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# Variable Herz–Morrey estimates for rough fractional Hausdorff operator

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## Abstract

As a first attempt, we obtain the boundedness of the rough fractional Hausdorff operator on variable exponent Herz-type spaces. The method used in this paper enables us to study the operator on some other function spaces with variable exponents.

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## 1 Introduction

The Hausdorff operator has a fascinating history of its development from simple to present form. The laying stone of this development is the one-dimensional Hausdorff operator

$$\mathcal{H}_\Phi g(y) = \int_0^\infty \frac{\Phi(t)}{t} g(yt^{-1}) dt, \quad (1.1)$$

where  $\Phi \in L^1_{\text{loc}}(\mathbb{R})$ . A change of variables in (1.1) results in the following form:

$$\mathcal{H}_\Phi g(y) = \int_0^\infty \frac{\Phi(yt^{-1})}{t} g(t) dt. \quad (1.2)$$

The boundedness of these operators, an essential part of the analysis, has been discussed in [10, 29, 30]. For a detailed history and the recent development of the Hausdorff operator, we refer the interested reader to the review papers [6, 28].

The Hausdorff operator finds its applications in the study of one-dimensional Fourier analysis. In particular, it plays a significant role in the summability of the classical Fourier series. Therefore extensions of the one-dimensional Hausdorff operators to multidimensional spaces become extremely important. Several authors have spared their studies for such an extension. In this regard, some contributions are [5, 26, 27, 31]. Here we are mainly interested in the rough Hausdorff operator [5]

$$H_{\Phi, \Omega} g(y) = \int_{\mathbb{R}^n} \frac{\Phi(y|t|^{-1})}{|t|^n} \Omega(t') g(t) dt \quad (1.3)$$

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and the fractional Hausdorff operator [31]

$$H_{\Phi}^{\beta}g(y) = \int_{\mathbb{R}^n} \frac{\Phi(y|t|^{-1})}{|t|^{n-\beta}} g(t) dt, \quad 0 \leq \beta < n. \quad (1.4)$$

The rough fractional Hausdorff operator, which is a combination of (1.3) and (1.4),

$$H_{\Phi,\Omega}^{\beta}g(y) = \int_{\mathbb{R}^n} \frac{\Phi(y|t|^{-1})}{|t|^{n-\beta}} \Omega(t') g(t) dt, \quad 0 \leq \beta < n, \quad (1.5)$$

was first studied in [14]. It is obvious that by taking  $\beta = 0$  in (1.5) we obtain  $H_{\Phi,\Omega}$ , and by taking  $\Omega = 1$  in the same identity, we get the fractional Hausdorff operator  $H_{\Phi}^{\beta}$ . Thus discussing the boundedness of  $H_{\Phi,\Omega}^{\beta}$  on variable exponent Herz-type spaces automatically includes the same results for  $H_{\Phi,\Omega}$  and  $H_{\Phi}^{\beta}$  on these spaces. Some studies containing boundedness results regarding  $H_{\Phi,\Omega}$ ,  $H_{\Phi,1}$ , and  $H_{\Phi}^{\beta}$  along with their commutators on different function spaces include [2, 3, 7, 11–13, 17, 19, 20, 35].

Another important aspect of the Hausdorff operator is that it contains other classical operators as its particular cases. For example, in  $H_{\Phi,1}^{\beta}$ , if we take  $\Phi(z) = \Phi_1(z) = z^{-n+\beta} \chi_{(1,\infty)}(z)$  and  $\Phi(z) = \Phi_2(z) = \chi_{(0,1]}(z)$ , respectively, then we obtain the  $n$ -dimensional fractional Hardy operator

$$P_{\beta}g(y) = \frac{1}{|y|^{n-\beta}} \int_{|y|<|t|} g(t) dt$$

and its adjoint operator:

$$Q_{\beta}g(y) = \int_{|y|\geq|t|} \frac{g(t)}{|t|^{n-\beta}} dt.$$

During the recent past, the boundedness of Hardy-type operators on variable exponent function spaces also drew great attention of the research community [18, 39]. Besides the Hardy operators, the Hausdorff operator also contains the  $n$ -dimensional version of the Caldron operator

$$Cg(y) = \frac{1}{|y|^n} \int_{|y|<|t|} g(t) dt + \int_{|t|\geq|y|} \frac{g(t)}{|t|^n} dt$$

if we choose  $\Phi(t) = \min\{1, \frac{1}{|t|^n}\}$  in the definition of  $H_{\Phi,1}$ .

Variable exponent function spaces have an increasing impact on the recent advances in harmonic analysis. This is mainly because of their frequent appearance in analysis and applications of different functional analysis tools for partial differential equations. This results in an increase in research publications in this field. The theory of variable Lebesgue spaces first appeared in the pioneering work of Kováčik and Rákosník [25]. Later on, several monographs appeared in the literature to strengthen the theory. Operator theory on function spaces also finds new dimensions, and many researchers discuss the boundedness of different operators on variable-exponent function spaces [24, 37]. In this paper, we inquire about the continuity of the rough fractional Hausdorff operator from this perspective.

On function spaces with constant exponents, the boundedness of the Hausdorff operator is accomplished by a scaling argument followed by the polar decomposition of integral on  $\mathbb{R}^n$ . However, such a direct approach on function spaces with variable exponent does not work well. It needs some modification in the later case. In this paper, we aim to tackle this problem by establishing the boundedness of the rough fractional Hausdorff operator on variable-exponent Herz-type spaces. Our strategy enables us to study the operator on other function spaces with variable exponents. As corollaries of our main results, we prove the boundedness of the rough Hausdorff operator on variable-exponent Herz-type spaces, which are also new to the best of the author's knowledge.

