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Generalized fixed points for fuzzy and nonfuzzy mappings in strong b-metric spaces

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Abstract

The main purpose of this research article is to generalize Kannan-type fixed-point (FP) theorems for single-valued mappings and Chatterjea-type FP result for fuzzy mappings (FMs) in the context of complete strong b-metric spaces (MSs). Moreover, fuzzy FPs are established for Suzuki-type fuzzy contraction in the setting of complete strong b-MSs. The conclusions are supported by nontrivial examples to enhance the validity of the results obtained in this study. In addition, previous findings have been made as corollaries from the relevant literature. The numerous implications that this technique has across the literature improve and integrate our findings. Applications of some of the results obtained are also incorporated.

Mathematics Subject Classification: 47H09; 47H10

Keywords: Fuzzy mappings; Strong *b*-metric space; Multivalued mappings; Fixed point

1 Introduction

In the past few decades, a noteworthy interest in FP theory has been directed to interchanging recent metric FP results from the usual MSs to some generalized MSs, like quasi-MSs usually called b-MSs introduced by Bakhtin [3] and Czerwik [8]. The class of strong b-MSs lying between the class of b-MSs and the class of MSs was introduced by Kirk and Shahzad [16]. As compared with b-MSs, strong b-MSs have the advantage that open balls are open in the induced topology and, hence, they have given many properties that are similar to the properties of classic MSs. In 1965, Zadeh [32] introduced the notion of fuzzy logic. In the theory of traditional logic, some element does or does not belong to the set, but in fuzzy logic a number from the interval [0, 1] expresses the affiliation of the element to the set. Zadeh started to research the theory of fuzzy sets (FSs) in order to deal with the issue of indeterminacy, which is a real problem that is fundamentally characterized by uncertainty. The concept of the FM was given by Heilpern [13] and for fuzzy contraction mapping in a metric linear space, a theorem was proved by him that is a fuzzy generalization of Banach's contraction principle. Many authors such as Banach [4], Benavides et al. [5], Ciric [7], Kirk [17], Meir and Keeler [18], Nadler [23], Subrahmanyam [26], and Suzuki [27, 28] proved theorems in which every contraction mapping was a continuous function.

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Then, in 1968, Kannan [15] was the first who introduced the contraction mapping that was not necessarily continuous.

Fuzzy common FPs for generalized mappings were obtained by Abbas et al. [1], fuzzy FPs and common FPs were established by Azam et al. [2] and fuzzy FPs for FMs were constructed by Estruch and Vidal [10] and Frigon and O'Regan [11]. Işık et al. [14] and Mohammadi et al. [19–22] have established valuable fixed-point and common fixed-point results using various contractive conditions for fuzzy and nonfuzzy mappings in the generalizations of metric spaces.

Also, some other authors [24, 25, 30, 31] worked on the existence of FPs and common FPs of FMs satisfying a contractive-type condition. Fuzzy theory has been applied in several fields, for example quantum physics, nonlinear dynamical systems, population dynamics, computer programming, fuzzy stability problems, statistical convergence, functional equation, approximation theory, nonlinear equations, and many others.

Theorem 1.1 [15] *Suppose (S, d) is a complete MS, and $\theta : S \to S$ is a mapping. If there exists $x \epsilon [0, \frac{1}{2})$, satisfying*

$$d(\theta s, \theta u) \leq x\{d(s, \theta s) + d(u, \theta u)\},$$

for all $s, u \epsilon S$, then θ has a unique FP $r \epsilon S$ and for any $s \epsilon S$ the sequence of iterates $\{\theta^n s\}$ converges to r.

After Kannan, Chatterjea [6] also proved a theorem with contraction mapping not necessarily continuous.

Theorem 1.2 [15] *Suppose (S, d) is a complete MS, and $\theta : S \to S$ is a mapping. If there exists $x \epsilon [0, \frac{1}{2})$, satisfying*

$$d(\theta s, \theta u) \leq x\{d(s, \theta u) + d(u, \theta s)\},$$

for all $s, u \epsilon S$, then θ has a unique FP $r \epsilon S$ and for any $s \epsilon S$ the sequence of iterates $\{\theta^n s\}$ converges to r.

Further, Gornicki [12] introduced various extensions of the Kannan FP theorem. He proved the following results:

Assume ζ denotes the class of functions that satisfy the condition $\zeta = \{\phi : (0, \infty) \to [0, \frac{1}{2}) : \phi(t_n) \to \frac{1}{2}$ implies $t_n \to 0$ as $n \to \infty\}$.

Theorem 1.3 [12] *Suppose (S, d) is a complete MS and $\theta : S \to S$ is a mapping. Also, assume there exists $\phi \epsilon \zeta$ such that for each $s, u \epsilon S$ with $s \neq u$,*

$$d(\theta s, \theta u) \leq \phi(d(s, u))\{d(s, \theta s) + d(u, \theta u)\}.$$

Then, θ has a unique FP $r \epsilon S$ and for any $s \epsilon S$ the sequence of iterates $\{\theta^n s\}$ converges to r.

In 2021, Doan [9] extended the results in [12] for a class of contractive mappings in strong b-MSs. He proved a new version of FP theorems for single-valued and multivalued mappings by combining the results in [15] and [29].

Theorem 1.4 [9] *Suppose (S,ϖ,σ) is a complete strong b-MS with $\sigma \geq 1$ and $T: S \to S$ is a mapping. Assume there exists $\phi \epsilon \zeta$ such that for each $s, u \epsilon S$ with $s \neq u$,*

$$\varpi(\theta s,\theta u) \leq \phi\big(\varpi(s,u)\big)\big\{\varpi(s,\theta s) + \varpi(u,\theta u)\big\}.$$

Then, θ has a unique FP $r \epsilon S$ and for any $s \epsilon S$ the sequence of iterates $\{\theta^n s\}$ converges to r.

