Aliyev *Journal of Inequalities and Applications* (2024) 2024:29
https://doi.org/10.1186/s13660-024-03107-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

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The best constant for inequality involving the sum of the reciprocals and product of positive numbers with unit sum

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Abstract

In this paper, we study a special algebraic inequality containing a parameter, the sum of reciprocals and the product of positive real numbers whose sum is 1. Using a new optimization argument the best values of the parameter are determined. In the case of three numbers the algebraic inequality has some interesting geometric applications involving a generalization of Euler's inequality about the ratio of radii of circumscribed and inscribed circles of a triangle.

Mathematics Subject Classification: Primary 26D05; secondary 52A40

Keywords: Best constant; Symmetric polynomials; Algebraic inequality; Geometric inequality; Euler's inequality

1 Introduction

Inequalities with sharp constants, or at least when good estimates can be given of the sharp constants, are of special interest both in themselves and when they are used for various applications. Just as one example, we mention the recent paper [20] in this Journal. Concerning the importance for various applications we refer to the recent books [21] and [19], and the references therein.

Consider the inequality

$$\sum_{i=1}^{n}\frac{1}{x_i} \geq \frac{\lambda}{1+n^{n-2}(\lambda-n^2)\prod_{i=1}^{n}x_i}, \tag{1}$$

where $x_1,x_2,\dots,x_n>0$; $\sum_{i=1}^{n}x_i=1$, for $n\geq 2$. Here, $\lambda>0$ is a real number and we are asked to find the best (maximal possible) λ for each n (see [7]). If such a λ exists, then we will denote it by λ_n. Note that the right-hand side of the inequality (1)

$$f(\lambda)=\frac{\lambda}{1+n^{n-2}(\lambda-n^2)\prod_{i=1}^{n}x_i},$$

where $x_1,x_2,\dots,x_n>0$; $\sum_{i=1}^{n}x_i=1$, is a nondecreasing function of $\lambda>0$. Hence, if (1) is true for a certain $\lambda=\lambda_n$, then it is also true for all $0<\lambda\leq\lambda_n$.

By the Cauchy–Schwarz inequality $\sum_{i=1}^{n} \frac{1}{x_i} \geq n^2 = f(n^2)$. Since the inequality holds true for $\lambda = n^2$, it also holds true for all $0 < \lambda \leq n^2$. Hence, the best constant $\lambda = \lambda_n$, if it exists, satisfies $\lambda_n \geq n^2$.

Case $n = 2$. For the case $n = 2$ there is no best constant. If $n = 2$, then we obtain the inequality

$$\frac{1}{x_1} + \frac{1}{x_2} \geq \frac{\lambda}{1 + (\lambda - 2^2)x_1x_2},$$

where $x_1, x_2 > 0$; $x_1 + x_2 = 1$. This inequality is true for any $\lambda > 0$. Indeed, if we multiply both sides by $(1 + (\lambda - 4)x_1x_2)$, then we obtain

$$\frac{1}{x_1} + \frac{1}{x_2} + (\lambda - 4)(x_1 + x_2) \geq \lambda.$$

Since $x_1 + x_2 = 1$, the parameter λ cancels out, and we obtain

$$\frac{1}{x_1} + \frac{1}{x_2} \geq 4,$$

which is always true.

Case $n = 3$. For case $n = 3$ the best constant is $\lambda_3 = 25$. We obtain the inequality

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \geq \frac{\lambda}{1 + 3(\lambda - 9)x_1x_2x_3},$$

where $x_1, x_2, x_3 > 0$; $x_1 + x_2 + x_3 = 1$. This inequality is true only for $0 < \lambda \leq 25$. We can show this by substituting $x_1 = x_2 = \frac{1}{4}$, $x_3 = \frac{1}{2}$ in this inequality. On the other hand, we can prove that

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \geq \frac{25}{1 + 48x_1x_2x_3},$$

holds true. Hence, $\lambda = 25$ is the maximum possible value for this inequality (see [2]). In the solution to problem [2] it was noted by D.B. Leep that the case $\lambda = 25$ is equivalent to a more general inequality $s_1^3s_2 + 48s_2s_3 - 25s_1^2s_3 \geq 0$ for symmetric polynomials $s_1 = x_1 + x_2 + x_3$, $s_2 = x_1x_2 + x_2x_3 + x_3x_1$, $s_3 = x_1x_2x_3$, which can also be written as

$$\begin{aligned} &x_1(x_2 - x_3)^2(3x_1 - x_2 - x_3)^2 \\ &\quad + x_2(x_1 - x_3)^2(3x_2 - x_1 - x_3)^2 + x_3(x_2 - x_1)^2(3x_3 - x_2 - x_1)^2 \geq 0, \end{aligned}$$

making case $n = 3$ almost trivial. Inequality (1) can also be written using symmetric polynomials, but as the results for cases $n = 4$ and $n = 5$ below suggest, there is no simple solution for $n > 3$. Let

$$s_1 = \sum_{i=1}^{n} x_i, \qquad s_{n-1} = \sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} x_j, \qquad s_n = \prod_{i=1}^{n} x_i.$$

If $\lambda > 0$, then (1) is equivalent to the inequality

$$s_1^n s_{n-1} + n^{n-2}\left(\lambda - n^2\right)s_{n-1}s_n - \lambda s_1^{n-1}s_n \geq 0,$$

which is homogeneous with respect to its variables $x_1, \ldots, x_n$.

