# An application of decision theory on the approximation of a generalized Apollonius-type quadratic functional equation 

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#### Abstract

To make better decisions on approximation, we may need to increase reliable and useful information on different aspects of approximation. To enhance information about the quality and certainty of approximating the solution of an Apollonius-type quadratic functional equation, we need to measure both the quality and the certainty of the approximation and the maximum errors. To measure the quality of it, we use fuzzy sets, and to achieve its certainty, we use the probability distribution function. To formulate the above problem, we apply the concept of $Z$-numbers and introduce a special matrix of the form $\operatorname{diag}(A, B, C)$ (named the generalized $Z$-number) where $A$ is a fuzzy time-stamped set, $B$ is the probability distribution function, and $C$ is a degree of reliability of $A$ that is described as a value of $A * B$. Using generalized $Z$-numbers, we define a novel control function to investigate H-U-R stability to approximate the solution of an Apollonius-type quadratic functional equation with quality and certainty of the approximation.


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## 1 Introduction

In many practical problems, the fuzzy probability approach can be an important component of decision making. In the real world, we consider various aspects of uncertainty that are not always well represented in fuzzy sets of information uncertainty. To overcome this problem, Zadeh introduced the Z-number (Z-N) in 2011 [25]; for more on the subject, see Aliev et al. [3] and Allahviranloo et al. [4]. A $Z-\mathrm{N}$ is an ordered binary of the form $(A, B)$ where the first component shows the fuzzy value and the second shows the uncertainty of the first. Based on the $Z-N$ theory, we provide a model which considers both certainty and quality for the solution of an Apollonius-type quadratic functional equation.

The question of stability of functional equations was first raised by Ulam, and then Hyers investigated stability for mappings from one Banach space to another. Stability analysis

[^0]in the sense of Ulam-Hyers can be used to find an approximate solution for a wide selection of functional equations such as integral equations, differential equations, fractional equations, etc.; see $[2,5,6,8-11,17-20,22,24,26]$. The functional equation
\[

$$
\begin{equation*}
\Upsilon(\tau-x)+\Upsilon(\tau-y)=\frac{1}{2} \Upsilon(x-y)+2 \Upsilon\left(\tau-\frac{x+y}{2}\right) \tag{1.1}
\end{equation*}
$$

\]

is called a quadratic functional equation of Apollonius-type, and Kim et al. [16] studied the stability of generalized Hyers-Ulam type for equation (1.1). Park-Rassias [21] and Wang [23] introduced the generalized quadratic functional equation of Apollonius-type:

$$
\begin{align*}
& \Upsilon\left(\sum_{i=1}^{n} \tau_{i}-\sum_{i=1}^{n} x_{i}\right)+\Upsilon\left(\sum_{i=1}^{n} \tau_{i}-\sum_{i=1}^{n} y_{i}\right) \\
& \quad=\frac{1}{2} \Upsilon\left(\sum_{i=1}^{n} x_{i}-\sum_{i=1}^{n} y_{i}\right)+2 \Upsilon\left(\sum_{i=1}^{n} \tau_{i}-\frac{\sum_{i=1}^{n} x_{i}+\sum_{i=1}^{n} y_{i}}{2}\right) . \tag{1.2}
\end{align*}
$$

Note that every solution of Eq. (1.2) is said to be a generalized quadratic mapping of Apollonius-type. Since Eq. (1.2) includes a quadratic function, we will need a bilinear space, so here we consider such equations in a $\mathbb{C}$-module space. In Sect. 2 of this article, we provide appropriate concepts and results, and in Sect. 3, we prove the H-U-R stability for (1.2) which shows both certainty and quality for the solution of a quadratic functional equation of Apollonius-type.

## 2 Basic concepts

Let $\Theta_{1}=[0,1]$, and let $x_{\Theta_{1}}$ be given as follows:

$$
x_{\Theta_{1}}=\operatorname{diag} \Theta_{1}=\left\{\left[\begin{array}{lll}
\theta_{1} & & \\
& \ddots & \\
& & \theta_{n}
\end{array}\right]=\operatorname{diag}\left[\theta_{1}, \ldots, \theta_{n}\right], \theta_{1}, \ldots, \theta_{n} \in \Theta_{1}\right\} .
$$

We write $\operatorname{diag}\left[\theta_{1}, \ldots, \theta_{n}\right] \preceq \operatorname{diag}\left[\kappa_{1}, \ldots, \kappa_{n}\right]$ when $\theta_{i} \leq \kappa_{i}$ for every $i=1, \ldots, n$.

Definition 2.1 ([13]) A mapping $\circledast: x_{\Theta_{1}} \times x_{\Theta_{1}} \rightarrow x_{\Theta_{1}}$ is called a generalized continuous t-norm (GCTN) if for all $\varrho, \kappa, \varpi, y, \kappa_{n}, \varpi_{n} \in x_{\Theta_{1}}, \mathbf{1}=\operatorname{diag}[1, \ldots, 1]$ the following conditions are satisfied:
(t1) $\varpi \circledast \mathbf{1}=\varpi$;
(t2) $\varpi \circledast \kappa=\kappa \circledast \varpi$;
$(\mathrm{t} 3) \varpi \circledast(\kappa \circledast \varrho)=(\varpi \circledast \kappa) \circledast \varrho$;
(t4) $\varrho \preceq \kappa$ and $\varpi \preceq y$ imply that $\varrho \circledast \varpi \preceq \kappa \circledast y$;
(t4) If $\lim _{n \rightarrow \infty} \kappa_{n}=\kappa$ and $\lim _{n \rightarrow \infty} \varpi_{n}=\varpi$, we have $\lim _{n \rightarrow \infty}\left(\kappa_{n} \circledast \varpi_{n}\right)=\kappa \circledast \varpi$.