In this article, we obtained the idea from [9] and extended it to [6, 29]. We prove FP theorems for single-valued FMs in strong b-MS by combining the results in [6] and [9].

2 Basic concepts

We recall some results and concepts, which are necessary to understand our results.

Definition 2.1 [16] Suppose S is a nonempty set and $\sigma \geq 1$. A mapping $\varpi : S \times S \to [0,+\infty)$ is called a strong b-metric on S if

$sb_1)$ $\varpi(s,u) \geq 0$, $\forall s,u \in S$;
$sb_2)$ $\varpi(s,u) = 0$ iff $s = u$;
$sb_3)$ $\varpi(s,u) = \varpi(u,s)$ $\forall s,u \in S$;
$sb_4)$ $\varpi(s,u) \leq \varpi(s,t) + \sigma\varpi(t,u)$, $\forall s,u,t \in S$.

Then, (S,ϖ,σ) is called strong b-MS.

Theorem 2.2 [29] *Suppose (S,d) is a complete MS and $T: S \to S$ is a mapping. Define a nonincreasing function $\psi : [0,1) \to (\frac{1}{2},1]$ by*

$$\psi(x) = \begin{cases} 1, & 0 \leq x < \frac{\sqrt{5}-1}{2}; \\ (1-x)x^{-2}, & \frac{\sqrt{5}-1}{2} \leq x < 2^{-\frac{1}{2}}; \\ (1+x)x^{-1}, & 2^{-\frac{1}{2}} \leq x < 1. \end{cases}$$

Assume that there exists $x \in [0,1)$ such that $\psi(x)d(s,\theta s) \leq d(s,u)$ implies $d(\theta s,\theta u) \leq xd(s,u)$ for all $s,u \in S$. Then, θ has a unique FP $r \in S$ and for any $s \in S$, the sequence of iterates $\{\theta^n s\}$ converges to r.

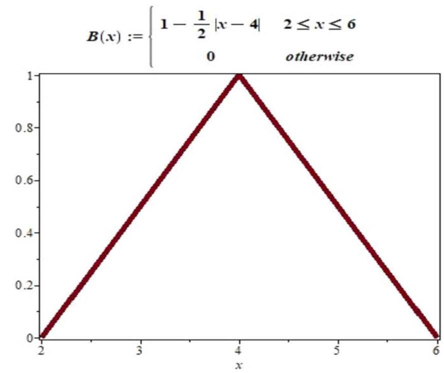
Definition 2.3 [16] Suppose (S,ϖ,σ) is a strong b-MS, $\{s_n\}$ is a sequence in S, and $s \in S$. Then,

(i) If $\lim_{n\to\infty} \varpi(s_n,s) = 0$, then $\{s_n\}$ is called convergent to s. This means $\lim_{n\to\infty} s_n = s$ or $s_n \to s$ as $n \to \infty$.
(ii) If $\lim_{n,m\to\infty} \varpi(s_n,s_m) = 0$, then $\{s_n\}$ is called a Cauchy sequence (CS) in S.
(iii) If every CS in S converges in S then (S,ϖ,σ) is complete.

Proposition 2.4 [16] *Suppose (S,ϖ,σ) is a strong b-MS and $\{s_n\}$ is a sequence in S. Then,*

(i) *If $\{s_n\}$ converges to $s \in S$ and $u \in S$, then $s = u$.*
(ii) *If $\lim_{n\to\infty} s_n = s \in S$ and $\lim_{n\to\infty} u_n = u \in S$, then $\lim_{n,m\to\infty} \varpi(s_n,u_n) = \varpi(s,u)$.*

Figure 1 Graph of FS $B(x)$



Proposition 2.5 [16] *Suppose* $\{s_n\}$ *is a sequence in strong b-MS* (S, ϖ, σ) *and*

$$\sum_{n=1}^{\infty} \varpi(s_n, s_{n+1}) < +\infty.$$

Then, $\{s_n\}$ *is a CS in* S.

Definition 2.6 [32] Suppose S is any arbitrary set and a function $A : S \to [0,1]$ is a FS. The functional value $A(s)$ is called the grade of membership of s in A. The collection of all FSs in S is denoted by $F(S)$.

The α-cut of A is denoted by A_α and is defined as follows:

$$A_\alpha = \big\{s; A(s) \ge \alpha \text{ if } \alpha \in (0,1]\big\}.$$

Example 2.7 Consider a FS B defined by the following membership function:

$$B(x) = \begin{cases} 1 - \frac{|x-4|}{2}, & \text{when } 2 \le x \le 6; \\ 0, & \text{otherwise.} \end{cases}$$

FS B can be seen in Fig. 1.

Here, for any $\alpha \in (0,1]$, the α-cut of B is

$$B_\alpha = \big[2(1+\alpha), 2(3-\alpha)\big].$$

Definition 2.8 [13] Suppose (S,d) is any MS and P is an arbitrary set. θ is called FM if $\theta : W \to F(S)$ is a function, i.e., $\theta(p) \in F(S)$ for each $p \in P$.

Example 2.9 Let $P = [-9,9]$ and $S = [-4,4]$. Define $T_1 : P \longrightarrow F(S)$ by

$$T_1(x)(y) = \frac{x^2 + y^2}{100}.$$

Then, T_1 is a FM. Note that $T_1(x)(y) \in [0,1]$, for all $x \in P$ and $y \in S$. The graphical representation $T_1(x)(y)$ showing the possible membership values of y in $T_1(x)$ is given in Fig. 2.