There are some geometric applications of case $n = 3$. The inequality

$$\frac{R}{r} \geq 2 + \mu \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{(a+b+c)^2}, \tag{2}$$

where R and r are, respectively, the circumradius and inradius, and a, b, c are the sides of a triangle, holds true if $\mu \leq 8$. Indeed, substituting $a = b = 3$, $c = 2$, and the corresponding values of $R = \frac{9}{4\sqrt{2}}$ and $r = \frac{\sqrt{2}}{2}$ in (2) we obtain $\mu \leq 8$. Hence, again, if we can prove (2) for $\mu = 8$, then $\mu = 8$ will the best constant for the inequality (2). For $\mu = 8$ we obtain

$$\frac{R}{r} \geq 2 + 8 \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{(a+b+c)^2},$$

which is a refinement of Euler's inequality $\frac{R}{r} \geq 2$ and follows directly from the case $n = 3$ (see [4, 5]).

Another geometric application is the following inequality about the sides a, b, c of a triangle that follows directly from the case $n = 3$ (see [6]):

$$\frac{a^3}{b+c-a} + \frac{b^3}{a+c-b} + \frac{c^3}{a+b-c} + 7(ab+bc+ca) \geq 8\bigl(a^2+b^2+c^2\bigr).$$

This inequality can also be written as a quintic inequality of symmetric polynomials

$$9\sum^{3} a^5 - 15\sum^{6} a^4 b + 6\sum^{6} a^3 b^2 + 25\sum^{3} a^3 bc - 16\sum^{3} ab^2c^2 \geq 0,$$

which is a special case ($\nu = 3$) of the following inequality mentioned in [10] (see p. 244, where $\nu = u + 1$)

$$\begin{aligned} &\nu^2 \sum^{3} a^5 - \nu(\nu+2)\sum^{6} a^4 b + 2\nu \sum^{6} a^3 b^2 \\ &\quad + (\nu+2)^2 \sum^{3} a^3 bc - 4(\nu+1)\sum^{3} ab^2c^2 \geq 0. \end{aligned}$$

This general inequality is also easily proved if we put $a = x_2 + x_3$, $b = x_1 + x_3$, $c = x_1 + x_2$, and simplify to obtain

$$\begin{aligned} &4x_1(x_2-x_3)^2(\nu x_1 - x_2 - x_3)^2 \\ &\quad + 4x_2(x_1-x_3)^2(\nu x_2 - x_1 - x_3)^2 + 4x_3(x_2-x_1)^2(\nu x_3 - x_2 - x_1)^2 \geq 0. \end{aligned}$$

Similar quartic and sextic inequalities were studied in [8, 23], and their references (see also [16], Chap. 3).

One more geometric application of case $n = 3$ is about the areas of triangles and needs the introduction of some notations. Let M be a point in a triangle ABC. Extend lines AM, BM, and CM to intersect the sides of triangle ABC at A_0, B_0, and C_0, respectively (see Fig. 1). Next, construct the parallel to A_0C_0 through M, which intersects BA and BC at C_1 and A_2, respectively. Analogously, draw the parallel through M to B_0A_0 (and to B_0C_0) to find A_1 and B_2 (and B_1 and C_2). Denote

$$T_1 = [MC_1B_2], \qquad T_2 = [MA_1C_2], \qquad T_3 = [MB_1A_2],$$

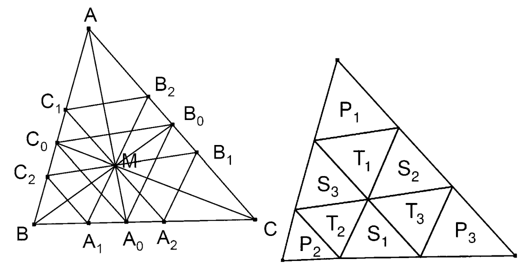


Figure 1 Geometric application of case $n = 3$

$$S_1 = [MA_1A_2], \qquad S_2 = [MB_1B_2], \qquad S_3 = [MC_1C_2],$$

$$P_1 = [AB_2C_1], \qquad P_2 = [BC_2A_1], \qquad P_3 = [CA_2B_1],$$

where the square brackets stand for the area of the triangles (see [3, 6]). Then,

$$P_1 + P_2 + P_3 + 7(S_1 + S_2 + S_3) \geq 8(T_1 + T_2 + T_3).$$

Case n = 4. For the case $n = 4$ the best constant is $\lambda_4 = \frac{582\sqrt{97}-2054}{121} \approx 30.4$ (see [4]). In this case, we obtain