In this paper, we choose the minimum t-norm $\circledast_{M}=x_{\Theta_{1}} \times x_{\Theta_{1}} \rightarrow x_{\Theta_{1}}$ which is defined as follows:

$$
\varpi \circledast_{M} \kappa=\operatorname{diag}\left[\varpi_{1}, \ldots, \varpi_{n}\right] \circledast_{M} \operatorname{diag}\left[\kappa_{1}, \ldots, \kappa_{n}\right]=\operatorname{diag}\left[\min \left\{\varpi_{1}, \kappa_{1}\right\}, \ldots, \min \left\{\varpi_{n}, \kappa_{n}\right\}\right] .
$$

Definition 2.2 ([7]) Let $\wp \in \mathbb{R}$ and $\wp \in(0,1]$, and let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Let $X$ be a linear space over $\mathbb{K}$. A fuzzy set $\aleph_{\wp}: X \times(0, \infty) \rightarrow \Theta_{1}$ is a $\wp$-fuzzy norm ( $\left.\wp-\mathrm{FN}\right)$ on $X$ if and only if we have

$$
\begin{aligned}
& (\wp N 1) \aleph_{\wp}(x, \zeta)=1 \text { if and only if } x=0 \text { for } \zeta \in(0, \infty) \text {; } \\
& (\wp N 2) \aleph_{\wp}(\gamma x, \zeta)=\aleph_{\wp}\left(x, \frac{\zeta}{\mid \gamma \gamma}\right) \text { for all } \gamma \neq 0 \in \mathbb{K} \text {, all } x \in X \text { and for } \zeta \in(0, \infty) \text {; } \\
& (\wp N 3) \aleph_{\wp}(x+y, \zeta+\delta) \geq \aleph_{\wp}(x, \zeta) \circledast \aleph_{\wp}(y, \delta) \text { for all } x, y \in X \text { and any } \zeta, \delta \in(0, \infty) \text {; } \\
& (\wp N 4) \lim _{\zeta \rightarrow+\infty} \aleph_{\wp}(x, \zeta)=1 \text { for any } \zeta \in(0, \infty) .
\end{aligned}
$$

A $\wp$-Banach FN space is a complete $\wp$-FN space.

Now, we use the concept of probability distribution functions to measure the certainty of a vector [1], where we put

$$
\epsilon_{0}(\zeta)= \begin{cases}0, & \text { if } \zeta \leq 0  \tag{2.1}\\ 1, & \text { if } \zeta>0\end{cases}
$$

Definition 2.3 Assume $\wp \in(0,1)$ and let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. A $\wp$-random normed space ( $\wp$ RNS) is a triple $\left(X,{ }^{\diamond} \mu, \circledast^{\prime}\right)$, where $X$ is a vector space over $\mathbb{K}, \circledast^{\prime}$ is a continuous t-norm, and $\wp \mu$ is a mapping from $X$ into $D^{+}$such that the following conditions hold:
$(\mu 1) \wp \mu_{x}(\zeta)=\epsilon_{0}(\zeta)$ for all $\zeta>0$ if and only if $x=0$;
( $\mu 2) \wp \mu_{\alpha x}(\zeta)=\varnothing \mu_{x}\left(\frac{\zeta}{|\alpha| \wp}\right)$ for all $x \in X$ and $\alpha \neq 0$;
( $\mu 3) \wp \mu_{x+y}(\zeta+\delta) \geq{ }^{\wp} \mu_{x}(\zeta) \circledast{ }^{\prime} \wp \mu_{y}(\delta)$ for all $x, y \in X$ and $\zeta, \delta \geq 0$, where ${ }^{\wp} \mu_{x}$ denotes the value of ${ }^{\triangleright} \mu$ at a point $x \in X$.

Let $\wp \in \mathbb{R}$ and $\wp \in(0,1]$, and let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. We define a matrix-valued function $\widetilde{Z}: X \times$ $\mathbb{R}^{+} \rightarrow x_{\Theta_{1}}$ with $\widetilde{Z}(x, \zeta)=\operatorname{diag}\left[\aleph_{\wp}(x, \zeta), \wp \mu_{x}(\zeta), \aleph_{\wp}(x, \zeta) \circledast \wp \mu_{x}(\zeta)\right]$, and call it a generalized $Z$-number (GZ-N), when for all $x, y \in X, \zeta, \delta>0$, and $\alpha \neq 0$ the following conditions are satisfied:
(Z1) $\widetilde{Z}(x, \zeta)=\operatorname{diag}[1,1,1]$ if and only if $x=0$;
(Z2) $\widetilde{Z}(\alpha x, \zeta)=\widetilde{Z}\left(x, \frac{\zeta}{|\alpha|^{\natural}}\right)$;
(Z3) $\widetilde{Z}(x+y, \zeta+\delta) \succeq \widetilde{Z}(x, \zeta) \circledast_{M} \widetilde{Z}(y, \delta)$.
Now we prove the above conditions. According to the conditions ( $\wp N 1$ ) and ( $\mu 1$ ), we have

$$
\widetilde{Z}(x, \zeta)=\operatorname{diag}\left[\aleph_{\wp}(x, \zeta),{ }^{\wp} \mu_{x}(t), \aleph_{\wp}(x, \zeta) \circledast \wp \mu_{x}(\zeta)\right]=\operatorname{diag}[1,1,1 \circledast 1]=\operatorname{diag}[1,1,1]
$$

if and only if $x=0$, therefore (Z1) is established.
According to the conditions ( $\wp N 2$ ) and ( $\mu 2$ ), we have

$$
\begin{aligned}
\widetilde{Z}(\alpha x, \zeta) & =\operatorname{diag}\left[\aleph_{\wp}(\alpha x, \zeta),{ }^{\wp} \mu_{\alpha x}(t), \aleph_{\wp}(\alpha x, \zeta) \circledast \circledast^{\wp} \mu_{\alpha x}(t)\right] \\
& =\operatorname{diag}\left[\aleph_{\wp}\left(x, \frac{\zeta}{|\alpha|^{\wp}}\right), \wp \mu_{x}\left(\frac{\zeta}{|\alpha|^{\wp}}\right), \aleph_{\wp}\left(x, \frac{\zeta}{|\alpha|^{\wp}}\right) \circledast{ }^{\wp} \mu_{x}\left(\frac{\zeta}{|\alpha|^{\wp}}\right)\right] \\
& =\widetilde{Z}\left(x, \frac{\zeta}{|\alpha|^{\wp}}\right),
\end{aligned}
$$

therefore (Z2) is established.
According to the conditions ( $\wp N 3$ ) and ( $\mu 3$ ), we have

$$
\widetilde{Z}(x+y, \zeta+\delta)=\operatorname{diag}[\aleph_{\wp}(x+y, \zeta+\delta), \wp \mu_{x+y}(\zeta+\delta), \aleph_{\wp}(x+y, \zeta+\delta) \circledast \overbrace{}^{\wp} \mu_{x+y}(\zeta+\delta)]
$$

$$
\begin{aligned}
\succeq & \operatorname{diag}\left[\aleph_{\wp}(x, \zeta) \circledast \aleph_{\wp}(y, \delta),{ }^{\wp} \mu_{x}(\zeta) \circledast{ }^{\wp} \mu_{y}(\delta),\left(\aleph_{\wp}(x, \zeta) \circledast \aleph_{\wp}(y, \delta)\right)\right. \\
& \left.\circledast\left({ }^{\wp} \mu_{x}(\zeta) \circledast \wp \mu_{y}(\delta)\right)\right] \\
= & \operatorname{diag}\left[\aleph_{\wp}(x, \zeta) \circledast \aleph_{\wp}(y, \delta), \wp \mu_{x}(\zeta) \circledast \wp \mu_{y}(\delta),\left(\aleph_{\wp}(x, \zeta)\right.\right. \\
& \left.\left.\circledast{ }^{\wp} \mu_{x}(\zeta)\right) \circledast\left(\aleph_{\wp}(y, \delta) \circledast{ }^{\wp} \mu_{y}(\delta)\right)\right] \\
= & \widetilde{Z}(x, \zeta) \circledast_{M} \widetilde{Z}(y, \delta),
\end{aligned}
$$

therefore (Z3) is established.

