ as the stopping criterion, we obtain the following table and graphs using the following as starting points:

Case A: $q_0(t) = 2t^3 + t$, $q_1(t) = \sin(t^2)$,
Case B: $q_0(t) = 2e^t + t^2$, $q_1(t) = \sin(2t)$,
Case C: $q_0(t) = 2e^{t+1} + t$, $q_1(t) = \sin(2t^2)$,
Case D: $q_0(t) = e^{t^2} + t^3$, $q_1(t) = \cos(2t^3) + e^t - 2t^3$.

Table 2 Numerical results for Example 5.2

Case		Zhao et al.	Thong et al.	Gang et al.	Abuchu et al.
Case A	Sec.	121.5575	39.1899	26.7381	25.7266
	No. of Iter.	206	46	22	20
Case B	Sec.	236.0356	72.8521	44.1703	36.5203
	No. of Iter.	88	38	19	18
Case C	Sec.	110.5115	35.1408	20.4326	16.7390
	No. of Iter.	206	50	24	20
Case D	Sec.	324.9522	41.5406	29.8858	18.5931
	No. of Iter.	488	52	29	21

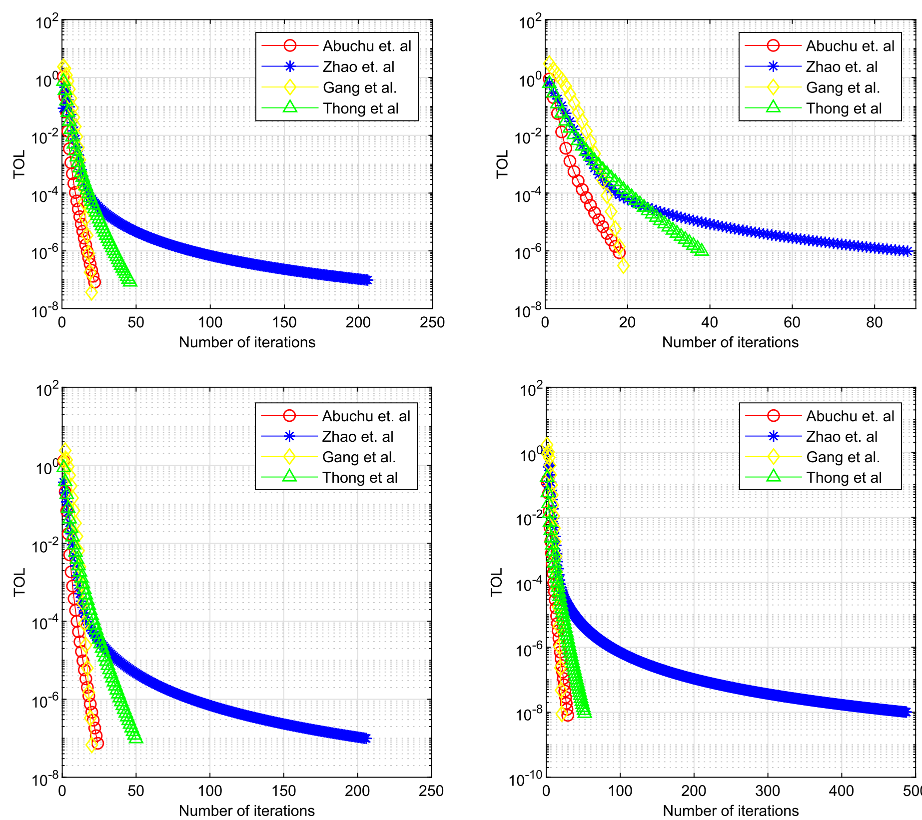

Figure 2 Plot of Errors against number of iterations for Example 5.2; Top Left: Case A; Top Right: Case B; Bottom Left: Case C; Bottom Right: Case D

6 Conclusion

A modified Tseng inertial iterative algorithm for solving quasi-monotone variational inequality and fixed point problems when the underlying operator is quasi-nonexpansive in real Hilbert spaces was introduced and studied. We established strong convergence of the proposed Algorithm 3.2 without prior knowledge of the Lipschitz constant of the cost operator under the adaptive step size condition and other certain mild assumptions on the algorithm parameters. The algorithm, which is embellished with inertial extrapolation and viscosity terms as well as good relaxation of the cost operator (quasi-monotone operator), generally exhibited enhanced efficiency and applicability. Finally, we presented some numerical experiments to demonstrate the applicability and the advantages of our algorithm. Tables 1 and 2 and Figs. 1 and 2 reveal that our algorithm perform more favourable in comparison to some related methods in literature. Our results complement and extend some recent results in the literature.

Funding
This work did not receive any external funding.

Data availability
Not applicable.

Declarations

Ethics approval and consent to participate
This study does not involve any experiments with human participants or animals.