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} \geq \frac{\lambda}{1 + 16(\lambda - 16)x_1x_2x_3x_4},$$

where $x_1, x_2, x_3, x_4 > 0$; $x_1 + x_2 + x_3 + x_4 = 1$. Again, this inequality is true only for $\lambda \leq \frac{582\sqrt{97}-2054}{121}$. Indeed, substituting in this inequality $x_1 = x_2 = x_3 = \frac{5+\sqrt{97}}{72}$, $x_4 = \frac{19-\sqrt{97}}{24}$, we obtain $0 < \lambda \leq \frac{582\sqrt{97}-2054}{121}$. On the other hand, we can prove that the inequality holds true for $\lambda = \frac{582\sqrt{97}-2054}{121}$. Hence, $\lambda = \frac{582\sqrt{97}-2054}{121}$ is the maximum possible value for this inequality.

Case n = 5. For the case $n = 5$ we obtain the inequality

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} \geq \frac{\lambda}{1 + 125(\lambda - 25)x_1x_2x_3x_4x_5},$$

where $x_1, x_2, x_3, x_4, x_5 > 0$; $x_1 + x_2 + x_3 + x_4 + x_5 = 1$, and it was conjectured in [4] that the best constant is

$$\lambda_5 = \frac{12{,}933{,}567 - 93{,}093\sqrt{22{,}535}}{4{,}135{,}801}\alpha + \frac{17{,}887{,}113 + 560{,}211\sqrt{22{,}535}}{996{,}728{,}041}\alpha^2 - \frac{288{,}017}{17{,}161} \approx 40.09,$$

where $\alpha = \sqrt[3]{8119 + 48\sqrt{22{,}535}}$. This conjecture for λ_5 will be proved in the current paper. Also, it will be proved that the equality cases in this inequality occur when $x_1 = x_2 = x_3 = x_4 = x_5 = \frac{1}{5}$ and when, for example, $x_1 = x_2 = x_3 = x_4 = x = \frac{\alpha}{240} + \frac{241}{240\alpha} + \frac{7}{240} \approx 0.173$, $x_5 = 1 - 4x \approx 0.308$.

Case n = 6. This case was not studied before. Using Maple, the exact value of λ_6 is calculated. Case $n = 6$ is possibly the last case for which these calculations of the exact value are possible.

Case $n \geq 7$. In view of the fact that quintic and higher-order equations are, in general, not solvable in radicals, it is unlikely that there is a precise formula for the best constant in the cases $n \geq 7$. Therefore, for the greater values of n ($n \geq 7$), instead of the exact value, it is reasonable to find some bounds or approximations for λ_n. In the current paper, it is proved that

$$\frac{n^3}{n-1} \leq \lambda_n \leq \frac{n^3}{n-2}. \tag{3}$$

Some possible improvements for this symmetric double inequality are also discussed.

It is interesting to compare the results of the current paper with the results for the following similar inequality

$$\sum_{i=1}^{n} \frac{1}{x_i} \leq \nu + \frac{n^2 - \nu}{n^n \prod_{i=1}^{n} x_i}, \tag{4}$$

where $x_1, x_2, \dots, x_i > 0$; $\sum_{i=1}^{n} x_i = 1$. The best constant ν_n for this inequality is known for all $n > 1$. See Corollary 2.13 in [12], where it is proved that $\nu \leq \nu_n = n^2 - \frac{n^n}{(n-1)^{n-1}}$. In particular, if $\nu = 0$, then we obtain

$$\sum_{i=1}^{n} \frac{1}{x_i} \leq \frac{1}{n^{n-2} \prod_{i=1}^{n} x_i}, \tag{5}$$

with the equality case possible only when $x_1 = \cdots = x_n = \frac{1}{n}$. Inequality (5) also follows from the following inequality for $E_i = \frac{1}{\binom{n}{i}} s_i$ (averages of s_i),

$$E_1^{\alpha_1} \cdots E_n^{\alpha_n} \leq E_1^{\beta_1} \cdots E_n^{\beta_n},$$

which holds if and only if

$$\alpha_m + 2\alpha_{m+1} + \cdots + (n - m + 1)\alpha_n \geq \beta_m + 2\beta_{m+1} + \cdots + (n - m + 1)\beta_n,$$

for each $1 \leq m \leq n$ (see [12], Theorem 1.1; [9], p. 94, item 77). Indeed, it is sufficient to note that inequality (5) can be written as $E_{n-1} \leq E_1^{n-1}$. This means that the above conditions for α_i, β_i ($i = 1, \dots, n$) are satisfied as

$$\begin{aligned} &\alpha_1 = \cdots = \alpha_{n-2} = 0, \qquad \alpha_{n-1} = n - 1, \qquad \alpha_n = 0, \\ &\beta_1 = n - 1, \qquad \beta_2 = \cdots = \beta_n = 0. \end{aligned}$$

Since (5) will be essential in the following text, an independent proof of (5) and some generalizations will be given in the Appendix. Note also that

$$\lim_{\lambda \to +\infty} \frac{\lambda}{1 + n^{n-2}(\lambda - n^2) \prod_{i=1}^{n} x_i} = \frac{1}{n^{n-2} \prod_{i=1}^{n} x_i}.$$

Using this and by comparing (1) and (5), we obtain that if $n > 2$, then $\lambda_n < +\infty$.