Figure 2 Graph of fuzzy mapping T_1

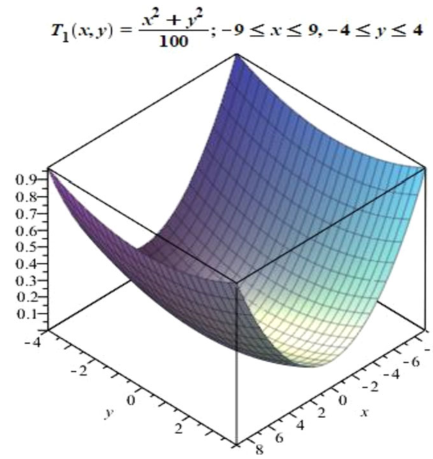
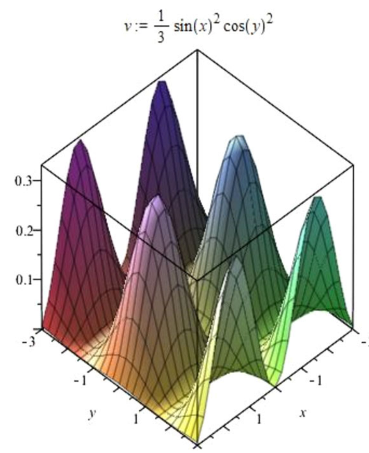


Figure 3 Graph of fuzzy mapping T_2



Example 2.10 Let $S = [-3, 3]$. Define $T_2 : S \longrightarrow F(S)$ by

$$T_2(x)(y) = \frac{\sin^2 x \cos^2 y}{3}.$$

Then, T_2 is a fuzzy mapping. Note that $T_2(x)(y) \in [0, 1]$, for all $x, y \in S$. The graphical representation $v = T_2(x)(y)$ showing the possible membership values of y in $T_2(x)$ is shown in Figure 3.

Definition 2.11 Suppose (S, d) is a MS and $CB(S)$ denotes the collection of all nonempty, closed, and bounded subsets of S. Consider a map $H : CB(S) \times CB(S) \to \mathbb{R}$. For $C, E \in CB(S)$ define

$$H(C, E) = \max\left\{\sup_{c\in C} d(c, E), \sup_{e\in E} d(e, C)\right\},$$

where $d(c, E) = \{\inf d(c, e) : e \in E\}$ is the distance of c to meet E. This H is a metric on $CB(S)$ is called the Hausdorff metric induced by the metric d.

Definition 2.12 Let (S,ϖ,K) be a strong b-MS. Let $\theta : S \to F(S)$ be a FM on S:

$$H\big([\theta s]_{\alpha_{\theta s}},[\theta u]_{\alpha_{\theta u}}\big) = \max\Big\{\sup_{s\in[\theta s]_{\alpha_{\theta s}}} d\big(s,[\theta u]_{\alpha_{\theta u}}\big), \sup_{u\in[\theta u]_{\alpha_{\theta u}}} d\big([\theta s]_{\alpha_{\theta s}},u\big)\Big\},$$

where H is the Hausdroff metric on $F(S)$ induced by ϖ, $[\theta s]_{\alpha_{\theta s}},[\theta u]_{\alpha_{\theta u}} \in F(S)$ and $d(s,[Lu]_{\alpha_{Lu}}) = \inf_{u\in[Lu]_{\alpha_{Lu}}} \varpi(s,u)$.

Lemma 2.13 [2] *Suppose (S,d,b) is a b-MS. Then, for $C,E \in CB(S)$,*
- (i) $d(c,E) \le H(C,E)$, $c \in C$;
- (ii) *For $\varepsilon > 0$ and $c \in C$, $\exists e \in E$ such that*

$$d(c,e) \le H(C,E) + \varepsilon.$$

Theorem 2.14 *Suppose (S,d) is a complete MS. If $\theta : S \to F(S)$ is a continuous FM such that $[\theta s]_{\alpha_{\theta s}}$ and $[\theta u]_{\alpha_{\theta u}}$ are closed and bounded subsets of S satisfying*

$$H\big([\theta s]_{\alpha_{\theta s}},[\theta u]_{\alpha_{\theta u}}\big) \le x\big\{d\big(s,[\theta s]_{\alpha_{\theta s}}\big) + d\big(u,[\theta u]_{\alpha_{\theta u}}\big)\big\},$$

$\forall s,u \in S$, where $0 \le x < \frac{1}{2}$. Then, θ has at least one FP.

3 Main results

In this section, we establish our main results.

Theorem 3.1 *Suppose (S,ϖ,σ) is a complete strong b-MS and $\theta : S \to S$ is a mapping. Suppose there exists $\phi \in \zeta$ such that for each $s,u \in S$ with $s \ne u$,*

$$\varpi(\theta s,\theta u) \le \phi\big(\varpi(s,u)\big)\big\{\varpi(s,\theta u) + \varpi(u,\theta s)\big\}.$$

Then, θ has a unique FP $r \in S$ and for any $s \in S$ the sequence of iterates $\{\theta^n s\}$ converges to r.

Proof Fix $s_0 \in S$ and define a sequence $\{s_n\}$ in S by $s_{n+1} = \theta s_n$ for all integers $n \ge 0$. Assume that there exists n such that $s_{n+1} = s_n$, then s_n is a FP of θ. Therefore, suppose that $s_{n+1} \ne s_n$ for all $n \ge 0$. Set $\varpi_n = \varpi(s_n,s_{n+1})$ for all $n \ge 0$. By hypothesis, we have

$$\begin{aligned}
\varpi_{n+1} &= \varpi(s_{n+1},s_{n+2}) \\
&= \varpi(\theta s_n,\theta s_{n+1}) \\
&\le \phi\big(\varpi(s_n,s_{n+1})\big)\big\{\varpi(s_n,\theta s_{n+1}) + \varpi(s_{n+1},\theta s_n)\big\} \\
&< \frac{1}{2k}\big\{\varpi(s_n,\theta s_{n+1}) + \varpi(s_{n+1},\theta s_n)\big\} \\
&= \frac{1}{2k}\big\{\varpi(s_n,s_{n+2}) + \varpi(s_{n+1},s_{n+1})\big\} \\
&\le \frac{1}{2k}\big\{\varpi(s_n,s_{n+1}) + \sigma\varpi(s_{n+1},s_{n+2})\big\} \\
&= \frac{1}{2k}\big\{\varpi(s_n,\theta s_n) + \sigma\varpi(s_{n+1},\theta s_{n+1})\big\} \\
&= \frac{1}{2k}\{\varpi_n + \sigma\varpi_{n+1}\}.
\end{aligned}$$