In the next section, we present some definitions and preliminary lemmas. Section 3 contains the theorems stating the boundedness criterion for rough fractional Hausdorff operators on variable-exponent Herz-type spaces and their detailed proofs.

## 2 Variable-exponent function spaces

Let  $O \subseteq \mathbb{R}^n$  be an open set, and let  $q(\cdot) : O \rightarrow [1, \infty)$  be a measurable function. We denote by  $q'(\cdot)$  the conjugate exponent of  $q(\cdot)$  satisfying

$$\frac{1}{q(\cdot)} + \frac{1}{q'(\cdot)} = 1.$$

The set  $\mathcal{P}(O)$  consists of all  $q(\cdot)$  such that

$$1 < q^- = \text{essinf}\{q(y) : y \in O\} \leq q^+ = \text{esssup}\{q(y) : y \in O\} < \infty.$$

The Lebesgue space with variable exponent  $L^{q(\cdot)}(O)$  is a set of all measurable functions  $g$  such that for a positive  $\sigma$ ,

$$\int_O \left( \frac{|g(y)|}{\sigma} \right)^{q(y)} dy < \infty.$$

Equipped with the Luxemburg norm

$$\|g\|_{L^{q(\cdot)}(O)} = \inf \left\{ \sigma > 0 : \int_O \left( \frac{|g(y)|}{\sigma} \right)^{q(y)} dy \leq 1 \right\},$$

the space  $L^{q(\cdot)}(O)$  becomes a Banach function space. The set of functions

$$L_{\text{loc}}^{q(\cdot)}(O) = \{g : g \in L^{q(\cdot)}(E) \text{ } \forall \text{ compact subset } E \subset O\}$$

serves to define local version of the variable-exponent Lebesgue space  $L_{\text{loc}}^{q(\cdot)}(O)$ .

Variable-exponent function spaces bear a deep connection with the boundedness of Hardy–Littlewood maximal operator  $\mathcal{M}$  defined by

$$\mathcal{M}g(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |g(t)| dt$$

on  $L^{q(\cdot)}(\mathbb{R}^n)$ . We denote by  $\mathcal{B}(\mathbb{R}^n)$  the set consisting of  $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  such that  $\mathcal{M}$  is bounded on  $L^{q(\cdot)}(\mathbb{R}^n)$ .

**Proposition 2.1** ([8, 34]) Let  $O \subset \mathbb{R}^n$  be an open set, and let  $q(\cdot) \in \mathcal{P}(O)$  satisfy

$$|q(\xi) - q(\eta)| \leq \frac{-C}{\ln(|\xi - \eta|)}, \quad \frac{1}{2} \geq |\xi - \eta|, \quad (2.1)$$

$$|q(\xi) - q(\eta)| \leq \frac{C}{\ln(|\xi| + e)}, \quad |\xi| \leq |\eta|, \quad (2.2)$$

then  $q(\cdot) \in \mathcal{B}(O)$ , where  $C$  is a positive constant independent of  $\xi$  and  $\eta$ .

**Lemma 2.2** ([25] Generalized Hölder inequality) Let  $q(\cdot), q_1(\cdot), q_2(\cdot) \in \mathcal{P}(O)$ .

(a) If  $g \in L^{q(\cdot)}(O)$  and  $h \in L^{q'(\cdot)}(O)$ , then

$$\int_O |g(y)h(y)| dy \leq r_q \|g\|_{L^{q(\cdot)}(O)} \|h\|_{L^{q'(\cdot)}(O)},$$

where  $r_q = 1 + \frac{1}{q_-} - \frac{1}{q_+}$ .

(b) If  $g \in L^{q_1(\cdot)}(O)$ ,  $h \in L^{q_2(\cdot)}(O)$ , and  $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$ , then

$$\|gh\|_{L^{q(\cdot)}(O)} \leq r_{q,q_1} \|g\|_{L^{q_1(\cdot)}(O)} \|h\|_{L^{q_2(\cdot)}(O)},$$

where  $r_{q,q_1} = (1 + \frac{1}{(q_1)_-} - \frac{1}{(q_1)_+})^{1/q_-}$ .

**Lemma 2.3** ([22]) If  $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ , then there exist constants  $0 < \delta < 1$  and  $C > 0$  such that for all balls  $B$  in  $\mathbb{R}^n$  and all measurable subsets  $S \subset B$ ,

$$\frac{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leq C \left( \frac{|B|}{|S|} \right)^\delta.$$

**Lemma 2.4** ([22]) Assuming that  $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ , for all balls  $B \subset \mathbb{R}^n$  and for a positive constant  $C$ , we have

$$C^{-1} \leq \frac{1}{|B|} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \leq C.$$

**Lemma 2.5** ([9]) Define a variable exponent  $\tilde{p}(\cdot)$  such that  $\frac{1}{q(t)} = \frac{1}{\tilde{p}(t)} + \frac{1}{p}$  ( $t \in \mathbb{R}^n$ ). Then we have

$$\|gh\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C \|g\|_{L^{\tilde{p}(\cdot)}(\mathbb{R}^n)} \|h\|_{L^p(\mathbb{R}^n)}.$$

**Lemma 2.6** ([33]) Let  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  satisfy conditions (2.1) and (2.2) in Proposition 2.1. Then

$$\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{\frac{1}{q(x)}} & \text{if } |Q| < 2^n \text{ and } x \in Q, \\ |Q|^{\frac{1}{q(\infty)}} & \text{if } |Q| \geq 1, \end{cases}$$

for all cubes (or balls)  $Q \subset \mathbb{R}^n$ , where  $q(\infty) = \lim_{x \rightarrow \infty} q(x)$ .