Special cases $n = 3$ and $n = 4$ of inequality (4) are also of interest for comparison with the corresponding cases of inequality (1). If $n = 3$, then the best constant inequality is

$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\le\frac{9}{4}+\frac{1}{4x_1x_2x_3}$, where $x_1,x_2,x_3>0$; $x_1+x_2+x_3=1$. Surprisingly, this inequality is also equivalent to a geometric inequality. One can show that it simplifies to $p^2\ge 16Rr-5r^2$, where p is the semiperimeter of a triangle. The last geometric inequality also follows from the formula for the distance between the incenter I and the centroid G of a triangle: $|IG|^2=\frac{1}{9}(p^2+5r^2-16Rr)$ (see [4]). If $n=4$, then the best constant inequality is $\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\le\frac{176}{27}+\frac{1}{27x_1x_2x_3x_4}$, where $x_1,x_2,x_3,x_4>0$; $x_1+x_2+x_3+x_4=1$ (see [25], Example 3).

The literature about symmetric polynomial inequalities is extensive [13–15, 17, 18, 24, 26]. Some of the results of the current paper were presented at the Maple Conference 2021 [1].

2 Main results

Let us consider all cases for $n\ge 3$ in a unified way. Assume first that $(x_1,x_2,\dots,x_n)\ne(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n})$. Then, by using (5), inequality (1) can be written as

$$\frac{n^2(1-n^n\prod_{i=1}^n x_i)}{\frac{n^2}{\sum_{i=1}^n\frac{1}{x_i}}-n^n\prod_{i=1}^n x_i}\ge\lambda, \tag{6}$$

where $x_1,x_2,\dots,x_n>0$; $\sum_{i=1}^n x_i=1$, for $n\ge 3$. Let us denote the left-hand side of (6) by $g(x_1,\dots,x_n)$, which is defined for all points of the bounded set

$$C=\left\{\mathbf{x}\middle|\mathbf{x}=(x_1,x_2,\dots,x_n);x_1,x_2,\dots,x_n\ge 0;\sum_{i=1}^n x_i=1\right\},$$

except for point $P_0(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n})$. For the points of boundary

$$\partial C=\left\{\mathbf{x}\middle|\mathbf{x}=(x_1,x_2,\dots,x_n);x_1,x_2,\dots,x_n\ge 0;\sum_{i=1}^n x_i=1,\prod_{i=1}^n x_i=0\right\},$$

function g is undefined and, obviously, for each $i=1,\dots,n$,

$$\lim_{x_i\to 0}g(x_1,\dots,x_n)=+\infty.$$

Lemma 1 *If* $x>0$, *then*

$$\lim_{(x_1,x_2,\dots,x_n)\to(x,x,\dots,x)}\frac{(\sum_{i=1}^n x_i)-\frac{n^2}{\sum_{i=1}^n\frac{1}{x_i}}}{(\sum_{i=1}^n x_i)^n-n^n\prod_{i=1}^n x_i}=\frac{2}{n^nx^{n-1}}.$$

Proof The limit can be interpreted as a single variable limit if we take

$$(x_1,x_2,\dots,x_n)=(x+\gamma_1t,x+\gamma_2t,\dots,x+\gamma_nt),$$

where not all constants γ_i are equal and $t\to 0$. Hence, we calculate

$$\lim_{t\to 0}\frac{(\sum_{i=1}^n(x+\gamma_it))-\frac{n^2}{\sum_{i=1}^n\frac{1}{x+\gamma_it}}}{(\sum_{i=1}^n(x+\gamma_it))^n-n^n\prod_{i=1}^n(x+\gamma_it)}$$

$$= \lim_{t\to 0} \frac{(\sum_{i=1}^n \gamma_i) - \frac{n^2}{(\sum_{i=1}^n \frac{1}{x+\gamma_i t})^2} \sum_{i=1}^n \frac{\gamma_i}{(x+\gamma_i t)^2}}{n(\sum_{i=1}^n (x+\gamma_i t))^{n-1}(\sum_{i=1}^n \gamma_i) - n^n(\prod_{i=1}^n (x+\gamma_i t))(\sum_{i=1}^n \frac{\gamma_i}{x+\gamma_i t})}$$

$$= \lim_{t\to 0} \frac{\frac{-2n^2}{(\sum_{i=1}^n \frac{1}{x+\gamma_i t})^3}(\sum_{i=1}^n \frac{\gamma_i}{(x+\gamma_i t)^2})^2 + \frac{n^2}{(\sum_{i=1}^n \frac{1}{x+\gamma_i t})^2}(\sum_{i=1}^n \frac{2\gamma_i^2}{(x+\gamma_i t)^3})}{n(n-1)(\sum_{i=1}^n (x+\gamma_i t))^{n-2}(\sum_{i=1}^n \gamma_i)^2 - n^n(\prod_{i=1}^n (x+\gamma_i t))((\sum_{i=1}^n \frac{\gamma_i}{x+\gamma_i t})^2 - \sum_{i=1}^n \frac{\gamma_i^2}{(x+\gamma_i t)^2})}$$