Hence, $\varpi_{n+1} < \varpi_n$ for all $n \geq 0$ and so $\{\varpi_n\}$ is monotonic decreasing and bounded below, so there exists $\eta \geq 0$ such that

$$\lim_{n\to\infty} \varpi_n = \eta.$$

Let $\eta > 0$. Then, by hypothesis,

$$\varpi(s_{n+1}, s_{n+2}) \leq \phi\big(\varpi(s_n, s_{n+1})\big)\big\{\varpi(s_n, s_{n+1}) + \sigma\varpi(s_{n+1}, s_{n+2})\big\}, \quad \forall n \geq 0,$$

which deduces

$$\varpi_{n+1} \leq \phi(\varpi_n)\{\varpi_n + \sigma\varpi_{n+1}\}.$$

This implies that $\frac{\varpi_{n+1}}{\varpi_n + \sigma\varpi_{n+1}} \leq \phi(\varpi_n)$ for all $n \geq 0$.

By letting $n \to \infty$, we obtain $\lim_{n\to\infty} \phi(\varpi_n) \leq \frac{1}{2k}$, and since $\phi \in \zeta$ this in turn gives $\eta = 0$. Hence, $\lim_{n\to\infty} \varpi_n = 0$.

On the other hand, for positive integers m, n with $m \neq n$ we obtain

$$\varpi(s_{n+1}, s_{m+1}) \leq \phi\big(\varpi(s_n, s_m)\big)\big\{\varpi(s_n, s_{n+1}) + \sigma\varpi(s_m, s_{m+1})\big\} < \frac{1}{2k}\{\varpi_n + \sigma\varpi_m\} \to 0,$$

as $n, m \to \infty$, so $\{s_n\}$ is a CS in S. By the completeness of S, there is $r \in S$ such that $\lim_{n\to\infty} s_n = r$, since

$$\begin{aligned}
\varpi(\theta r, r) &\leq \varpi(\theta s_n, \theta r) + \sigma\varpi(\theta s_n, r) \\
&\leq \phi\big(\varpi(s_n, r)\big)\big\{\varpi(s_n, \theta r) + \varpi(r, \theta s_n)\big\} + \sigma\varpi(s_{n+1}, r) \\
\varpi(\theta r, r) &\leq \phi\big(\varpi(s_n, r)\big)\big\{\varpi(s_n, \theta r) + \varpi(r, s_{n+1})\big\} + \sigma\varpi(s_{n+1}, r)
\end{aligned}$$

implies $\varpi(\theta r, r) \to 0$ as $n \to \infty$.

Hence, $\theta r = r$. Assume $\bar{r}$ is another FP of θ. By hypothesis,

$$\begin{aligned}
\varpi(r, \bar{r}) &= \varpi(\theta r, \theta\bar{r}) \\
&\leq \phi\big(\varpi(r, \bar{r})\big)\big\{\varpi(r, \theta\bar{r}) + \varpi(\bar{r}, \theta r)\big\} \\
&= \phi\big(\varpi(r, \bar{r})\big)\big\{\varpi(r, \bar{r}) + \varpi(\bar{r}, r)\big\} \\
&= 2\phi\big(\varpi(r, \bar{r})\big)\big\{\varpi(r, \bar{r})\big\}
\end{aligned}$$

and hence

$$\big(1 - 2\phi\big(\varpi(r, \bar{r})\big)\big)\varpi(r, \bar{r}) \leq 0.$$

Since $(1 - 2\phi(\varpi(r, \bar{r}))) \neq 0$, then $\varpi(r, \bar{r}) = 0$ and so $r = \bar{r}$. Hence, θ has a unique FP $r \in S$. □

Example 3.2 Suppose $S = \{1, 2, 3\}$ and let $\varpi : S \times S \to [0, +\infty)$ by

$$\varpi(1,1) = \varpi(2,2) = \varpi(3,3) = 0,$$

$$\varpi(1,2) = \varpi(2,1) = \frac{1}{5},$$

$$\varpi(1,3) = \varpi(3,1) = 7,$$
$$\varpi(2,3) = \varpi(3,2) = 2.$$

Then, $(S,\varpi,\sigma = 2)$ is a strong b-MS, but it is not MS, because $7 = \varpi(3,1) > \varpi(3,2) + \varpi(2,1) = \frac{11}{5}$. Let $\theta : S \to S$ by $\theta 1 = 1$, $\theta 2 = 1$, $\theta 3 = 2$, and the function $\phi \in \zeta$ given by $\phi(t) = t\sin(t)$, $t > 0$ and $\phi(0) \in [0, \frac{1}{2})$. Then,

$$\varpi(\theta 1,\theta 2) = \varpi(1,1) = 0 < \frac{1}{25}\sin\left(\frac{1}{5}\right) = \phi\big(\varpi(1,2)\big)\big\{\varpi(1,\theta 2) + \varpi(2,\theta 1)\big\},$$
$$\varpi(\theta 2,\theta 3) = \varpi(1,2) = \frac{1}{5} < 14\sin(2) = \phi\big(\varpi(2,3)\big)\big\{\varpi(2,\theta 3) + \varpi(3,\theta 2)\big\},$$
$$\varpi(\theta 3,\theta 1) = \varpi(2,1) = \frac{1}{5} < \frac{252}{5}\sin(7) = \phi\big(\varpi(3,1)\big)\big\{\varpi(3,\theta 1) + \varpi(1,\theta 3)\big\}.$$

Therefore, θ satisfies all the conditions of Theorem 3.1. Hence, 0 is a fixed point of θ.

If we take $\sigma = 1$ in Theorem 3.1, the strong b-MS is a usual MS, then we obtain the following corollary.