The boundedness of the fractional integral operator  $I_\beta$  defined by

$$I_\beta(g)(z) = \int_{\mathbb{R}^n} \frac{g(t)}{|z-t|^{n-\beta}} dt \quad (2.3)$$

on variable Lebesgue space (see [4]) takes a crucial part in proving our main result.

**Proposition 2.7** *Let  $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  and  $0 < \beta < \frac{n}{(q_1)_+}$ , and define  $q_2(\cdot)$  by*

$$\frac{1}{q_2(\cdot)} = \frac{1}{q_1(\cdot)} - \frac{\beta}{n}. \quad (2.4)$$

*Then*

$$\|I_\beta f\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}. \quad (2.5)$$

Using Proposition (2.7), Wu [38] established the following result.

**Lemma 2.8** *Let  $\beta, q_1(\cdot)$ , and  $q_2(\cdot)$  be as defined in Proposition 2.7. Then*

$$\|\chi_{B_l}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \leq C 2^{-l\beta} \|\chi_{B_l}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \quad (2.6)$$

for all balls  $B_l = \{x \in \mathbb{R}^n : |x| \leq 2^l\}$  with  $l \in \mathbb{Z}$ .

Let  $C_l = \{x \in \mathbb{R}^n : 2^{l-1} < |x| \leq 2^l\}$  and  $\chi_l = \chi_{C_l}$  for  $l \in \mathbb{Z}$ . Then the homogenous Herz space with variable exponent is defined as follows.

**Definition 2.9** ([23]) *Let  $\alpha \in \mathbb{R}$ ,  $0 < q < \infty$ , and  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ . The homogenous Herz space with variable exponent  $\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$  is the set of all measurable functions  $f$  such that*

$$\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n) = \left\{ f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)} = \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha q} \|f \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{\frac{1}{q}}.$$

If  $p(\cdot) = p$ , then we have the classical Herz space  $\dot{K}_q^{\alpha,p}$  defined in [32]. Some generalizations of Herz spaces were made in [1, 15, 16, 36] shortly after their first appearance.

**Definition 2.10** *Let  $\alpha \in \mathbb{R}$ ,  $0 < q < \infty$ ,  $\lambda \in [0, \infty)$ , and  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ . The space  $M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$  is the set of all measurable functions  $f$  given by*

$$M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n) = \left\{ f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k=-\infty}^{k_0} \|2^{k\alpha} f \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q}.$$

Obviously,  $M\dot{K}_{q,p(\cdot)}^{\alpha,0}(\mathbb{R}^n) = \dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$  is the Herz space with variable exponent. The Herz–Morrey spaces with variable exponent  $M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}$  are first defined in [21, 22].

**Remark 2.11** Let  $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  and meet conditions (2.1) and (2.2) in Proposition 2.1, then so does  $p'(\cdot)$  and  $p'_1(\cdot)$ . This implies that  $p_1(\cdot), p'_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ . Therefore, using Lemma 2.2, we have the constants  $\delta_2 \in (0, \frac{1}{(p_2)_+})$ ,  $\delta_1 \in (0, \frac{1}{(p'_1)_+})$ ,  $\delta_2^* \in (0, \frac{1}{(p)_+})$ , and  $\delta_1^* \in (0, \frac{1}{(p')_+})$  such that the inequalities

$$\begin{aligned} \frac{\|\chi_S\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}} &\leq C \left( \frac{|S|}{|B|} \right)^{\delta_2}, & \frac{\|\chi_S\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}} &\leq C \left( \frac{|S|}{|B|} \right)^{\delta_1}, \\ \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} &\leq C \left( \frac{|S|}{|B|} \right)^{\delta_2^*}, & \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} &\leq C \left( \frac{|S|}{|B|} \right)^{\delta_1^*}, \end{aligned}$$

hold for all balls  $B \subset \mathbb{R}^n$  and  $S \subset B$ .

### 3 Main results and proofs

In this section, we present theorems on the boundedness of the rough fractional Hausdorff operator on the Herz-type spaces. We denote

$$A_{\Phi,s} = \left( \int_0^\infty |\Phi(t)|^s t^{(n-\beta)s-n} \frac{dt}{t} \right)^{\frac{1}{s}}.$$

**Theorem 3.1** Let  $0 < \beta < \frac{n}{(p_1)_+}$ ,  $0 < q_1 \leq q_2 < \infty$ , and  $\Omega \in L^s(S^{n-1})$ , and let  $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  satisfy conditions (2.1) and (2.2) in Proposition 2.1 with

$$\frac{1}{p_1(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{\beta}{n},$$

where  $p_1(\cdot) < \frac{n}{\beta}$ . Suppose that for  $0 < \delta_1, \delta_2 < 1$ ,  $\alpha$  satisfies  $\frac{n}{p_1} - n\delta_2 - \beta < \alpha < n\delta_1 - \frac{n}{(p'_1)_+}$  and  $0 \leq \lambda < \alpha + \beta + n\delta_2 - \frac{n}{p_1}$ . If  $\Phi$  is a radial function and  $A_{\Phi,s} < \infty$ , then for  $f \in M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$ , we have

$$\|H_{\Phi,\Omega}^\beta f\|_{M\dot{K}_{q_2,p_2(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)} \leq CA_{\Phi,s} \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}. \quad (3.1)$$