Theorem $2.4([12,14,15])$ Let $(\Omega, d)$ be a complete generalized metric space $(G M S)$ and $\Xi: \Omega \rightarrow \Omega$ such that for all $x, y \in \Omega$,

$$
d(\Xi x, \Xi y) \leq L d(x, y)
$$

with $L \in(0,1)$.
Then, for each given $x \in \Omega$, either $d\left(\Xi^{n} x, \Xi^{n+1} x\right)=\infty$ for all $n \geq 0$, or there exists a nonnegative integer $n_{0}$ such that $d\left(\Xi^{n} x, \Xi^{n+1} x\right)<\infty$ for all $n \geq n_{0}, \lim _{n \rightarrow \infty} \Xi^{n} x=y^{*}$, and $y^{*}$ is the fixed point of $\Xi$ in the set $\Omega^{*}=\left\{y \in \Omega \mid d\left(\Xi^{n_{0}} x, y\right)<\infty\right\}$ and $d\left(y, y^{*}\right) \leq \frac{1}{1-L} d(y, \Xi y)$ for all $y \in \Omega^{*}$.

Lemma 2.5 ([21]) If $\Upsilon: \mathbf{K} \rightarrow \mathbf{H}$ with $\Upsilon(0)=0$ is a solution of (1.2) for all $\tau_{1}, \ldots, \tau_{n}, x_{1}, \ldots$, $x_{n}, y_{1}, \ldots, y_{n} \in \mathbf{K}$, where $\mathbf{K}$ and $\mathbf{H}$ are real vector spaces, then $\Upsilon: \mathbf{K} \rightarrow \mathbf{H}$ is a quadratic function, i.e., $\Upsilon(\kappa \omega)=\kappa^{2} \Upsilon(\omega)$ for $\kappa \in \mathbb{K}$ and $\omega \in \mathbf{K}$.

## 3 Measure of the quality and the certainty of the approximation of the solution of quadratic functional equation of Apollonius-type with GZ-N

We consider a function $\Upsilon: \mathbf{K} \rightarrow \mathbf{H}$ such that $\mathbf{K}$ is a $\wp-\mathrm{N}$ left $\mathbb{C}$-module and $\mathbf{H}$ is a $\wp-\mathrm{N}$ left Banach $\mathbb{C}$-module, and let $n \geq 2$ be a fixed integer. We show the H-U-R stability of (1.2) via Theorem 2.4 to measure both the quality and certainty of the approximation of (1.2). For the function $\Upsilon: \mathbf{K} \rightarrow \mathbf{H}$, we define the difference operator

$$
\begin{aligned}
D_{b} \Upsilon\left(\tau_{1}, \ldots, \tau_{n}, x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}\right):= & \Upsilon\left(\sum_{i=1}^{n} b \tau_{i}-\sum_{i=1}^{n} b x_{i}\right)+\Upsilon\left(\sum_{i=1}^{n} b \tau_{i}-\sum_{i=1}^{n} b y_{i}\right) \\
& -\frac{1}{2} \Upsilon\left(\sum_{i=1}^{n} b x_{i}-\sum_{i=1}^{n} b y_{i}\right) \\
& -2 b^{2} \Upsilon\left(\sum_{i=1}^{n} \tau_{i}-\frac{\sum_{i=1}^{n} x_{i}+\sum_{i=1}^{n} y_{i}}{2}\right)
\end{aligned}
$$

for all $\tau_{i}, x_{i}, y_{i} \in \mathbf{K}, b \in C_{1}=\{b \in \mathbb{C}| | b \mid=1\}$, and $i=1, \ldots, n$.

Theorem 3.1 (i) Let $\Upsilon: \mathbf{K} \rightarrow \mathbf{H}$ with $\Upsilon(0)=0$, and assume there exists a function $\mho(x, \zeta)=$ $\operatorname{diag}[\varphi(x, \zeta), \psi(x, \zeta), \varphi(x, \zeta) \circledast \psi(x, \zeta)]$ with $\varphi: \mathbf{K}^{3 n} \times \mathbb{R}^{+} \rightarrow \Theta_{1}, \psi: \mathbf{K}^{3 n} \rightarrow D^{+}$such that

$$
\begin{equation*}
\tilde{Z}\left(D_{b} \Upsilon\left(\tau_{1}, \ldots, \tau_{n}, x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}\right), \zeta\right) \succeq \mho\left(\left(\tau_{1}, \ldots, \tau_{n}, x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}\right), \zeta\right) \tag{3.1}
\end{equation*}
$$

and

$$
\begin{align*}
& \mho\left(\left(2 \tau_{1}, \ldots, 2 \tau_{n}, 2 x_{1}, \ldots, 2 x_{n}, 2 y_{1}, \ldots, 2 y_{n}\right), \zeta\right) \\
& \quad \succeq \mho\left(\left(\tau_{1}, \ldots, \tau_{n}, x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}\right), \frac{\zeta}{4^{\diamond} L}\right) \tag{3.2}
\end{align*}
$$

for all $\tau_{i}, x_{i}, y_{i} \in \mathbf{K}, b \in C_{1}, i=1, \ldots, n$ and for some $L \in(0,1)$, then there exists a unique generalized quadratic mapping of Apollonius-type $\Phi: \mathbf{K} \rightarrow \mathbf{H}$ such that

$$
\begin{equation*}
\widetilde{Z}(\Upsilon(x)-\Phi(x), \zeta) \succeq \mho((\underbrace{x / n, \ldots, x / n}_{2 n}, \underbrace{0, \ldots, 0}_{n}), \frac{(1-L) \zeta}{2^{\wp} L}), \tag{3.3}
\end{equation*}
$$