Corollary 3.3 *Suppose (S,d) is a complete MS and $\theta : S \to S$ is a mapping. Assume there exists $\phi \in \zeta$ such that for each $s,u \in S$ with $s \neq u$,*

$$\varpi(\theta s,\theta u) \le \phi\big(\varpi(s,u)\big)\big\{\varpi(s,\theta u) + \varpi(u,\theta s)\big\}.$$

Then, θ has a unique FP $r \in S$ and for any $s \in S$ the sequence of iterates $\{\theta^n s\}$ converges to r.

Theorem 3.4 *Suppose (S,ϖ,σ) is a complete strong b-MS with $\sigma \ge 1$ and $\theta : S \to F(S)$ is a fuzzy map. Suppose $[\theta s]_{\alpha_{\theta s}}$ and $[\theta u]_{\alpha_{\theta u}}$ are closed and bounded subsets of S such that*

$$H\big([\theta s]_{\alpha_{\theta s}}, [\theta u]_{\alpha_{\theta u}}\big) \le \beta\varpi(s,u),$$

for all $s,u \in S$ and $\beta \in [0,1)$. Then, there exists r such that $r \in [\theta r]_{\alpha_{\theta r}}$.

Proof Let $s_1 \in [\theta s_0]_{\alpha_{\theta s_0}}$, with $[\theta s_1]_{\alpha_{\theta s_1}} \neq \phi$, where $s_0 \in S$, $[\theta s_0]_{\alpha_{\theta s_0}}$ are closed and bounded subsets of S. By using Lemma 2.13, $\exists s_2 \in [\theta s_1]_{\alpha_{\theta s_1}}$ such that

$$\varpi(s_1,s_2) \le H\big([\theta s_0]_{\alpha_{\theta s_0}}, [\theta s_1]_{\alpha_{\theta s_1}}\big) + \beta. \tag{3.1}$$

Now, $\exists\, s_3 \in [\theta s_2]_{\alpha_{\theta s_2}}$ for $[\theta s_2]_{\alpha_{\theta s_2}} \neq \phi$ are closed and bounded subsets of S such that

$$\varpi(s_2,s_3) \le H\big([\theta s_1]_{\alpha_{\theta s_1}}, [\theta s_2]_{\alpha_{\theta s_2}}\big) + \beta^2. \tag{3.2}$$

Given the contracting condition implies:

$$\begin{aligned}
\varpi(s_2,s_3) &\le \beta\varpi(\varpi(s_1,s_2) + \beta^2,\\
\varpi(s_3,s_4) &\le H\varpi\big([\theta s_2]_{\alpha_{\theta s_2}}, [\theta s_3]_{\alpha_{\theta s_3}}\big) + \beta^3,\\
&\le \beta\varpi(s_2,s_3) + \beta^3.
\end{aligned}$$

By utilizing (3.2), we obtain

$$\begin{aligned}
\varpi(s_3,s_4) &\leq \beta\big[\beta\varpi(s_1,s_2)+\beta^2\big]+\beta^3,\\
&\leq \beta^2\varpi(s_1,s_2)+2\beta^3,\\
&\leq \beta^2\big[H([\theta s_0]_{\alpha_{\theta s_0}},[\theta s_2]_{\alpha_{\theta s_2}}\big]+2\beta^3,\\
&\leq \beta^2\big[\beta\varpi(s_0,s_1)+\beta\big]+2\beta^3,\\
&\leq \beta^3\varpi(s_0,s_1)+\beta^3+2\beta^3,\\
&\leq \beta^3\varpi(s_0,s_1)+3\beta^3.
\end{aligned}$$

Generally,

$$\varpi(s_n,s_{n+1})=\beta^n\varpi(s_0,s_1)+n\beta^n.$$

For convenience, we set $\varpi(s_n,s_{n+1})=\varpi_n$, so it is possible to write the above result as

$$\varpi_n\leq\beta^n\varpi_0+n\beta^n. \tag{3.3}$$

Consider positive integers m, n. Without loss of generality we suppose that $m\geq n$. Now,

$$\varpi(s_n,s_m)\leq\varpi(s_n,s_{n+1})+\sigma\varpi(s_{n+1},s_{n+2})+\sigma^2\varpi(s_{n+2},s_{n+3})+\cdots+\sigma^{m-n-1}\varpi(s_{m-1},s_m).$$

By utilizing (3.3), we obtain

$$\begin{aligned}
\varpi(s_n,s_m) &\leq \varpi(s_n,s_{n+1})+\sigma\varpi(s_{n+1},s_{n+2})+\sigma^2\varpi(s_{n+2},s_{n+3})\\
&\quad+\cdots+\sigma^{m-n-1}\beta^{m-1}\varpi(s_{m-1},s_m)+\sigma^{m-n-1}(m-1)\beta^{m-1}\\
&\leq \beta^n\varpi_0\big(1+\beta\sigma+(\beta\sigma)^2+(\beta\sigma)^3+\cdots+\sigma^{m-n-1}\beta^{m-n-1}\big)+\sum_{i=n}^{m-i}i\sigma^{i-n}\beta^i
\end{aligned}$$

and hence

$$\varpi(s_n,s_m)\leq\beta\varpi_0\left[\frac{1+(\sigma\beta)^{m-n-1}}{1-\sigma\beta}\right]+\sum_{i=n}^{m-i}i\sigma^{i-n}\beta^i.$$

In the limiting case, $m,n\to\infty$,

$$\varpi(s_n,s_m)=0.$$

This implies that $\{s_n\}$ is a CS in S. The completeness of S implies that there exists $r\in S$ such that $s_n\to r$. We will now demonstrate that r is a FP of θ. By utilizing Lemma 2.13,

$$\begin{aligned}
\varpi\big(r,[\theta r]_{\alpha_{\theta r}}\big) &\leq \varpi(r,s_n)+\sigma\varpi\big(s_n,[\theta r]_{\alpha_{\theta r}}\big)\\
&\leq \varpi(r,s_n)+\sigma H\big([\theta s_{n-1}]_{\alpha_{\theta s_{n-1}}},[\theta r]_{\alpha_{\theta r}}\big)\\
&\leq \varpi(r,s_n)+\sigma\beta\varpi(s_{n-1},r),
\end{aligned}$$

when $n\to\infty$, $\varpi(r,[\theta r]_{\alpha_{\theta r}})\leq 0$. Thus, $r\in[\theta r]_{\alpha_{\theta r}}$ and, hence, r is a FP of θ. □