*Proof* Since  $q_1 \leq q_2$ , by the definition of the Morrey–Herz space we have

$$\begin{aligned} \|H_{\Phi,\Omega}^\beta f\|_{M\dot{K}_{q_2,p_2(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{q_1} &\leq \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha q_1} \| (H_{\Phi,\Omega}^\beta f) \chi_k \|_{L^{p_2(\cdot)}(\mathbb{R}^n)}^{q_1} \\ &\leq \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha q_1} \left( \sum_{j=-\infty}^{\infty} \| (H_{\Phi,\Omega}^\beta(f\chi_j)) \chi_k \|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\ &\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha q_1} \left( \sum_{j=-\infty}^{k-1} \| (H_{\Phi,\Omega}^\beta(f\chi_j)) \chi_k \|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \end{aligned}$$

$$\begin{aligned}
& + C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k \alpha q_1} \left( \sum_{j=k}^{\infty} \| (H_{\Phi, \Omega}^\beta(f \chi_j)) \chi_k \|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\
& =: I_1 + I_2.
\end{aligned}$$

For both  $I_1$  and  $I_2$ , we have to approximate the inner norm  $\| (H_{\Phi, \Omega}^\beta(f \chi_j)) \chi_k \|_{L^{p_2(\cdot)}(\mathbb{R}^n)}$ . For this, we proceed as follows:

$$\begin{aligned}
|H_{\Phi, \Omega}^\beta(f \chi_j)(x)| & \leq \int_{C_j} \left| \frac{\Phi(x|y|^{-1})}{|y|^{n-\beta}} \Omega(y') f(y) \right| dy \\
& \leq \left\| \left( \frac{\Phi(x|y|^{-1})}{|y|^{n-\beta}} \Omega(y') \right) \chi_j \right\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Since  $p'_1(\cdot) \in P(\mathbb{R}^n)$ , we can fix  $s$  such that  $s > p'_1(\cdot)$ . We define a new variable  $p(\cdot)$  such that  $\frac{1}{p'_1(\cdot)} = \frac{1}{s} + \frac{1}{p(\cdot)}$ . So by Lemma 2.5 we obtain

$$|H_{\Phi, \Omega}^\beta(f \chi_j)(x)| \leq \left\| \left( \frac{\Phi(x|y|^{-1})}{|y|^{n-\beta}} \Omega(y') \right) \chi_j \right\|_{L^s(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \quad (3.2)$$

By polar decomposition it is easy to see that

$$\begin{aligned}
\left\| \left( \frac{\Phi(x|y|^{-1})}{|y|^{n-\beta}} \Omega(y') \right) \chi_j \right\|_{L^s(\mathbb{R}^n)}^s & = \int_{C_j} \left| \frac{\Phi(x|y|^{-1})}{|y|^{n-\beta}} \Omega(y') \right|^s dy \\
& = \int_{2^{j-1}}^{2^j} \int_{S^{n-1}} \left| \frac{\Phi(|x|r^{-1})}{r^{n-\beta}} \right|^s |\Omega(y')|^s d\sigma(y') r^n \frac{dr}{r},
\end{aligned}$$

where  $d\sigma(y')$  denotes the normalized Lebesgue measure on the unit sphere  $S^{n-1}$ . A change of variables results in the following inequality:

$$\begin{aligned}
\left\| \left( \frac{\Phi(x|y|^{-1})}{|y|^{n-\beta}} \Omega(y') \right) \chi_j \right\|_{L^s(\mathbb{R}^n)}^s & = \int_{S^{n-1}} |\Omega(y')|^s d\sigma(y') \int_{\frac{|x|}{2^j}}^{\frac{|x|}{2^{j-1}}} |\Phi(t)|^s (|x|t^{-1})^{n-(n-\beta)s} \frac{dt}{t} \\
& \leq \|\Omega\|_{L^s(S^{n-1})}^s |x|^{n-(n-\beta)s} \int_0^\infty |\Phi(t)|^s t^{(n-\beta)s-n} \frac{dt}{t} \\
& = A_{\Phi, s}^s \|\Omega\|_{L^s(S^{n-1})}^s |x|^{s\beta+n-ns}.
\end{aligned}$$

Therefore

$$\left\| \left( \frac{\Phi(x|y|^{-1})}{|y|^{n-\beta}} \Omega(y') \right) \chi_j \right\|_{L^s(\mathbb{R}^n)} \leq C A_{\Phi, s} |x|^{\beta - \frac{n}{s'}}. \quad (3.3)$$

Also, when  $|B_j| \leq 2^n$  and  $y \in B_j$ , by  $\frac{1}{p'_1(y)} = \frac{1}{s} + \frac{1}{p(y)}$  and Lemma 2.6 we have

$$\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx |B_j|^{\frac{1}{p(\cdot)}} \approx |B_j|^{-\frac{1}{s}} \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}.$$

When  $|B_j| \geq 1$ , we have

$$\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx |B_j|^{\frac{1}{p(\infty)}} \approx |B_j|^{-\frac{1}{s}} \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}.$$

Hence we obtain

$$\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx |B_j|^{-\frac{1}{s}} \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}. \quad (3.4)$$

Substituting (3.3) and (3.4) into (3.2), we get

$$|H_{\Phi,\Omega}^\beta(f\chi_j)(x)| \leq CA_{\Phi,s}|x|^{\beta-\frac{n}{s'}}|B_j|^{-\frac{1}{s}}\|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}\|f\chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.$$

Multiplying both sides with  $\chi_k(x)$  and taking the  $L^{p_2(\cdot)}$  norm, the last inequality becomes

$$\begin{aligned} & \| (H_{\Phi,\Omega}^\beta(f\chi_j))\chi_k \|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ & \leq CA_{\Phi,s}|B_k|^{\frac{\beta}{n}-\frac{1}{s'}}|B_j|^{-\frac{1}{s}}\|\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}\|f\chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \end{aligned} \quad (3.5)$$