$$= \frac{\frac{-2n^2}{(\frac{n}{x})^3}(\frac{\sum_{i=1}^n \gamma_i}{x^2})^2 + \frac{n^2}{(\frac{n}{x})^2}\frac{\sum_{i=1}^n 2\gamma_i^2}{x^3}}{n(\cancel{n}-1)(nx)^{n-2}(\sum_{i=1}^n \gamma_i)^2 - n^n x^n(\cancel{(\frac{\sum_{i=1}^n \gamma_i}{x})^2} - \frac{\sum_{i=1}^n \gamma_i^2}{x^2})} = \frac{2}{n^n x^{n-1}},$$

where we used L'Hôpital's rule twice and the fact that $n\sum_{i=1}^n \gamma_i^2 > (\sum_{i=1}^n \gamma_i)^2$ (the Cauchy–Schwarz inequality, the equality case is not possible as not all γ_i are equal). The proof is completed. □

In particular, if $\sum_{i=1}^n x_i = 1$, then $x = \frac{1}{n}$, and therefore, by Lemma 1,

$$\lim_{(x_1,x_2,\dots,x_n)\to(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n})} \frac{n^2(1 - n^n\prod_{i=1}^n x_i)}{\frac{n^2}{\sum_{i=1}^n \frac{1}{x_i}} - n^n\prod_{i=1}^n x_i}$$

$$= \frac{n^2}{1 - \lim_{(x_1,x_2,\dots,x_n)\to(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n})} \frac{1-\frac{n^2}{\sum_{i=1}^n \frac{1}{x_i}}}{1-n^n\prod_{i=1}^n x_i}} = \frac{n^2}{1 - \frac{2}{n^n(\frac{1}{n})^{n-1}}} = \frac{n^3}{n-2}.$$

As an immediate consequence of this and (6), we obtain an upper bound for the best constant

$$\lambda_n \le \frac{n^2}{n-2}. \tag{7}$$

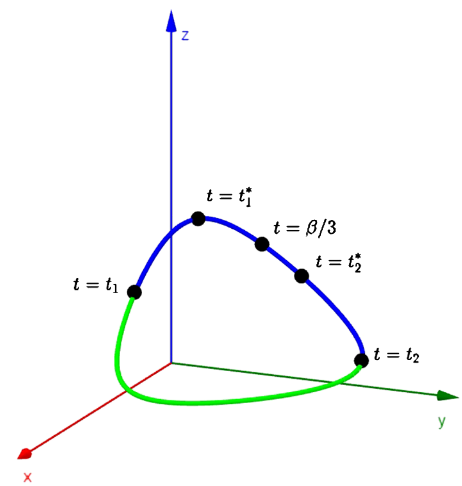
We want to use a well-known result in the analysis, which states that a continuous function over a compact set achieves its minimum (and maximum) values at certain points. For this purpose, let us change function $g(x_1,\dots,x_n)$, to a new function g_1 so that g_1 is defined also at point $P_0(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n})$ and points of ∂C, and g_1 is continuous in the compact set $\overline{C} = C \cup \partial C$:

$$g_1(x_1,\dots,x_n) = \begin{cases} \frac{\pi}{2}, & \text{if } \prod_{i=1}^n x_i = 0; \\ \tan^{-1}\frac{n^3}{n-2}, & \text{if } (x_1,x_2,\dots,x_n) = (\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}); \\ \tan^{-1} g(x_1,\dots,x_n), & \text{otherwise.} \end{cases}$$

Since g_1 is a continuous function in compact $\overline{C}$, g_1 reaches its extreme values somewhere in $\overline{C}$. Obviously, g_1 reaches its maximum value $\frac{\pi}{2}$ at the boundary points ∂C where $\prod_{i=1}^n x_i = 0$, and the minimum value at a point of C. The minimum of g is achieved at the same point of C if the minimum point is different from $P_0(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n})$. In any case, $\inf_{\mathbf{x}\in C} g = \tan(\min_{\mathbf{x}\in\overline{C}} g_1)$. We use an optimization argument similar to [12, 25] but with 3 variables, to determine where these points must lie. This method can also be used for other inequalities involving only symmetric polynomials s_1, s_{n-1}, and s_n.