Example 3.5 Consider a set $S = \{3,4,5\}$. A mapping $\varpi : S \times S \to [0,\infty)$ defined by

$$\varpi(4,3) = 2 = \varpi(3,4),$$
$$\varpi(3,5) = 3 = \varpi(5,3),$$
$$\varpi(5,4) = 6 = \varpi(4,5),$$
$$\varpi(5,5) = \varpi(3,3) = \varpi(4,4) = 0$$

is a strong b-metric. The triplet $(S,\varpi,\sigma = 5)$ is a complete strong b-MS.

For any $\alpha \in (0,1]$, define a mapping $\theta : S \to F(S)$ and $\theta(s) : S \to [0,1]$ by

$$\theta(3)(t) = \begin{cases} \frac{\alpha}{4}, & t = 3; \\ \frac{\alpha}{5}, & t = 4, \\ \alpha, & t = 5; \end{cases}$$

$$\theta(4)(t) = \begin{cases} \frac{\alpha}{2}, & t = 3,4; \\ \alpha, & t = 5, \end{cases}$$

$$\theta(5)(t) = \begin{cases} \alpha, & t = 5; \\ \frac{\alpha}{3}, & t = 3,4 \end{cases}$$

and

$$[\theta 3]_{\alpha_{\theta 3}} = \big\{t \in S : \theta(3)(t) \geq \alpha\big\} = \{5\},$$
$$[\theta 5]_{\alpha_{\theta 5}} = \big\{t \in S : \theta(5)(t) \geq \alpha\big\} = \{5\},$$
$$[\theta 4]_{\alpha_{\theta 4}} = \big\{t \in S : \theta(4)(t) \geq \alpha\big\} = \{5\}.$$

Then,

$$H\big([\theta 3]_{\alpha_{\theta 3}}, [\theta 4]_{\alpha_{\theta 4}}\big) = H\big(\{5\},\{5\}\big) = 0,$$
$$H\big([\theta 4]_{\alpha_{\theta 4}}, [\theta 5_{\alpha_{\theta 5}}\big) = H\big(\{5\},\{5\}\big) = 0,$$
$$H\big([\theta 3]_{\alpha_{\theta 3}}, [\theta 5]_{\alpha_{\theta 5}}\big) = H\big(\{5\},\{5\}\big) = 0.$$

We also have,

$$0 = H\big([\theta 3]_{\alpha_{\theta 3}}, [\theta 4]_{\alpha_{\theta 4}}\big) \leq \beta\varpi(3,4) \leq 2\beta,$$
$$0 = H\big([\theta 3]_{\alpha_{\theta 3}}, [\theta 5]_{\alpha_{\theta 5}}\big) \leq \beta\varpi(3,5) \leq 3\beta.$$
$$0 = H\big([\theta 5]_{\alpha_{\theta 5}}, [\theta 4]_{\alpha_{\theta 4}}\big) \leq \beta\varpi(5,4) \leq 6\beta.$$

Thus, all hypotheses of Theorem 3.4 are satisfied and $r = 5$ is a unique FP of θ.

Corollary 3.6 *Suppose (S,ϖ) is a complete MS with and $\theta : S \to F(S)$ is a fuzzy map. Suppose $[\theta s]_{\alpha_{\theta s}}$ and $[\theta u]_{\alpha_{\theta u}}$ are closed and bounded subsets of S defined as*

$$H\big([\theta s]_{\alpha_{\theta s}}, [\theta u]_{\alpha_{\theta u}}\big) \leq \beta\varpi(s,u),$$

for all $s,u \in S$ and $\beta \in [0,1)$. Then, there exist r such that $r \in [\theta r]_{\alpha_{\theta r}}$.

Theorem 3.7 *Suppose (S,ϖ,σ) is a complete strong b-MS with $\sigma \geq 1$ and $\theta : S \to F(S)$ is a fuzzy map. Suppose $[\theta s]_{\alpha_{\theta s}}$ and $[\theta u]_{\alpha_{\theta u}}$ are closed and bounded subsets of S defined by*

$$H\big([\theta s]_{\alpha_{\theta s}},[\theta u]_{\alpha_{\theta u}}\big) \leq \beta\big[\varpi\big(s,[\theta u]_{\alpha_{\theta u}}\big)+\varpi\big(u,[\theta s]_{\alpha_{\theta s}}\big)\big], \tag{3.4}$$

for all $s,u \in S$ and $\beta \in [0,1)$. Then, there exist r in S such that $r \in [\theta r]_{\alpha_{\theta r}}$.