Having estimating the inner norm, we are now in position to approximate  $I_1$  and  $I_2$ . Let us first approximate  $I_1$ . For  $j < k$ , in view of Remark 2.11, from (3.5) we obtain the following inequality:

$$\begin{aligned} & \| (H_{\Phi,\Omega}^\beta(f\chi_j))\chi_k \|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ & \leq CA_{\Phi,s}|B_k|^{\frac{\beta}{n}-1}2^{n(j-k)(\delta_1-\frac{1}{s})}\|\chi_{B_k}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}\|f\chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

The conditions  $0 < \beta < \frac{n}{(p_1)_+}$  and  $\frac{1}{p_2(\cdot)} + \frac{\beta}{n} = \frac{1}{p_1(\cdot)}$  ensure the applicability of Lemma 2.8 to obtain

$$\begin{aligned} & \| (H_{\Phi,\Omega}^\beta(f\chi_j))\chi_k \|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ & \leq CA_{\Phi,s}2^{n(j-k)(\delta_1-\frac{1}{s})}|B_k|^{-1}\|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}\|f\chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Finally, Lemma 2.4 helps us in establishing the following inequality:

$$\| (H_{\Phi,\Omega}^\beta(f\chi_j))\chi_k \|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq CA_{\Phi,s}2^{n(j-k)(\delta_1-\frac{1}{s})}\|f\chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \quad (3.6)$$

Hence, using (3.6), we get

$$\begin{aligned} I_1 & \leq CA_{\Phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha q_1} \left( \sum_{j=-\infty}^{k-1} 2^{n(j-k)(\delta_1-\frac{1}{s})} \|f\chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1} \\ & \leq CA_{\Phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} \left( \sum_{j=-\infty}^{k-1} 2^{(k-j)(\alpha-n\delta_1+\frac{n}{s})} 2^{j\alpha} \|f\chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1}. \end{aligned} \quad (3.7)$$

The condition  $\alpha < n\delta_1 - \frac{n}{(p_1)_+}$  implies that  $\alpha < n(\delta_1 - \frac{1}{s})$ . So for  $1 < q_1 < \infty$ , we use the Hölder inequality to obtain

$$\begin{aligned} I_1 & \leq CA_{\Phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda q_1} \\ & \times \sum_{k=-\infty}^{k_0} \sum_{j=-\infty}^{k-1} 2^{\frac{q_1}{2}(k-j)(\alpha-n\delta_1+\frac{n}{s})} 2^{j\alpha q_1} \|f\chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \left( \sum_{j=-\infty}^{k-1} 2^{\frac{q'_1}{2}(k-j)(\alpha-n\delta_1+\frac{n}{s})} \right)^{\frac{q_1}{q'_1}} \end{aligned}$$

$$\begin{aligned}
&\leq CA_{\Phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=-\infty}^{k-1} 2^{\frac{q_1}{2}(k-j)(\alpha-n\delta_1+\frac{n}{s})} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&= CA_{\Phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \sum_{k=j+1}^{k_0} 2^{\frac{nq_1}{2}(k-j)(\alpha-n\delta_1+\frac{n}{s})} \\
&\leq CA_{\Phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\leq CA_{\Phi,s}^{q_1} \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{q_1}.
\end{aligned}$$

Similarly, for  $0 < q_1 \leq 1$ , from (3.7) we have

$$\begin{aligned}
I_1 &\leq CA_{\phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=-\infty}^{k-1} 2^{q_1(k-j)(\alpha-n\delta_1+\frac{n}{s})} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&= CA_{\phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \sum_{k=j+1}^{k_0} 2^{q_1(k-j)(\alpha-n\delta_1+\frac{n}{s})} \\
&\leq CA_{\phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\leq CA_{\phi,s}^{q_1} \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{q_1}.
\end{aligned}$$

Next, we turn to estimate  $I_2$ . For  $j \geq k$ , we again use inequality (3.5) and Remark 2.11 to obtain

$$\begin{aligned}
&\|(H_{\Phi,\Omega}^\beta(f \chi_j)) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\leq CA_{\Phi,s} |B_k|^{\frac{\beta}{n}-\frac{1}{s'}} |B_j|^{-\frac{1}{s}} \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} 2^{n(k-j)\delta_2} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Making use of the Lemma 2.8 once again, we get

$$\begin{aligned}
&\|(H_{\Phi,\Omega}^\beta(f \chi_j)) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\leq CA_{\Phi,s} |B_k|^{\frac{\beta}{n}-\frac{1}{s'}} |B_j|^{-\frac{1}{s}-\frac{\beta}{n}} \|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} 2^{n(k-j)\delta_2} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)},
\end{aligned}$$

which in view of Lemma 2.4 results in

$$\begin{aligned}
\|(H_{\Phi,\Omega}^\beta(f \chi_j)) \chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leq CA_{\Phi,s} |B_k|^{\frac{\beta}{n}-\frac{1}{s'}} |B_j|^{\frac{1}{s'}-\frac{\beta}{n}} 2^{n(k-j)\delta_2} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&= CA_{\Phi,s} 2^{(k-j)(\beta+n\delta_2-\frac{n}{s'})} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Hence  $I_2$  can be approximated as

$$I_2 \leq CA_{\Phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} \left( \sum_{j=k}^{\infty} 2^{(k-j)(\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{j\alpha} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right)^{q_1}. \quad (3.8)$$