Competing interests
The authors declare no competing interests.

Author contributions
J.A.A. and A.E.O. made conceptualization, methodology and writing draft preparation. G.C.U. and H.I. performed the formal analysis, writing-review and editing. O.K.N. made investigation, review and validation. All authors read and approved the final version.

Author details
[1]School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa. [2]Department of Mathematics, University of Calabar, Calabar, Nigeria. [3]Department of Engineering Science, Bandırma Onyedi Eylül University, Bandırma 10200, Balıkesir, Turkey. [4]Department of Mathematics, University of Eswatini, Private Bag 4, Kwaluseni, Kingdom of Eswatini. [5]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa 0204, Pretoria, South Africa.

Received: 29 May 2023 Accepted: 7 March 2024 Published online: 21 March 2024

References

1. Abuchu, J.A., Ugwunnadi, G.C., Narain, O.K.: Inertial Mann-type iterative method for solving split monotone variational inclusion problem with applications. J. Ind. Manag. Optim. **19**(4), 3020–3043 (2023)
2. Abuchu, J.A., Ugwunnadi, G.C., Narain, O.K.: Iterative method for solving pseudomonotone variational inequality and fixed point problems. Adv. Nonlinear Var. Inequal. **26**(1), 1–24 (2023)
3. Anh, P.K., Anh, T.V., Muu, L.D.: On bilevel split pseudomonotone variational inequality problems with applications. Acta Math. Vietnam. **42**, 413–429 (2017)
4. Antipin, A.S.: On a method for convex programs using a symmetrical modification of the Lagrange function. Èkon. Mat. Metody **12**, 1164–1173 (1976)
5. Byrne, C.: A unified treatment of some iterative algorithms in signal processing and image reconstruction. Inverse Probl. **20**(103) (2003)
6. Censor, Y., Gibali, A., Reich, S.: The subgradient extragradient method for solving variational inequalities in Hilbert space. J. Optim. Theory Appl. **148**, 318–335 (2011)
7. Censor, Y., Gibali, A., Reich, S.: Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space. Optimization **61**, 1119–1132 (2012)
8. Cholamjiak, P., Thong, D.V., Cho, Y.J.: A Novel Inertial Projection and Contraction Method for Solving Pseudomonotone Variational Inequality Problems. Acta Appl. Math. (2019)
9. Cottle, R.W., Yao, J.C.: Pseudo-monotone complementarity problems in Hilbert space. J. Optim. Theory Appl. **75**, 281–295 (1992)
10. Dong, L.Q., Yang, J.F., Yuan, H.B.: The projection and contraction algorithm for solving variational inequality problems in Hilbert space. J. Nonlinear Convex Anal. **20**, 111–122 (2019)
11. Dong, Q.L., Cho, J.Y., Rassias, T.M.H.: The projection and contraction methods for finding common solutions to variational inequality problems. Optim. Lett. **12**, 1871–1896 (2018)
12. Ferris, M.C., Pang, J.S.: Engineering and economic applications of complementarity problems. SIAM Rev. **39**(4), 669–713 (1997)
13. Fichera, G.: Sul pproblem elastostatico di Signorini con ambigue condizioni al contorno. Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. **34**, 138–142 (1963)
14. Gang, C., Shehu, Y., Iyiola, O.S.: Inertial Tseng's extragradient method for solving variational inequality problems of pseudo-monotone and non-Lipschitz operators. J. Ind. Manag. Optim. **18**(4), 2873–2902 (2022)
15. Godwin, E.C., Alakoya, T.O., Mewomo, O.T., Yao, J.C.: Relaxed inertial Tsengextragradient method for variational inequality and fixed point problems. Appl. Anal. https://doi.org/10.1080/00036811.2022.2107913
16. Goldstein, A.A.: Convex programming in Hilbert space. Bull. Am. Math. Soc. **70**, 709–710 (1964)
17. Hadjisavvas, N., Schaible, S.: Quasimonotone variational inequalities in Banach spaces. J. Optim. Theory Appl. **90**, 95–111 (1996)
18. He, B.S., Yang, Z.H., Yuan, X.M.: An approximate proximal-extragradient type method for monotone variational inequalities. J. Math. Anal. Appl. **300**, 362–374 (2004)
19. Korpelevich, G.M.: An extragradient method for finding sadlle points and for other problems. Èkon. Mat. Metody **12**, 747–756 (1976)