Let $P(x_1,x_2,\dots,x_n)$ be a minimum point of g_1. Select any 3 of the coordinates of $(x_1,x_2,\dots,x_n)$, say x_1, x_2, and x_3. Let us assume that $x_1x_2x_3 = \alpha$ and $x_1 + x_2 + x_3 = \beta$. Since

Figure 2 Parametric space curve (blue and green) representing intersection of the plane $x + y + z = \beta$ (not shown) and the surface $xyz = \alpha$ (not shown)



$P \in C$, $\alpha, \beta > 0$. Also, by the AM-GM inequality $\beta^3 \geq 27\alpha$ and it is known that if $\beta^3 = 27\alpha$, then $x_1 = x_2 = x_3$. Hence, suppose that $\beta^3 > 27\alpha$. Let us now take arbitrary positive numbers x, y, z such that $xyz = \alpha$ and $x + y + z = \beta$. Without loss of generality we can assume that $x \leq y \leq z$. Since $x + z = \beta - y$ and $xz = \frac{\alpha}{y}$, the numbers x and z are the solutions of the quadratic equation $\delta^2 + (y - \beta)\delta + \frac{\alpha}{y} = 0$. If we take $y = t$, then we obtain parametrization of the curve obtained by intersection of the plane $x + y + z = \beta$ and the surface $xyz = \alpha$:

$$x = \frac{-t + \beta \pm \sqrt{(t - \beta)^2 - \frac{4\alpha}{t}}}{2}, \qquad y = t, \qquad z = \frac{-t + \beta \mp \sqrt{(t - \beta)^2 - \frac{4\alpha}{t}}}{2}.$$

Parameter t changes in the interval $[t_1, t_2]$, where t_1 and t_2 are the zeros of the cubic $\kappa(t) = t(t - \beta)^2 - 4\alpha$ in intervals $(0, \frac{\beta}{3})$ and $(\frac{\beta}{3}, \beta)$, respectively. The third zero t_3 of $\kappa(t)$ satisfies $t_3 > \beta$ and therefore $t_3 \notin [t_1, t_2]$. Since we are interested only with the case $x \leq y \leq z$, we will take one half of this curve (see Fig. 2)

$$x = \frac{-t + \beta - \sqrt{(t - \beta)^2 - \frac{4\alpha}{t}}}{2}, \qquad y = t, \qquad z = \frac{-t + \beta + \sqrt{(t - \beta)^2 - \frac{4\alpha}{t}}}{2},$$

and in a smaller interval $[t_1^*, t_2^*]$, where t_1^* and t_2^* are the zeros of the cubic $\kappa^*(t) = \kappa(t) - t(3t - \beta)^2$ in intervals $(t_1, \frac{\beta}{3})$ and $(\frac{\beta}{3}, t_2)$, respectively. Again, since the third zero t_3^* of $\kappa^*(t)$ satisfies $t_3^* > \beta$, $t_3^* \notin [t_1, t_2]$. Note that if $t = t_1^*$, then $x = y$, and if $t = t_2^*$, then $y = z$. Consider the sum $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ and note that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{y} + \frac{x + z}{xz} = \frac{1}{y} + \frac{\beta - y}{\alpha/y} = \frac{1}{t} + \frac{\beta t - t^2}{\alpha}.$$

Denote $h(t) = \frac{1}{t} + \frac{\beta t - t^2}{\alpha}$, where $t \in [t_1^*, t_2^*]$. Hence, if $t \in (t_1^*, t_2^*)$, then $h'(t) = -\frac{1}{t^2} + \frac{\beta}{\alpha} - \frac{2t}{\alpha} = \frac{(z-t)(t-x)}{xy^2 z} > 0$, and $h'(t_1^*) = h'(t_2^*) = 0$. Consequently, $h(t)$ attains its minimum and maximum in the interval $[t_1^*, t_2^*]$ at the endpoints t_1^*, t_2^*, respectively. We are interested in making h smaller, which happens when the sum $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is smaller. Hence, the minimum of $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$ is reached when $x_1 = x_2 \leq x_3$. Since the coordinates x_1, x_2, and x_3 were chosen arbitrarily, these results hold for any trio of coordinates. Therefore, the left-hand side of

(6) is minimal only when there are at most 2 distinct numbers in the set $\{x_1, x_2, \dots, x_n\}$ Furthermore, if the two numbers are distinct, then the smaller one is repeated $n-1$ times in $\{x_1, x_2, \dots, x_n\}$ i.e., $x_1 = x_2 = \cdots = x_{n-1} \le x_n$. Consequently, in (6) we can restrict ourselves only to the case where $x_1 = x_2 = \cdots = x_{n-1} = x$, $x_n = 1-(n-1)x$, where $0 < x \le \frac{1}{n}$. By substituting these in (6) and simplifying, we obtain

$$\frac{n^2}{1-(n-1)\frac{(nx-1)^2}{((n-1)-n(n-2)x)(1-n^n x^{n-1}(1-(n-1)x))}} \ge \lambda.$$

We will study the part of the denominator that is dependent on x, and for simplicity put $t = nx$. Hence,

$$\begin{aligned}&\frac{(t-1)^2}{((n-1)-(n-2)t)(1-nt^{n-1}+(n-1)t^n)}\\&\quad=\frac{1}{((n-1)-(n-2)t)(1+2t+3t^2+\cdots+(n-1)t^{n-2})}.\end{aligned}$$