Similarly, the condition  $0 \leq \lambda < \alpha + \beta + n\delta_2 - \frac{n}{p_1^-}$  implies that  $\lambda < \alpha + \beta + n\delta_2 - \frac{n}{s'}$  and  $\alpha + \beta + n\delta_2 - \frac{n}{s'} > 0$ . So we can choose a constant  $\epsilon > 1$  such that  $\lambda - \frac{1}{\epsilon}(\alpha + \beta + n\delta_2 - \frac{n}{s'}) < 0$ . For  $1 < q_1 < \infty$ , by the Hölder inequality we get

$$\begin{aligned}
I_2 &\leq CA_{\Phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \\
&\quad \times \sum_{k=-\infty}^{k_0} \sum_{j=k}^{\infty} 2^{\frac{q_1}{\epsilon}(k-j)(\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \left( \sum_{j=k}^{\infty} 2^{q'_1 \frac{(\epsilon-1)}{\epsilon}(k-j)(\alpha+\beta+n\delta_2-\frac{n}{s'})} \right)^{\frac{q_1}{q'_1}} \\
&\leq CA_{\Phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k}^{\infty} 2^{\frac{q_1}{\epsilon}(k-j)(\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\leq CA_{\Phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k}^{k_0-1} 2^{\frac{q_1}{\epsilon}(k-j)(\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\quad + CA_{\Phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k_0}^{\infty} 2^{\frac{q_1}{\epsilon}(k-j)(\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&= M_1 + M_2.
\end{aligned}$$

In view of the same condition  $\frac{n}{s'} - n\delta_2 - \beta < \alpha$ ,  $M_1$  is approximated as

$$\begin{aligned}
M_1 &\leq CA_{\Phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \sum_{k=-\infty}^j 2^{\frac{q_1}{\epsilon}(k-j)(\alpha+\beta+n\delta_2-\frac{n}{s'})} \\
&\leq CA_{\Phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\
&\leq CA_{\Phi,s}^{q_1} \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{q_1}.
\end{aligned}$$

Since  $\lambda < \frac{1}{\epsilon}(\alpha + \beta + n\delta_2 - \frac{n}{s'})$  and  $\alpha + \beta + n\delta_2 - \frac{n}{s'} > 0$ , we get

$$\begin{aligned}
M_2 &\leq CA_{\Phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k_0}^{\infty} 2^{\frac{q_1}{\epsilon}(k-j)(\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{j\lambda q_1} 2^{-j\lambda q_1} \\
&\quad \times \left( \sum_{l=-\infty}^j 2^{l\alpha q_1} \|f \chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \right) \\
&\leq CA_{\Phi,s}^{q_1} \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} 2^{\frac{q_1}{\epsilon} k (\alpha+\beta+n\delta_2-\frac{n}{s'})} \sum_{j=k_0}^{\infty} 2^{q_1 j (\lambda - \frac{1}{\epsilon}(\alpha+\beta+n\delta_2-\frac{n}{s'}))} \\
&\leq CA_{\Phi,s}^{q_1} \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} 2^{\frac{q_1}{\epsilon} k_0 (\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{q_1 k_0 (\lambda - \frac{1}{\epsilon}(\alpha+\beta+n\delta_2-\frac{n}{s'}))} \\
&\leq CA_{\Phi,s}^{q_1} \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{q_1}.
\end{aligned}$$

This completes the approximation of  $I_2$  in the case  $1 < q_1 < \infty$ .

Finally, it remains to estimate  $I_2$  for the case  $0 < q_1 \leq 1$ . For this, from (3.8) we get

$$\begin{aligned} I_2 &\leq CA_{\phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k}^{\infty} 2^{q_1(k-j)(\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\ &\leq CA_{\phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k}^{k_0-1} 2^{q_1(k-j)(\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\ &\quad + CA_{\phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k_0}^{\infty} 2^{q_1(k-j)(\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\ &= L_1 + L_2. \end{aligned}$$

Since  $\alpha + \beta + n\delta_2 - \frac{n}{s'} > 0$ , for  $L_1$ , we get

$$\begin{aligned} L_1 &\leq CA_{\phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k}^{k_0-1} 2^{q_1(k-j)(\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\ &= CA_{\phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \sum_{k=-\infty}^j 2^{q_1(k-j)(\alpha+\beta+n\delta_2-\frac{n}{s'})} \\ &\leq CA_{\phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\ &\leq CA_{\phi,s}^{q_1} \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{q_1}. \end{aligned}$$

Lastly, for the estimation of  $L_2$ , we proceed as follows:

$$\begin{aligned} L_2 &\leq CA_{\phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k_0}^{\infty} 2^{q_1(k-j)(\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{j\alpha q_1} \|f \chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \\ &\leq CA_{\phi,s}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} \sum_{j=k_0}^{\infty} 2^{q_1(k-j)(\alpha+\beta+n\delta_2-\frac{n}{s'})} \left( \sum_{l=-\infty}^j 2^{l\alpha q_1} \|f \chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1} \right) \\ &\leq CA_{\phi,s}^{q_1} \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{q_1} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} 2^{q_1 k (\alpha+\beta+n\delta_2-\frac{n}{s'})} \sum_{j=k_0}^{\infty} 2^{q_1 j (\lambda-\alpha+\beta+n\delta_2-\frac{n}{s'})}. \end{aligned}$$

Since  $0 \leq \lambda < \alpha + \beta + n\delta_2 - \frac{n}{s'}$ , we have

$$L_2 \leq CA_{\phi,s}^{q_1} \|f\|_{M\dot{K}_{q_1,p_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{q_1}.$$

Combining all the estimates we arrive at (3.1). Thus we have completed the proof of the theorem.  $\square$

Taking  $\lambda = 0$  in Theorem 3.1, we obtain the boundedness of rough fractional Hausdorff operator on variable Herz space.