20. Maingé, P.E.: Convergence theorems for inertial KM-type algorithms. J. Comput. Appl. Math. **219**(1), 223–236 (2008)
21. Nadezhkina, N., Takahashi, W.: Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl. **128**, 191–201 (2006)
22. Noor, M.: Extragradient methods for pseudomonotone variational inequalities. J. Optim. Theory Appl. **117**, 475–488 (2003)
23. Ogwo, G.N., Izuchukwu, C., Shehu, Y., Mewomo, O.T.: Convergence of relaxed inertial subgradient extragradient methods for quasimonotone variational inequality problems. J. Sci. Comput. **90** (2022)
24. Polyak, B.T.: Some methods of speeding up the convergence of iteration methods. Comput. Math. Math. Phys. **4**(5), 1–17 (1964)
25. Popov, L.D.: A modification of the Arrow–Hurwicz method for searching for saddle points. Mat. Zametki **28**, 777–784 (1980)
26. Saejung, S., Yotkaew, P.: Approximation of zeros of inverse strongly monotone operators in Banach spaces. Nonlinear Anal. **75**, 742–750 (2012)
27. Shehu, Y., Vuong, P.T., Zemkoho, A.: An inertial extrapolation method for convex simple bilevel optimization. Optim. Methods Softw. (2019)
28. Stampacchia, G.: Variational inequalities. In: Theory and Applications of Monotone Operators. Proceedings of the NATO Advanced Study Institute, pp. 102–192 (1968). Venice, Italy (Edizioni Odersi, Gubbio, Italy)
29. Tan, B., Liu, L., Qin, X.: Self adaptive inertial extragradient algorithms for solving bilevel pseudomonotone variational inequality problems. Jpn. J. Ind. Appl. Math. (2020)
30. Thong, D.V., Hieu, D.V.: Weak and strong convergence theorems for variational inequality problems. Numer. Algorithms **78**, 1045–1060 (2018)
31. Thong, D.V., Hieu, D.V.: New extragradient methods for solving variational inequality problems and fixed point problems. J. Fixed Point Theory Appl. **20**, 1–29 (2018)
32. Thong, D.V., Hieu, D.V.: Modified Tseng's extragradient algorithms for variational inequality problems. J. Fixed Point Theory Appl. **20** (2018)
33. Thong, D.V., Hieu, D.V.: Inertial subgradient extragradient algorithms with line-search process for solving variational inequality problems and fixed point problems. Numer. Algorithms **80**, 1283–1307 (2019)
34. Thong, D.V., Li, X.H., Dong, Q.L., Cho, Y.J., Rassias, T.M.: A projection and contraction method with adaptive step sizes for solving bilevel pseudomonotone variational inequality problems. Optimization (2020)
35. Thong, D.V., Shehu, Y., Iyiola, O.S.: Weak and strong convergence theorems for solving pseudo-monotone variational inequalities with non-Lipschitz mappings. Numer. Algorithms **84**, 795–823 (2020)
36. Thong, D.V., Triet, N.A., Li, X.-H., Dong, Q.-L.: Strong convergence of extragradient methods for solving bilevel pseudo-monotone variational inequality problems. Numer. Algorithms **83**, 1123–1143 (2020)
37. Tseng, P.: A modified forward-backward splitting method for maximal monotone mappings. SIAM J. Control Optim. **38**, 431–446 (2000)
38. Ugwunnadi, G.C., Izuchukwu, C., Jolaoso, L.O., Okeke, C.C., Aremu, K.O.: A modified inertial projection and contraction method for solving bilevel split variational inequality problems. Appl. Set-Valued Anal. Optim. **4**(1), 55–71 (2022)
39. Wang, F., Xu, H.-K.: Weak and strong convergence theorems for variational inequality and fixed point problems with Tseng's extragradient method. Taiwan. J. Math. **16**, 1125–1136 (2012)
40. Yang, J., Liu, H.: Strong convergence result for solving monotone variational inequalities in Hilbert space. Numer. Algorithms, 741–752 (2019)
41. Yao, Y., Iyiola, O.S., Shehu, Y.: Subgradient extragradient method with double inertial steps for variational inequalities. J. Sci. Comput. **90**(2), 71 (2022)
42. Yao, Y., Shahzad, N., Yao, J.C.: Convergence of Tseng-type self-adaptive algorithms for variational inequalities and fixed point problems. Carpath. J. Math. **37**, 541–550 (2021)
43. Zegeye, H., Shahzad, N., Yao, Y.: Minimum-norm solution of variational inequality and fixed point problem in Banach spaces. Optimization **64**, 453–471 (2015)
44. Zhao, T., Wang, D., Ceng, L., He, L., Wang, C., Fan, H.: Quasi-inertial Tseng's extragradient algorithms for pseudomonotone variational inequalities and fixed point problems of quasi-nonexpansive operators. Numer. Funct. Anal. Optim. **42**, 69–90 (2021)
45. Zhao, X., Yao, Y.: Modified extragradient algorithms for solving monotonevariational inequalities and fixed point problems. Optimization **69**, 1987–2002 (2020)
46. Zheng, L.: A double projection algorithm for quasimonotone variational inequalities in Banach spaces. J. Inequal. Appl. (2018)

Publisher's Note