Proof Suppose $\{s_n : n \in \mathbb{N}\}$ is a sequence such that $s_{n+1} \in [\theta s_n]_{\alpha_{\theta s_n}}$. By using Lemma 2.13, for each $s_1 \in [\theta s_0]_{\alpha_{\theta s_0}}$, $\exists s_2 \in [\theta s_1]_{\alpha_{\theta s_1}}$ such that

$$\begin{aligned}
\varpi(s_1,s_2) &\leq H\big([\theta s_0]_{\alpha_{\theta s_0}},[\theta s_1]_{\alpha_{\theta s_1}}\big)+\beta,\\
&\leq \beta\big[\varpi\big(s_0,[\theta s_1]_{\alpha_{\theta s_1}}\big)+\varpi\big(s_1,[\theta s_0]_{\alpha_{\theta s_0}}\big)\big]+\beta,\\
&\leq \beta\big[\varpi(s_0,s_2)+\varpi(s_1,s_1)\big]+\beta,\\
\varpi(s_1,s_2) &\leq \beta\varpi(s_0,s_2)+\beta.
\end{aligned}$$

By using sb_4,

$$\begin{aligned}
&\varpi(s_1,s_2) \leq \beta\varpi(s_0,s_1)+\beta\sigma\varpi(s_1,s_2)+\beta,\\
&(1-\beta\sigma)\varpi(s_1,s_2) \leq \varpi(s_0,s_1)+\beta,\\
&\varpi(s_1,s_2) \leq \frac{\beta}{(1-\beta\sigma)}[\varpi(s_0,s_1)+\frac{\beta}{(1-\beta\sigma)}, \qquad (3.5)\\
&\varpi(s_1,s_2) \leq \gamma\varpi(s_0,s_1)+\gamma.
\end{aligned}$$

Here, $\gamma = \frac{\beta}{(1-\beta\sigma)}$, where $\beta \in (0,\frac{1}{2\sigma})$, then $\gamma \in (0,\frac{1}{\sigma})$. By using Lemma 2.13 again,

$$\begin{aligned}
\varpi(s_2,s_3) &\leq H\big([\theta s_2]_{\alpha_{\theta s_2}},[\theta s_1]_{\alpha_{\theta s_1}}\big)+\beta\gamma,\\
&\leq \beta\big[\varpi\big(s_1,[\theta s_2]_{\alpha_{\theta s_2}}\big)+\varpi\big(s_2,[\theta s_1]_{\alpha_{\theta s_1}}\big)\big]+\beta\gamma,\\
&\leq \beta\big[\varpi(s_1,s_3)+\varpi(s_2,\theta s_2)\big]+\beta\gamma,\\
&\leq \beta\big[\varpi(s_1,s_3)\big]+\beta\gamma.
\end{aligned}$$

By using sb_4,

$$\begin{aligned}
\varpi(s_2,s_3) &\leq \beta\big[\varpi(s_1,s_2)+\sigma\varpi(s_2,s_3)\big]+\beta\gamma,\\
&= \beta\varpi(s_1,s_2)+\beta\sigma\varpi(s_2,s_3)+\beta\gamma,\\
(1-\beta\sigma)\varpi(s_2,s_3) &\leq \beta\varpi(s_1,s_2)+\beta\gamma,\\
\Rightarrow \quad \varpi(s_2,s_3) &= \frac{\beta}{(1-\beta\sigma)}\varpi(s_1,s_2)+\frac{\beta\gamma}{(1-\beta\sigma)},\\
&= \gamma\varpi(s_2,s_3)+\gamma^2.
\end{aligned}$$

By using (3.5),

$$\begin{aligned}\varpi(s_2,s_3) &\leq \beta\big[\varpi(s_0,s_1)+\gamma\big]+\gamma^2,\\ &= \gamma^2\varpi(s_0,s_1)+2\gamma^2,\\ \Rightarrow \quad \varpi(s_2,s_3) &= \gamma^2\varpi(s_0,s_1)+2\gamma^2.\end{aligned}$$

Generally,

$$\varpi(s_n,s_{n+1}) \leq \gamma^n\varpi(s_0,s_1)+n\gamma^n. \tag{3.6}$$

To show $\{s_n\}_{n=1}^{\infty}$ is a CS, let $m,n\in\mathbb{N}$ with $m>n$ $\varpi(s_n,s_m)\leq\varpi(s_n,s_{n+1})+\sigma\varpi(s_{n+1},s_{n+2})+\sigma^2\varpi(s_{n+2},s_{n+3})+\cdots+\sigma^{m-n-1}\varpi(s_{m-1},s_m)$. By using (3.6), we have $\varpi(s_n,s_m)\leq\gamma^n\varpi(s_0,s_1)+n\gamma^n+\sigma\gamma^{n+1}\varpi(s_0,s_1)+\sigma(n+1)\gamma^{n+1}+\sigma^2\gamma^{n+2}\varpi(s_0,s_1)+\sigma^3(n+2)\gamma^{n+3}+\cdots+s^{m-n-1}\gamma^{m-1}\times\varpi(s_0,s_1)+s^{m-n-1}(m-1)\gamma^{m-1}$.

$$\varpi(s_n,s_m)\leq\gamma^n\varpi(s_0,s_1)\left[\frac{1-(\sigma\gamma)^{m-n-1}}{1-\sigma\gamma}\right]+\sum_{i=n}^{m-1}i\sigma^{i-n}\gamma^i.$$

Taking $m,n\to\infty$, $\Rightarrow\varpi(s_n,s_m)=0$. Hence, $\{s_n\}$ is a CS in S. Since S is complete, so $\exists r\in S$ such that $s_n\to r$.

$$\begin{aligned}\varpi\big(r,[\theta r]_{\alpha_{\theta r}}\big) &\leq \varpi(r,s_n)+\sigma\varpi\big(s_n,[\theta r]_{\alpha_{\theta r}}\big),\\ &\leq \varpi(r,s_n)+\sigma H\big([\theta s_{n-1}]_{\alpha_{\theta s_{n-1}}},[\theta r]_{\alpha_{\theta r}}\big),\\ &\leq \varpi(r,s_n)+\sigma\beta\big[\varpi\big(s_{n-1},[\theta r]_{\alpha_{\theta r}}\big)+\varpi\big(r,[\theta s_{n-1}]_{\alpha_{\theta s_{n-1}}}\big)\big],\end{aligned}$$

as $n\to\infty\Rightarrow\varpi(r,[\theta r]_{\alpha_{\theta r}})\leq 0$. Hence, $r\in[\theta r]_{\alpha_{\theta r}}$, i.e., r is the FP of θ. □

Corollary 3.8 *Suppose (S,ϖ) is a complete MS and $\theta:S\to F(S)$ is a fuzzy map. Suppose $[\theta s]_{\alpha_{\theta s}}$ and $[\theta u]_{\alpha_{\theta u}}$ are closed and bounded subsets of S defined as*

$$H\big([\theta s]_{\alpha_{\theta s}},[\theta u]_{\alpha_{\theta u}}\big)\leq\beta\big[\varpi\big(s,[\theta u]_{\alpha_{\theta u}}\big)+\varpi\big(u,[\theta s]_{\alpha_{\theta s}}\big)\big],$$

for all $s,u\in S$ and $\beta\in[0,\frac{1}{2})$. Then, there exists r in S such that $r\in[\theta r]_{\alpha_{\theta r}}$.