for all $x \in \mathbf{K}$.
(ii) If instead we assume that (3.1) holds and

$$
\begin{align*}
& \mho\left(\left(\tau_{1}, \ldots, \tau_{n}, x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}\right), \zeta\right) \\
& \quad \succeq \mho\left(\left(2 \tau_{1}, \ldots, 2 \tau_{n}, 2 x_{1}, \ldots, 2 x_{n}, 2 y_{1}, \ldots, 2 y_{n}\right), \frac{4^{\S} \zeta}{L}\right), \tag{3.4}
\end{align*}
$$

for all $\tau_{i}, x_{i}, y_{i} \in \mathbf{K}, i=1, \ldots, n, b \in C_{1}$, and some $L \in(0,1)$, then there exists a unique generalized quadratic mapping of Apollonius-type $\Phi: \mathbf{K} \rightarrow \mathbf{H}$ such that

$$
\begin{equation*}
\widetilde{Z}(\Upsilon(x)-\Phi(x), \zeta) \succeq \mho((\underbrace{x / n, \ldots, x / n}_{2 n}, \underbrace{0, \ldots, 0}_{n}), \frac{(1-L) \zeta}{2^{\wp}}) \tag{3.5}
\end{equation*}
$$

for all $x \in \mathbf{K}$.
Moreover, in both cases, if $\Upsilon(t x)$ is continuous in $t \in \mathbb{C}$ for each fixed $x \in \mathbf{K}$, then $\Phi$ is $\mathbb{C}$-Q, i.e., $\Phi(b x)=b^{2} \Phi(x)$ for all $x \in \mathbf{K}$ and all $b \in \mathbb{C}$.

Proof For all $\tau_{i}, x_{i}, y_{i} \in \mathbf{K}, i=1, \ldots, n$, using (3.2) we have

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \mho\left(\left(2^{k} \tau_{1}, \ldots, 2^{k} \tau_{n}, 2^{k} x_{1}, \ldots, 2^{k} x_{n}, 2^{k} y_{1}, \ldots, 2^{k} y_{n}\right), 4^{k \wp} \zeta\right)=1, \tag{3.6}
\end{equation*}
$$

and letting $\tau_{i}=x_{i}=\frac{x}{n}$ and $y_{i}=0$, as well as $b=1$, in (3.1) and using $\Upsilon(0)=0$, we have

$$
\begin{equation*}
\widetilde{Z}\left(\Upsilon(x)-4 \Upsilon\left(\frac{x}{2}\right), \zeta\right) \succeq \mho((\underbrace{x / n, \ldots, x / n}_{2 n}, \underbrace{0, \ldots, 0}_{n}), \frac{\zeta}{2^{\gamma}}) \tag{3.7}
\end{equation*}
$$

for all $x \in$ and for all $i=1, \ldots, n$. Consider the set $\Omega:=\{\mu \mid \mu: \mathbf{K} \rightarrow \mathbf{H}, \mu(0)=0\}$ and

$$
\begin{align*}
\nabla(\mu, \varpi)= & \inf \{K \in(0, \infty) \mid \\
& \widetilde{Z}(\mu(x)-\varpi(x), \zeta) \succeq \mho(\underbrace{(x / n, \ldots, x / n}_{2 n}, \underbrace{0, \ldots, 0}_{n}), \frac{\zeta}{2^{\wp} K}) \forall x \in \mathbf{K}\} . \tag{3.8}
\end{align*}
$$

We now show the completeness of $(\Omega, \nabla)$. We begin by showing $\nabla(\mu, \varpi)=0$ if and only if $\mu=\varpi$. Assume that $\nabla(\mu, \varpi)=0$. Then

$$
\begin{align*}
\nabla( & \mu, \varpi)  \tag{3.9}\\
= & \inf \{K \geq 0: \widetilde{Z}(\mu(x)-\varpi(x), \zeta) \succeq \mho((\underbrace{x / n, \ldots, x / n}_{2 n}, \underbrace{0, \ldots, 0}_{n}), \frac{\zeta}{2^{\wp} K}), \\
& \forall \mu, \varpi \in \Omega, x \in \mathbf{K}, \zeta \in(0,+\infty)\} \\
= & 0
\end{align*}
$$

and so

$$
\begin{equation*}
\widetilde{Z}(\mu(x)-\partial(x), \zeta) \succeq \mho((\underbrace{x / n, \ldots, x / n}_{2 n}, \underbrace{0, \ldots, 0}_{n}), \frac{\zeta}{2^{\wp} K}) \tag{3.10}
\end{equation*}
$$

for all $x \in \mathbf{K}, \zeta>0$ and $K \in[0,+\infty)$. Letting $K \rightarrow 0$ in (3.10), we get

$$
\widetilde{Z}(\mu(x)-\check{\partial}(x), \zeta)=\mathbf{1} .
$$

Thus $\mu(x)=\partial(x)$ for every $x \in \mathbf{K}$, and vice versa. In addition, we have $\nabla(\mu, \partial)=\nabla(\nearrow, \mu)$ for every $\mu, \varnothing \in \Omega$. Now we show $\nabla(\mu, \mathrm{w}) \leq \nabla(\mu, \nearrow)+\nabla(\varnothing, \mathrm{w})$. Let $\nabla(\mu, \nearrow)=\rho_{1} \in(0,+\infty)$ and $\nabla(\partial, \mathrm{w})=\rho_{2} \in(0,+\infty)$. Then, we have

$$
\begin{aligned}
& \widetilde{Z}(\mu(x)-\partial(x), \zeta) \succeq \vartheta((\underbrace{x / n, \ldots, x / n}_{2 n}, \underbrace{0, \ldots, 0}_{n}), \frac{\zeta}{2^{\wp} \rho_{1}}), \\
& \widetilde{Z}(\partial(x)-\mathrm{w}(x), \zeta) \succeq \vartheta((\underbrace{x / n, \ldots, x / n}_{2 n}, \underbrace{0, \ldots, 0}_{n}), \frac{\zeta}{2^{\wp} \rho_{2}}),
\end{aligned}
$$

for all $x \in \mathbf{K}$ and $\zeta \in(0,+\infty)$. Now, the triangle inequality, (Z3), implies that