**Corollary 3.2** Let  $0 < \beta < \frac{n}{(p_1')^+}$ ,  $0 < q_1 \leq q_2 < \infty$ , and  $\Omega \in L^s(S^{n-1})$ , and let  $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  satisfy conditions (2.1) and (2.2) in Proposition 2.1 with

$$\frac{1}{p_1(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{\beta}{n},$$

where  $p_1(\cdot) < \frac{n}{\beta}$ . Suppose that for  $0 < \delta_1, \delta_2 < 1$ ,  $\alpha$  satisfies  $\frac{n}{p_1^-} - n\delta_2 - \beta < \alpha < n\delta_1 - \frac{n}{(p_1')^+}$ . If  $\Phi$  is a radial function and  $A_{\Phi,s} < \infty$ , then for  $f \in \dot{K}_{p_1(\cdot)}^{\alpha,q_1}(\mathbb{R}^n)$ , we have

$$\|H_{\Phi,\Omega}^\beta f\|_{\dot{K}_{p_2(\cdot)}^{\alpha,q_2}(\mathbb{R}^n)} \leq CA_{\Phi,s} \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1}(\mathbb{R}^n)}.$$

The following theorem establishes the boundedness of the rough Hausdorff operator on the Morrey–Herz space with variable exponent.

**Theorem 3.3** Let  $0 < q_1 \leq q_2 < \infty$  and  $\Omega \in L^s(S^{n-1})$ , and let  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  satisfy conditions (2.1) and (2.2) in Proposition 2.1. Suppose that for  $0 < \delta_1^*, \delta_2^* < 1$ ,  $\alpha$  satisfies  $\frac{n}{p_1^-} - n\delta_2^* < \alpha < n\delta_1^* - \frac{n}{(p_1')^+}$  and  $0 \leq \lambda < \alpha + n\delta_2 - \frac{n}{p_1^-}$ . If  $\Phi$  is a radial function and  $A_{\Phi,s} < \infty$ , then for  $f \in M\dot{K}_{q_1,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$ , we have

$$\|H_{\Phi,\Omega} f\|_{M\dot{K}_{q_2,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)} \leq CA_{\Phi,s} \|f\|_{M\dot{K}_{q_1,p(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}.$$

*Proof* The proof is similar to that of Theorem 3.1, so we omit the details.  $\square$

As a corollary of Theorem 3.3, we obtain the boundedness of the rough Hausdorff operator on the variable Herz space.

**Corollary 3.4** Let  $0 < q_1 \leq q_2 < \infty$  and  $\Omega \in L^s(S^{n-1})$ , and let  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$  satisfy conditions (2.1) and (2.2) in Proposition 2.1. Suppose that for  $0 < \delta_1^*, \delta_2^* < 1$ ,  $\alpha$  satisfies  $\frac{n}{p_1^-} - n\delta_2^* < \alpha < n\delta_1^* - \frac{n}{(p_1')^+}$ . If  $\Phi$  is a radial function and  $A_{\Phi,s} < \infty$ , then for  $f \in \dot{K}_{p(\cdot)}^{\alpha,q_1}(\mathbb{R}^n)$ , we have

$$\|H_{\Phi,\Omega} f\|_{\dot{K}_{p(\cdot)}^{\alpha,q_2}(\mathbb{R}^n)} \leq CA_{\Phi,s} \|f\|_{\dot{K}_{p(\cdot)}^{\alpha,q_1}(\mathbb{R}^n)}.$$

## 4 Conclusions

In this paper, we have shown that the rough fractional Hausdorff operator is bounded on Herz-type spaces. The scaling argument commonly employed to establish the boundedness of the Hausdorff operator on function spaces with constant exponents makes the study of Hausdorff operators on function spaces with variable exponents unsuitable. We overcome this problem by employing a new strategy. Furthermore, this strategy will be helpful in studying Hausdorff-type operators on other function spaces with variable exponents.

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The authors declare no competing interests.

**Author contributions**

Amjad Hussain: Methodology, Writing- Reviewing and Editing, Conceptualization. Ilyas Khan: Validation, Supervision, Investigation. Abdullah Mohamed: Visualization, Preparation, Funding acquisition