Denote the polynomial in the denominator by

$$p_n(t) = \bigl((n-1)-(n-2)t\bigr)\bigl(1+2t+3t^2+\cdots+(n-1)t^{n-2}\bigr),$$

where $0 \le t \le 1$. By taking the derivative and simplifying, we obtain

$$\begin{aligned}p_n'(t) &= (n-1)\bigl(1\cdot 2+2\cdot 3t+3\cdot 4t^2+\cdots+(n-2)\cdot(n-1)t^{n-3}\bigr)\\&\quad-(n-2)\bigl(1^2+2^2t+3^2t^2+\cdots+(n-1)^2t^{n-2}\bigr)\\&=1\cdot n+2\cdot(n+1)t+3\cdot(n+2)t^2+\cdots+(n-2)\cdot(2n-3)t^{n-3}\\&\quad-(n-2)(n-1)^2t^{n-2}.\end{aligned}$$

Since $p_n'(0) = n > 2$ and $p_n'(1) = -\frac{n(n-1)(n-2)}{6} < 0$, there is at least 1 zero of the polynomial $p_n'(t)$ in the interval $(0,1)$. On the other hand, by Descartes' rule of signs (see p. 247 in [11], or p. 28 in [22]) the number of positive zeros of $p_n'(t)$ does not exceed the number of sign changes in the sequence of coefficients of $p_n'(t)$, which is 1. Hence, $p_n'(t)$ has exactly one zero t_n in $[0,1]$, which is also the maximum point of $p_n(t)$. This means that there is exactly one point $x = \frac{t_n}{n}$ in $(0, \frac{1}{n})$, such that $x_1 = x_2 = \cdots = x_{n-1} = x$, $x_n = 1-(n-1)x$ makes the left-hand side of (6) minimal. This minimal value is also the best constant for (1):

$$\lambda_n = \frac{n^2}{1-\frac{n-1}{p_n(t_n)}}. \tag{8}$$

For $n = 3, 4, 5$, and 6 it is possible to find the exact values of t_n and the corresponding λ_n.

- if $n=3$, then $p_3(t) = (2-t)(1+2t)$, and $p_3'(t) = 3-4t$. Therefore, $t_3 = \frac{3}{4}$. By (8), the best constant is $\lambda_3 = \frac{3^2}{1-\frac{3-1}{p_3(t_3)}} = 25$ (see [2]).
- if $n=4$, then $p_4(t) = (3-2t)(1+2t+3t^2)$ and $p_4'(t) = 4+10t-18t^2$. Therefore, $t_4 = \frac{5+\sqrt{97}}{18}$. By (8), the best constant is $\lambda_4 = \frac{4^2}{1-\frac{4-1}{p_4(t_4)}} = \frac{582\sqrt{97}-2054}{121} \approx 30.423077$ (see [4]).

- if $n = 5$, then $p_5(t) = (4 - 3t)(1 + 2t + 3t^2 + 4t^3)$ and $p_5'(t) = 5 + 12t + 21t^2 - 48t^3$. Using Cardano's formula and Maple, we find that $t_5 = \frac{\theta+7+241\theta^{-1}}{48}$, where $\theta = (8119 + 48\sqrt{22{,}535})^{\frac{1}{3}}$. By (8), the best constant is $\lambda_5 = \frac{5^2}{1-\frac{5-1}{p_5(t_5)}} \approx 40.090307$, which coincides with the value of λ_5 conjectured in [4].
- if $n = 6$, then

  $$p_6(t) = (5 - 4t)\bigl(1 + 2t + 3t^2 + 4t^3 + 5t^4\bigr), \qquad p_6'(t) = 6 + 14t + 24t^2 + 36t^3 - 100t^4.$$

  Using Ferrari's method and Maple, we find that

  $$t_6 = \frac{9 + \phi + \sqrt{50\psi + 962 - 11{,}300\psi^{-1} + 47{,}258\phi^{-1}}}{100},$$

  where

  $$\phi = \sqrt{-50\psi + 481 + 11{,}300\psi^{-1}}, \qquad \psi = (1473 + \sqrt{13{,}712{,}905})^{\frac{1}{3}}.$$

  By (8), the best constant is $\lambda_6 = \frac{6^2}{1-\frac{6-1}{p_6(t_6)}} \approx 52.358913$.

For larger values of n, we can give some bounds for λ_n. We already found an upper bound (7). We will now focus on a similar lower bound.