$$
\begin{aligned}
\tilde{Z}\left(\mu(x)-\mathrm{w}(x),\left(\rho_{1}+\rho_{2}\right) \zeta\right) & \succeq \widetilde{Z}\left(\mu(x)-\delta(x), \rho_{1} \zeta\right) \circledast \widetilde{Z}\left(\delta(x)-\mathrm{w}(x), \rho_{2} \zeta\right) \\
& \succeq \mho((\underbrace{x / n, \ldots, x / n}_{2 n}, \underbrace{0, \ldots, 0}_{n}), \frac{\zeta}{2^{\wp}}) \\
& \circledast \mho(\underbrace{(x / n, \ldots, x / n}_{2 n}, \underbrace{0, \ldots, 0}_{n}), \frac{\zeta}{2^{\wp}}) \\
= & \mho(\underbrace{x / n, \ldots, x / n}_{2 n}, \underbrace{0, \ldots, 0}_{n}), \frac{\zeta}{2^{\wp}}),
\end{aligned}
$$

for all $x \in \mathbf{K}$ and $\zeta \in(0,+\infty)$. Then, $\nabla(\mu, \mathrm{w}) \leq \rho_{1}+\rho_{2}$ and $\nabla(\mu, \mathrm{w}) \leq \nabla(\mu, \check{\partial})+\nabla(\check{\delta}, \mathrm{w})$. To prove the completeness of $(\Omega, \nabla)$, assume $\left\{\wp_{k}\right\}$ is a Cauchy sequence in $\Omega$ such that for every $x \in \mathbf{K}, \boldsymbol{\rho} \succ 0$, and $\zeta \in(0,+\infty)$, we have

$$
\mho(x, \zeta) \succ 1-\rho .
$$

Choose $q_{0} \in \mathbb{N}$ such that

$$
\nabla\left(\wp_{q}, \wp_{p}\right)<\rho \quad \forall q, p \geq q_{0} .
$$

Then

$$
\begin{aligned}
\widetilde{Z}\left(\wp_{q}(x)-\wp_{p}(x), \zeta\right) & \succeq \widetilde{Z}\left(\wp_{q}(x)-\wp_{p}(x), \rho \zeta\right) \\
& \succeq V((\underbrace{x / n, \ldots, x / n}_{2 n}, \underbrace{0, \ldots, 0}_{n}), \frac{\zeta}{2^{\wp} K}) \\
& \succ \mathbf{1}-\boldsymbol{\rho},
\end{aligned}
$$

for every $x \in \mathbf{K}$ and $\zeta \in(0,+\infty)$, i.e.,

$$
\widetilde{Z}\left(\wp_{q}(x)-\wp_{p}(x), \zeta\right) \succ \mathbf{1 - \rho} .
$$

For a fixed $x \in \mathbf{K},\left\{\wp_{q}(x)\right\}_{k}$ is Cauchy in the complete space $(\mathbb{R}, \widetilde{Z}, \circledast)$. Now, since $\mathbf{K}$ is compact, we conclude that $\left\{\wp_{q}(x)\right\}_{k}$ is uniformly convergent (say) $\wp$ from $\mathbf{K}$ to $(\mathbb{R}, \widetilde{Z}, \circledast)$. Note also that $\wp \in \Omega$. Thus we have proved the completeness of $(\Omega, \nabla)$.

We now define a function $\Xi: \Omega \rightarrow \Omega$ by

$$
\begin{equation*}
(\Xi \mu)(x)=\frac{1}{4} \mu(2 x), \quad \forall \mu \in \Omega, x \in \mathbf{K} . \tag{3.11}
\end{equation*}
$$

Let $\mu, \varpi \in \Omega$ and $K \in[0, \infty]$ such that $\nabla(\mu, \varpi)<K$, and by (3.8) we have

$$
\begin{equation*}
\widetilde{Z}(\mu(x)-\varpi(x), \zeta) \succeq \mho((\underbrace{x / n, \ldots, x / n}_{2 n}, \underbrace{0, \ldots, 0}_{n}), \frac{\zeta}{2^{\wp} K}), \quad \forall x \in \mathbf{K} . \tag{3.12}
\end{equation*}
$$

Now using (3.1), (3.2), and (3.12), we have

$$
\begin{equation*}
\widetilde{Z}\left(4^{-1} \mu(2 x)-4^{-1} \varpi(2 x), \zeta\right) \succeq \mho(\underbrace{x / n, \ldots, x / n}_{2 n}, \underbrace{0, \ldots, 0}_{n}), \frac{\zeta}{2^{\wp} K L}) \quad \forall x \in \mathbf{K} . \tag{3.13}
\end{equation*}
$$

Hence, we have that $\nabla(\Xi \mu, \Xi \varpi) \leq L \nabla(\mu, \varpi)$. It follows from (3.7) that $\nabla(\Xi \Upsilon, \Upsilon) \leq L<$ $\infty$. So, by Theorem 2.4, $\Xi$ has a unique fixed point $\Phi: \mathbf{K} \rightarrow \mathbf{H}$ in the set $\Omega^{*}=\{\mu \in$ $\Omega \mid \nabla(\Upsilon, \mu)<\infty\}$ such that

$$
\begin{equation*}
\Phi(x):=\lim _{k \rightarrow \infty}\left(\Xi^{k} \Upsilon\right)(x)=\lim _{k \rightarrow \infty} \frac{1}{4^{k}} \Upsilon\left(2^{k} x\right) \tag{3.14}
\end{equation*}
$$

and $\Phi(2 x)=2^{2} \Phi(x)$ for all $x \in \mathbf{K}$. Also,

$$
\begin{equation*}
\nabla(\Phi, \Upsilon) \leq \frac{1}{1-L} \nabla(\Xi \Upsilon, \Upsilon) \leq \frac{L}{1-L} \tag{3.15}
\end{equation*}
$$

Thus (3.3) holds for all $x \in \mathbf{K}$.