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**References**

1. Almeida, A., Drihem, D.: Maximal, potential and singular type operators on Herz spaces with variable exponents. *J. Math. Anal. Appl.* **394**, 781–795 (2012)
2. Bandaliyev, R., Safarov, K.: On two-weight inequalities for Hausdorff operators of special kind in Lebesgue spaces. *Hacet. J. Math. Stat.* **50**, 1334–1346 (2021)
3. Burenkov, V., Liflyand, E.: Hausdorff operators on Morrey-type spaces. *Kyoto J. Math.* **60**, 93–106 (2020)
4. Capone, C., Cruz-Uribe, D., Fiorenza, A.: The fractional maximal operator and fractional integrals on variable  $L^p(\mathbb{R})$  spaces. *Rev. Mat. Iberoam.* **23**, 743–770 (2007)
5. Chen, J., Fan, D., Li, J.: Hausdorff operators on function spaces. *Chin. Ann. Math.* **33B**, 537–556 (2012)
6. Chen, J., Fan, D., Wang, S.: Hausdorff operators on Euclidean spaces. *Appl. Math.* **28**, 548–564 (2013)
7. Chen, J., Fan, D., Zhang, C.: Multilinear Hausdorff operators and their best constants. *Acta Math. Sin.* **28B(8)**, 1521–1530 (2012)
8. Cruz-Uribe, D., Fiorenza, A., Neugebauer, C.J.: The maximal function on variable  $L^p$  spaces. *Ann. Acad. Sci. Fenn., Math.* **28**, 223–238 (2003)
9. Diening, L., Hästö, P., Hästö, P., Ružička, M.: Lebesgue and Sobolev Spaces with Variable Exponent. Lecture Notes Math., vol. 2017. Springer, Heidelberg (2011)
10. Fan, D., Lin, X.: Hausdorff operator on real Hardy spaces. *Analysis* **34**, 319–337 (2014)
11. Fan, D., Zhao, F.: Multilinear fractional Hausdorff operators. *Acta Math. Sin.* **30**, 1407–1421 (2014)
12. Gao, G., Wu, X., Hussain, A., Zhao, G.: Some estimates for Hausdorff operators. *J. Math. Inequal.* **9**, 641–651 (2015)
13. Gao, G., Zhao, F.: Sharp weak bounds for Hausdorff operators. *Anal. Math.* **41**, 163–173 (2015)
14. Gao, L., Hussain, A.:  $(L^p, L^q)$ -Boundedness of Hausdorff operators with power weights on Euclidean spaces. *Anal. Theory Appl.* **31**, 101–108 (2015)
15. Ho, K.-P.: Extrapolation to Herz spaces with variable exponents and applications. *Rev. Mat. Complut.* **33**, 437–463 (2020)
16. Ho, K.-P.: Spherical maximal function, maximal Bochner–Riesz mean and geometrical maximal function on Herz spaces with variable exponents. *Rend. Circ. Mat. Palermo (2) Suppl.* **70**, 559–574 (2021)
17. Hussain, A., Ahmed, M.: Weak and strong type estimates for the commutators of Hausdorff operator. *Math. Inequal. Appl.* **20**, 49–56 (2017)
18. Hussain, A., Asim, M., Jarad, F.: Variable  $\lambda$ -central Morrey space estimates for the fractional Hardy operators and commutators. *J. Math.* **2022**, Article ID 5855068 (2022)
19. Hussain, A., Gao, G.: Multidimensional Hausdorff operators and commutators on Herz-type spaces. *J. Inequal. Appl.* **2013**, 594 (2013)
20. Hussain, A., Gao, G.: Some new estimates for the commutators of  $n$ -dimensional Hausdorff operator. *Appl. Math.* **29**, 139–150 (2014)
21. Izuki, M.: Boundedness of vector-valued sublinear operators on Herz–Morrey spaces with variable exponent. *Math. Sci. Res. J.* **13**, 243–253 (2009)
22. Izuki, M.: Fractional integrals on Herz–Morrey spaces with variable exponent. *Hiroshima Math. J.* **40**, 343–355 (2010)
23. Izuki, M.: Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization. *Anal. Math.* **13**, 33–50 (2010)
24. Kokilashvili, V., Meskhi, A., Rafeiro, H., Samko, S.: Integral Operators in Nonstandard Function Spaces: Variable Exponent Lebesgue and Amalgam Spaces, vol. 1. Birkhäuser, Heidelberg (2016)
25. Kováčik, O., Rákosník, J.: On spaces  $L^{p(x)}$  and  $W^{k,p(x)}$ . *Czechoslov. Math. J.* **41**, 592–618 (1991)
26. Lerner, A., Liflyand, E.: Multidimensional Hausdorff operators on the real Hardy space. *J. Aust. Math. Soc.* **83**, 79–86 (2007)
27. Liflyand, E.: Boundedness of multidimensional Hausdorff operators on  $H^1(\mathbb{R}^n)$ . *Acta Sci. Math.* **74**, 845–851 (2008)
28. Liflyand, E.: Hausdorff operators on Hardy spaces. *Eurasian Math. J.* **4**, 101–141 (2013)
29. Liflyand, E., Miyachi, A.: Boundedness of the Hausdorff operators in  $H^p$  spaces  $0 < p < 1$ . *Stud. Math.* **194**, 279–292 (2009)
30. Liflyand, E., Móricz, F.: The Hausdorff operator is bounded on the real Hardy space  $H^1(\mathbb{R})$ . *Proc. Am. Math. Soc.* **128**, 1391–1396 (2000)
31. Lin, X., Sun, L.: Some estimates on the Hausdorff operator. *Acta Sci. Math.* **78**, 669–681 (2012)

32. Lu, S., Xu, L.: Boundedness of rough singular integral operators on the homogeneous Morrey–Herz spaces. *Hokkaido Math. J.* **34**, 299–314 (2005)
33. Nakai, E., Sawano, Y.: Hardy spaces with variable exponents and generalized Campanato spaces. *J. Funct. Anal.* **262**, 3665–3748 (2012)
34. Nekavinda, A.: Hardy–Littlewood maximal operator on  $L^{p(\cdot)}(\mathbb{R})$ . *Math. Inequal. Appl.* **7**, 255–265 (2004)
35. Ruan, J., Fan, D.: Hausdorff type operators on the power weighted Hardy spaces  $H_{1,\mu}(\mathbb{R}^n)$ . *Math. Nachr.* **290**, 2388–2400 (2017)
36. Samko, S.: Variable exponent Herz spaces. *Mediterr. J. Math.* **10**, 2007–2025 (2013)
37. Sawano, Y., Di-Fazio, G., Hakim, D.I.: Morrey Spaces. Introduction and Applications to Integral Operators and PDE's, Volumes I and II. CRC Press, Boca Raton (2020)
38. Wu, J.: Boundedness of some sublinear operators on Herz–Morrey spaces with variable exponent. *Georgian Math. J.* **21**, 101–111 (2014)
39. Yee, T.-L., Ho, K.-P.: Hardy's inequalities and integral operators on Herz–Morrey spaces. *Open Math.* **18**, 106–121 (2020)

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