By the AM-GM inequality,

$$\sum_{i=1}^{n} \frac{1}{x_i} \geq \frac{n}{G_n}, \tag{9}$$

where $x_1, x_2, \dots, x_n > 0$; $\sum_{i=1}^{n} x_i = 1$, $G_n = \sqrt[n]{\prod_{i=1}^{n} x_i}$, and $n \geq 2$. Let us show that if $\lambda = \frac{n^3}{n-1}$, then

$$\frac{n}{G_n} \geq \frac{\lambda}{1 + n^{n-2}(\lambda - n^2)G_n^n}, \tag{10}$$

where $(x_1, x_2, \dots, x_n) \neq (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$ and therefore $G_n = \sqrt[n]{\prod_{i=1}^{n} x_i} < \frac{1}{n}$. Indeed, we can simplify (10) to

$$\frac{n^2(1 - s^n)}{s(1 - s^{n-1})} \geq \frac{n^3}{n - 1},$$

where $s = nG_n < 1$. This is easily proved, as we can write it in the following form

$$1 + \frac{1}{s + s^2 + \dots + s^{n-1}} \geq \frac{n}{n - 1},$$

where noting $s < 1$ completes the proof. From (1), (9), and (10) it follows that if $\lambda \leq \frac{n^3}{n-1}$, then (1) holds true. This means that we have now a lower bound for the best constant:

$$\lambda_n \geq \frac{n^3}{n - 1}. \tag{11}$$

Combining (7) and (11) we obtain the following symmetric double inequality.

Theorem 1 *If $n > 2$, then*

$$\frac{n^3}{n-1} \le \lambda_n \le \frac{n^3}{n-2}. \tag{12}$$

It is possible to improve these estimates in exchange for a less-elegant formula. For example, if we put $x_1 = x_2 = \cdots = x_{n-1} = \frac{1}{n+1}$, $x_n = \frac{2}{n+1}$, then we obtain from (6) a new upper bound for the best constant:

$$\lambda_n \le (n+1)^2 \cdot \frac{\frac{1}{2} - \frac{n^n}{(n+1)^n}}{\frac{n+1}{2n-1} - \frac{n^{n-2}}{(n+1)^{n-2}}}. \tag{13}$$

One can check that (13) is sharper than (6) for all $n > 3$. We can also prove that $\frac{n}{n+1} \le t_n$ or equivalently, $p_n'(\frac{n}{n+1}) \ge 0$ for all $n \ge 3$. Indeed,

$$p_n'\left(\frac{n}{n+1}\right) = 3n(n+1)^2\left(\frac{n}{n+1}\right)^n\left(\left(1+\frac{1}{n}\right)^n - \frac{8}{3} + \frac{1}{n} - \frac{1}{3n^2}\right).$$

For $n = 3, 4, \dots, 25$ one can check directly that $(1+\frac{1}{n})^n - \frac{8}{3} + \frac{1}{n} - \frac{1}{3n^2} \ge 0$. For $n > 25$, one can use the fact that $(1+\frac{1}{n})^n > \frac{8}{3}$ and $\frac{1}{n} > \frac{1}{3n^2}$.

Appendix

We will give a proof of (5) here. We can use the optimization argument given after Lemma 1 of the current paper, to maximize the left-hand side of (5), while keeping the right-hand side of (5) fixed. This is achieved when for any 3 of the coordinates, say x_1, x_2, and x_3, of $(x_1, x_2, \dots, x_n)$, $x_1 \le x_2 = x_3$. Hence, we can restrict ourselves only to the case where $x_1 = x$ and $x_2 = \cdots = x_{n-1} = x_n = \frac{1-x}{n-1}$, where $0 < x \le \frac{1}{n}$. For this particular case (5) is transformed into

$$\frac{1}{x} + \frac{(n-1)^2}{1-x} \le \frac{(n-1)^{n-1}}{n^{n-2}x(1-x)^{n-1}},$$

which can be simplified to the correct inequality

$$(nx-1)^2\big((n-2)(nx)^{n-3} + (n-3)(nx)^{n-4} + \cdots + 1\big) \ge 0.$$

The equality case is possible only when $nx = 1$.

Inequality (5) can also be written as the homogeneous inequality $A_n^{n-1}H_n \ge G_n^n$, where A_n, H_n, and G_n are, respectively, the arithmetic, harmonic, and geometric means of arbitrary positive numbers $x_1, \dots, x_n$:

$$A_n = \frac{\sum_{i=1}^n x_i}{n}, \qquad H_n = \frac{n}{\sum_{i=1}^n \frac{1}{x_i}}, \qquad G_n = \sqrt[n]{\prod_{i=1}^n x_i}.$$

Since $A_n \ge G_n$, automatically $A_n^l H_n \ge G_n^{l+1}$ for any real number $l \ge n-1$. It would be natural to ask whether the general inequality $A_n^l H_n \ge G_n^{l+1}$ can hold true also for some real number $l < n-1$. The answer to this question is negative. A counterexample is found if one takes $x_1 = x_2 = \cdots = x_{n-1} = 1$ and $x_n = x$, where $x \to 0^+$. Indeed, if $l < n-1$, then $A_n^l H_n = O(x)$ and $G_n^{l+1} = O(x^{\frac{l+1}{n}}) \gg O(x)$.

Acknowledgements
This work was completed with the support of ADA University Faculty Research and Development Fund.

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
Y.A. wrote the main manuscript text and prepared Figs. 1–2.

Received: 17 December 2023 Accepted: 19 February 2024 Published online: 01 March 2024

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