Now, we prove part (ii). It follows from (3.4) that

$$
\begin{aligned}
& \lim _{k \rightarrow \infty} \mho\left(\left(\frac{\tau_{1}}{2^{k}}, \ldots, \frac{\tau_{n}}{2^{k}}, \frac{x_{1}}{2^{k}}, \ldots, \frac{x_{n}}{2^{k}}, \frac{y_{1}}{2^{k}}, \ldots, \frac{y_{n}}{2^{k}}\right), \frac{\zeta}{4^{k \wp}}\right)=1, \\
& \forall \tau_{1}, \ldots, \tau_{n}, x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n} \in \mathbf{K} .
\end{aligned}
$$

Let $\Xi: \Omega \rightarrow \Omega$ be the mapping defined by

$$
\Xi \mu(x)=4 \mu\left(\frac{x}{2}\right), \quad \forall \mu \in \Omega, x \in \mathbf{K}
$$

It follows from (3.7) that $\nabla(\Upsilon, \Xi \Upsilon) \leq 1$. Therefore, by Theorem 2.4, the sequence $\left\{\Xi^{k} \Upsilon\right\}$ converges to a fixed point $\Phi$ of $\Omega$. This implies

$$
\Phi(x)=\lim _{k \rightarrow \infty} 4^{k} \Upsilon\left(\frac{x}{2^{k}}\right)
$$

and $\Phi(2 x)=4 \Phi(x)$ for all $x \in \mathbf{K}$. Also

$$
\nabla(\Upsilon, \Phi) \leq \frac{1}{1-L} \nabla(\Upsilon, \Xi \Upsilon) \leq \frac{1}{1-L}
$$

Thus (3.5) holds.
Now we use (3.1), (3.6), and (3.14) to prove that $\Phi$ is a quadratic function. Note that

$$
\begin{aligned}
& \widetilde{Z}\left(D_{1} \Phi\left(\tau_{1}, \ldots, \tau_{n}, x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}\right), \zeta\right) \\
& \quad=\lim _{k \rightarrow \infty} \widetilde{Z}\left(D_{1} \Upsilon\left(2^{k} \tau_{1}, \ldots, 2^{k} \tau_{n}, 2^{k} x_{1}, \ldots, 2^{k} x_{n}, 2^{k} y_{1}, \ldots, 2^{k} y_{n}\right), 4^{k \wp} \zeta\right) \\
& \quad \succeq \lim _{k \rightarrow \infty} \mho\left(\left(2^{k} \tau_{1}, \ldots, 2^{k} \tau_{n}, 2^{k} x_{1}, \ldots, 2^{k} x_{n}, 2^{k} y_{1}, \ldots, 2^{k} y_{n}\right), 4^{k \wp} \zeta\right)=1,
\end{aligned}
$$

that is,

$$
\begin{aligned}
\Phi\left(\sum_{i=1}^{n} \tau_{i}-\sum_{i=1}^{n} x_{i}\right)+\Phi\left(\sum_{i=1}^{n} \tau_{i}-\sum_{i=1}^{n} y_{i}\right)= & \frac{1}{2} \Phi\left(\sum_{i=1}^{n} x_{i}-\sum_{i=1}^{n} y_{i}\right) \\
& +2 \Phi\left(\sum_{i=1}^{n} \tau_{i}-\frac{\sum_{i=1}^{n} x_{i}+\sum_{i=1}^{n} y_{i}}{2}\right)
\end{aligned}
$$

for all $\tau_{i}, x_{i}, y_{i} \in \mathbf{K}$ and $i=1, \ldots, n$. From Lemma $2.5, \Phi$ is a quadratic function. Now we show that $\Phi$ is unique. Assume that there exists a quadratic function $T: \mathbf{K} \rightarrow \mathbf{H}$ which satisfies (3.3). Since $\nabla(\Upsilon, T) \leq \frac{L}{1-L}$ and $T$ is a quadratic function, we get $T \in \Omega^{*}$ and $T$ is a fixed point of $\Xi$. Thus $T=\Phi$, because both of them are fixed points and the fixed point of $\Xi$ in $\Omega^{*}$ is unique.
In addition, assume $\Upsilon(t x)$ is continuous with respect to $t \in \mathbb{R}$ for every fixed $x \in \mathbf{K}$, then $\Phi$ is an $\mathbb{R}$-quadratic function. Now, we show that $\Phi$ is a $\mathbb{C}$-quadratic function. Putting $\tau_{1}=\cdots=\tau_{n}=x_{1}=\cdots=x_{n}=x$ and $y_{1}=\cdots=y_{n}=0$ in (3.1), we get

$$
\begin{equation*}
\widetilde{Z}\left(\frac{1}{2} \Upsilon(n b x)-2 b^{2} \Upsilon\left(\frac{n x}{2}\right), \zeta\right) \succeq \mho((\underbrace{x, \ldots, x}_{2 n}, \underbrace{0, \ldots, 0}_{n}), \zeta) \tag{3.16}
\end{equation*}
$$

for all $x \in \mathbf{K}$ and all $b \in C_{1}$. From the definition of $\Phi$, (3.6), and (3.16), we obtain

$$
\begin{aligned}
\tilde{Z}\left(\frac{1}{2} \Phi(n b x)-2 b^{2} \Phi\left(\frac{n x}{2}\right), \zeta\right) & =\lim _{k \rightarrow \infty} \frac{1}{2} \widetilde{Z}\left(\Upsilon\left(2^{k} n b x\right)-2 b^{2} \Upsilon\left(2^{k-1} n x\right), 4^{k \wp} \zeta\right) \\
& \succeq \lim _{k \rightarrow \infty} \mho(\underbrace{\left(2^{k} x, \ldots, 2^{k} x\right.}_{2 n}, \underbrace{0, \ldots, 0}_{n}), 4^{k \wp} \zeta)=1
\end{aligned}
$$

for all $x \in \mathbf{K}$ and all $b \in C_{1}$. So $\Phi(n b x)=4 b^{2} \Phi\left(\frac{n x}{2}\right)$ for all $x \in \mathbf{K}$ and all $b \in C_{1}$. Since $\Phi$ is an $\mathbb{R}$-quadratic function, we get $\Phi(b x)=b^{2} \Phi(x)$ for all $x \in \mathbf{K}$ and all $b \in C_{1} \cup\{0\}$. Now, let $a \in \mathbb{C} \backslash\{0\}$. Since $\Phi$ is an $\mathbb{R}$-quadratic function,

$$
\Phi(a x)=\Phi\left(|a| \cdot \frac{a}{|a|} x\right)=|a|^{2} \Phi\left(\frac{a}{|a|} x\right)=|a|^{2} \cdot\left(\frac{a}{|a|}\right)^{2} \Phi(x)=a^{2} \Phi(x)
$$

for all $x \in \mathbf{K}$ and all $a \in \mathbb{C}$. This proves that $\Phi$ is a $\mathbb{C}$-quadratic function.

In the following, we consider the result in Theorem 3.1. The first result is for the case when $r \in(0,2), \sigma \in \mathbb{R}^{+}$, and $\Upsilon: \mathbf{K} \rightarrow \mathbf{H}$ is a mapping with $\Upsilon(0)=0$ such that

$$
\begin{aligned}
& \widetilde{Z}\left(D_{b} \Upsilon\left(\tau_{1}, \ldots, \tau_{n}, x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}\right), \zeta\right) \\
& \quad \succeq \quad \operatorname{diag}\left(\frac{\frac{\zeta}{\sigma}}{\frac{\zeta}{\sigma}+\sum_{i=1}^{n}\left(\left\|x_{i}\right\|_{\wp}^{r}+\left\|y_{i}\right\|_{\wp}^{r}+\left\|\tau_{i}\right\|_{\wp}^{r}\right)}, e^{-\frac{\sum_{i=1}^{n}\left(\left\|x_{i}\right\|_{\beta}^{r}+\left\|y_{i}\right\|_{\wp}^{r}+\left\|\tau_{i}\right\|_{\wp}^{r}\right)}{\frac{\zeta}{\sigma}}},\right. \\
& \\
& \left.\quad \frac{\frac{\zeta}{\sigma}}{\frac{\zeta}{\sigma}+\sum_{i=1}^{n}\left(\left\|x_{i}\right\|_{\wp}^{r}+\left\|y_{i}\right\|_{\wp}^{r}+\left\|\tau_{i}\right\|_{\wp}^{r}\right)} \circledast e^{-\frac{\left.\sum_{i=1}^{n}\left\|x_{i}\right\|_{\wp}^{r}+\left\|y_{i} i\right\|_{\wp}^{r}+\left\|\tau_{i}\right\|_{\wp}^{r}\right)}{\frac{\zeta}{\sigma}}}\right)
\end{aligned}
$$

for all $\tau_{i}, x_{i}, y_{i} \in \mathbf{K}$ and $b \in C_{1}$, and $i=1, \ldots, n$, then for all $x \in \mathbf{K}$, there exists a unique generalized quadratic mapping of Apollonius-type $\Phi: \mathbf{K} \rightarrow \mathbf{H}$ satisfying

$$
\begin{aligned}
& \widetilde{Z}(\Upsilon(x)-\Phi(x), \zeta)
\end{aligned}
$$

The second result is for the case of $r, s, t \in \mathbb{R}^{+}$such that $\gamma:=r+s+t<2$ and $\delta, \sigma \in \mathbb{R}^{+}$, and when $\Upsilon: \mathbf{K} \rightarrow \mathbf{H}$ with $\Upsilon(0)=0$ is such that

$$
\begin{aligned}
& \widetilde{Z}\left(D_{b} \Upsilon\left(\tau_{1}, \ldots, \tau_{n}, x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}\right), \zeta\right) \\
& \quad \geq \operatorname{diag}\left(\frac{\frac{\zeta}{\sigma}}{\frac{\zeta}{\sigma}+\sum_{i=1}^{n}\left[\left\|x_{i}\right\|_{\wp}^{r}\left\|y_{i}\right\|_{\wp}^{s}\left\|\tau_{i}\right\|_{\wp}^{t}+\left(\left\|x_{i}\right\|_{\wp}^{\gamma}+\left\|y_{i}\right\|_{\wp}^{\gamma}+\left\|\tau_{i}\right\|_{\wp}^{\gamma}\right)\right]},\right. \\
& \\
& e^{-\frac{\left.\sum_{i=1}^{n}\| \| x_{i}\left\|_{\wp}^{r}\right\|_{i}\left\|_{\wp}^{s}\right\| \tau_{i} \|_{\wp}^{t}+\left(\left\|x_{i}\right\|_{\wp}^{\gamma}+\left\|y_{i}\right\|_{\wp}^{\gamma}+\left\|\tau_{i}\right\|_{\wp}^{\gamma}\right)\right]}{\frac{\zeta}{\sigma}}}, \\
& \\
& \frac{\frac{\zeta}{\sigma}}{\frac{\xi}{\sigma}+\sum_{i=1}^{n}\left[\left\|x_{i}\right\|_{\wp}^{r}\left\|y_{i}\right\|_{\wp}^{s}\left\|\tau_{i}\right\|_{\wp}^{t}+\left(\left\|x_{i}\right\|_{\wp}^{\gamma}+\left\|y_{i}\right\|_{\wp}^{\gamma}+\left\|\tau_{i}\right\|_{\wp}^{\gamma}\right)\right]} \\
& \\
& \left.\circledast e^{-\frac{\left.\sum_{i=1}^{n}\left[\left\|x_{i}\right\|_{\wp}^{r}\left\|y_{i}\right\|_{\wp}^{s}\left\|\tau_{i}\right\|_{\wp}^{t}+\| \| x_{i}\left\|_{\wp}^{\gamma}+\right\| y_{i}\left\|_{\wp}^{\gamma}+\right\| \tau_{i} \|_{\wp}^{\gamma}\right)\right]}{\frac{\zeta}{\sigma}}}\right)
\end{aligned}
$$

for all $\tau_{1}, \ldots, \tau_{n}, x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n} \in \mathbf{K}$ and $b \in C_{1}$. Then there exists a unique generalized quadratic mapping of Apollonius-type $\Phi: \mathbf{K} \rightarrow \mathbf{H}$ satisfying

$$
\begin{aligned}
& \widetilde{Z}(\Upsilon(x)-\Phi(x), \zeta)
\end{aligned}
$$

for all $x \in \mathbf{K}$.
Using Theorem 3.1(i), we get that $\Upsilon: \mathbf{K} \rightarrow \mathbf{H}$ is a mapping for which there exists a function $\mho: K^{3 n} \rightarrow[0, \infty)$ satisfying (3.1) and (3.4), and then $\mho(0, \ldots, 0)=0$. It follows from (3.1) that $\Upsilon(0)=0$.

If $r>2, \sigma \in \mathbb{R}^{+}$, and $\Upsilon: \mathbf{K} \rightarrow \mathbf{H}$ is a mapping such that

$$
\begin{aligned}
& \widetilde{Z}\left(D_{b} \Upsilon\left(\tau_{1}, \ldots, \tau_{n}, x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}\right), \zeta\right) \\
& \quad \succeq \operatorname{diag}\left(\frac{\frac{\zeta}{\sigma}}{\frac{\zeta}{\sigma}+\sum_{i=1}^{n}\left(\left\|x_{i}\right\|_{\wp}^{r}+\left\|y_{i}\right\|_{\wp}^{r}+\left\|\tau_{i}\right\|_{\wp}^{r}\right)}, e^{-\frac{\sum_{i=1}^{n}\left(\left\|x_{i}\right\|_{\wp}^{r}+\left\|y_{i}\right\|_{\wp}^{r}+\left\|\tau_{i}\right\|_{\wp}^{r}\right)}{\frac{\zeta}{\sigma}}},\right. \\
& \\
& \left.\frac{\frac{\zeta}{\sigma}}{\frac{\zeta}{\sigma}+\sum_{i=1}^{n}\left(\left\|x_{i}\right\|_{\wp}^{r}+\left\|y_{i}\right\|_{\wp}^{r}+\left\|\tau_{i}\right\|_{\wp}^{r}\right)} \circledast e^{-\frac{\sum_{i=1}^{n}\left(\left\|x_{i}\right\|_{\wp}^{r}+\left\|y_{j}\right\|_{\wp}^{r}+\left\|\tau_{i}\right\|_{\wp}^{r}\right)}{\frac{\zeta}{\sigma}}}\right)
\end{aligned}
$$

for all $\tau_{1}, \ldots, \tau_{n}, x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n} \in \mathbf{K}$ and $b \in C_{1}$, then Theorem 3.1(ii) implies that there exists a unique generalized quadratic mapping of Apollonius-type $\Phi: \mathbf{K} \rightarrow \mathbf{H}$ satisfying

$$
\begin{aligned}
& \widetilde{Z}(\Upsilon(x)-\Phi(x), \zeta)
\end{aligned}
$$

for all $x \in \mathbf{K}$.
Let $r, s, t \in \mathbb{R}^{+}, \gamma:=r+s+t>2, \sigma \in \mathbb{R}^{+}$and let $\Upsilon: \mathbf{K} \rightarrow \mathbf{H}$ be such that

$$
\begin{aligned}
& \widetilde{Z}\left(D_{b} \Upsilon\left(\tau_{1}, \ldots, \tau_{n}, x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}\right), \zeta\right) \\
& \succeq \operatorname{diag}\left(\frac{\frac{\zeta}{\sigma}}{\frac{\zeta}{\sigma}+\sum_{i=1}^{n}\left[\left\|x_{i}\right\|_{\wp}^{r}\left\|y_{i}\right\|_{\wp}^{s}\left\|\tau_{i}\right\|_{\wp}^{t}+\left(\left\|x_{i}\right\|_{\wp}^{\gamma}+\left\|y_{i}\right\|_{\wp}^{\gamma}+\left\|\tau_{i}\right\|_{\wp}^{\gamma}\right)\right]},\right. \\
& e^{-\frac{\left.\sum_{i=1}^{n}\left[\left\|x_{i}\right\|_{\beta}^{r}\left\|y_{i}\right\|_{\beta}^{s}\left\|\tau_{i}\right\|^{t}{ }^{t}+\| \| x_{i}\left\|_{\beta}^{\gamma}+\right\| y_{j}\left\|_{\phi}^{\gamma}+\right\| \tau_{i} \|_{\beta}^{\gamma}\right)\right]}{\sigma}}, \\
& \frac{\frac{\zeta}{\sigma}}{\frac{\zeta}{\sigma}+\sum_{i=1}^{n}\left[\left\|x_{i}\right\|_{\wp}^{r}\left\|y_{i}\right\|_{\wp}^{s}\left\|\tau_{i}\right\|_{\wp}^{t}+\left(\left\|x_{i}\right\|_{\wp}^{\gamma}+\left\|y_{i}\right\|_{\wp}^{\gamma}+\left\|\tau_{i}\right\|_{\wp}^{\gamma}\right)\right]} \\
& \left.\circledast e^{-\frac{\sum_{i=1}^{n}\left[\left\|x_{i}\right\|_{\rho}^{r}\left\|y_{i}\right\|_{\rho}^{\|}\left\|\tau_{i}\right\|_{\rho}^{\tau}+\left(\left\|x_{i}\right\|_{\infty}^{\gamma}+\left\|y_{i}\right\|_{\rho}^{\gamma}+\left\|\tau_{i}\right\|_{\rho}^{\gamma}\right)\right]}{\frac{\zeta}{\sigma}}}\right)
\end{aligned}
$$

for all $\tau_{1}, \ldots, \tau_{n}, x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n} \in \mathbf{K}$ and $b \in C_{1}$. Theorem 3.1(ii) implies that there exists a unique generalized quadratic mapping of Apollonius-type $\Phi: \mathbf{K} \rightarrow \mathbf{H}$ satisfying

$$
\begin{aligned}
& \widetilde{Z}(\Upsilon(x)-\Phi(x), \zeta) \\
& \quad \succeq \operatorname{diag}\left(\frac{\frac{n^{\wp \gamma}\left(2^{\wp \gamma}-4^{\wp}\right) \zeta}{2^{\wp(\gamma+1) \cdot 2 n \sigma}}}{\frac{n^{\wp \gamma}\left(2^{\wp \gamma}-4^{\wp}\right) \zeta}{2^{\wp(\gamma+1) \cdot 2 n \sigma}}+\|x\|_{\wp}^{\gamma}}, e^{-\frac{\|x\|_{\wp}^{\gamma}}{\frac{n^{\wp \gamma}\left(2^{\wp \gamma}-4 \wp\right) \zeta}{2^{\wp(\gamma+1) \cdot 2 n \sigma}}}}, \frac{\frac{n^{\wp \gamma}\left(2^{\wp \gamma}-4^{\wp}\right) \zeta}{2^{\wp(\gamma+1) \cdot 2 n \sigma}}}{\frac{n^{\wp \gamma}\left(2^{\wp \gamma}-4^{\wp}\right) \zeta}{2^{\wp(\gamma+1) \cdot 2 n \sigma}+\|x\|_{\wp}^{\gamma}} \circledast e^{-\frac{\|x\|_{\wp}^{\gamma}}{\left.n^{\wp \gamma} 2^{\wp \gamma}-4 \wp\right) \zeta}} 2^{\wp(\gamma+1) \cdot 2 n \sigma}}\right)
\end{aligned}
$$

for all $x \in \mathbf{K}$.
Also, in all the above results, for the continuous mapping $\Upsilon(t x)$ for each fixed $x \in \mathbf{K}$ and $t \in \mathbb{C}, \Phi$ is a $\mathbb{C}$-quadratic function.

## 4 Conclusion

Using the concept of GZ-N, we measured the quality and certainty of approximation of the solution of an Apollonius-type quadratic functional equation. Our method and technique can be applied to a wide range of functional equations to measure the quality and certainty of approximation of the solution.

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## Author contributions

A.A., methodology, writing-original draft preparation. R.S., supervision, and project administration. T.A., project administration. D.O., writing-original draft preparation, editing, and methodology. All authors read and approved the final manuscript.

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