Lemma 3.9 *Let (S,ϖ,σ) be a strong b-MS and $[C]_{\alpha_C},[E]_{\alpha_E}\in F(S)$. If $H([C]_{\alpha_C},[E]_{\alpha_E})>0$ then for each $g>1$ and $c\in[C]_{\alpha_C}$ there exists $e\in[E]_{\alpha_E}$ such that*

$$\varpi(c,e)<gH\big([C]_{\alpha_C},[E]_{\alpha_E}\big).$$

Proof Using the characteristics of infimum, with $\varepsilon=(g-1)H([C]_{\alpha_C},[E]_{\alpha_E})>0$ there exists $e\in[E]_{\alpha_E}$ such that

$$\varpi(c,e)<\varpi\big(c,[E]_{\alpha_E}\big)+\varepsilon.$$

On the other hand, by the definition of $H([C]_{\alpha_C},[E]_{\alpha_E})$,

$$\varpi\big(c,[E]_{\alpha_E}\big) \le H\big([C]_{\alpha_C},[E]_{\alpha_E}\big).$$

This deduces

$$\varpi(c,e) < g.H\big([C]_{\alpha_C},[E]_{\alpha_E}\big). \qquad \square$$

Theorem 3.10 *Suppose (S,ϖ,σ) is a complete strong b-MS and $\theta: S\to F(S)$ is a FM. Suppose $[\theta s]_{\alpha_{\theta s}}$ and $[\theta u]_{\alpha_{\theta u}}$ are closed and bounded subsets of S and there exists $x\in(0,k)$ with $0<k<\frac{1}{2}$ and $\alpha\in(0,1]$ satisfying $\frac{1}{\sigma+1}\varpi(s,[\theta s]_{\alpha_{\theta s}})\le\varpi(s,u)$ implies $H([\theta s]_{\alpha_{\theta s}},[\theta u]_{\alpha_{\theta u}})\le x\{\varpi(s,[\theta s]_{\alpha_{\theta s}})+\varpi(u,[\theta u]_{\alpha_{\theta u}})\}$, for all $s,u\in S$. Then, θ has a unique FP $r\in S$. Moreover, for each $s\in S$ the sequence of iterates $\{\theta^n s\}$ converges to r.*

Proof Assume $s_0\in S$ and choose $s_1\in[\theta s_0]_{\alpha_{\theta s_0}}$.

Step 1. If $H([\theta s_0]_{\alpha_{\theta s_0}},[\theta s_1]_{\alpha_{\theta s_1}})=0$ then $[\theta s_0]_{\alpha_{\theta s_0}}=[\theta s_1]_{\alpha_{\theta s_1}}$. θ. Thus, s_1 is a FP of θ. If $H([\theta s_0]_{\alpha_{\theta s_0}},[\theta s_1]_{\alpha_{\theta s_1}})>0$, by Lemma 3.9 then for each $g_1>1$, there exists $s_2\in[\theta s_1]_{\alpha_{\theta s_1}}$ such that

$$\varpi(s_1,s_2) < g_1 H\big([\theta s_0]_{\alpha_{\theta s_0}},[\theta s_1]_{\alpha_{\theta s_1}}\big).$$

Step 2. Similarly, if $H([\theta s_1]_{\alpha_{\theta s_1}},[\theta s_2]_{\alpha_{\theta s_2}})=0$ then $[\theta s_1]_{\alpha_{\theta s_1}}=[\theta s_2]_{\alpha_{\theta s_2}}$. Thus, s_2 is a FP of θ. If $H([\theta s_1]_{\alpha_{\theta s_1}},[\theta s_2]_{\alpha_{\theta s_2}})>0$, by Lemma 3.9 then for each $g_2>1$, there exists $s_3\in[\theta s_2]_{\alpha_{\theta s_2}}$ such that

$$\varpi(s_2,s_3) < g_2 H\big([\theta s_1]_{\alpha_{\theta s_1}},[\theta s_2]_{\alpha_{\theta s_2}}\big).$$

Step n. Continuing in this manner, if $H([\theta s_{n-1}]_{\alpha_{\theta s_{n-1}}},[\theta s_n]_{\alpha_{\theta s_n}})=0$. Thus, s_n is a FP of θ. If $H([\theta s_{n-1}]_{\alpha_{\theta s_{n-1}}},[\theta s_n]_{\alpha_{\theta s_n}})>0$, by Lemma 3.9 then for each $g_n>1$, there exists $s_{n+1}\in[\theta s_n]_{\alpha_{\theta s_n}}$ such that

$$\varpi(s_n,s_{n+1}) < g_n H\big([\theta s_{n-1}]_{\alpha_{\theta s_{n-1}}},[\theta s_n]_{\alpha_{\theta s_n}}\big).$$

The above process continues, if at step k satisfying $H([\theta s_{k-1}]_{\alpha_{\theta s_{k-1}}},[\theta s_k]_{\alpha_{\theta s_k}})=0$, then s_k is a FP of θ. If not, we obtain two sequences $\{s_n\}$ and $\{g_n\}$ such that $s_n\in[\theta s_{n-1}]_{\alpha_{\theta s_{n-1}}}$, $g_n>1$ and

$$\varpi(s_n,s_{n+1}) < g_n H\big([\theta s_{n-1}]_{\alpha_{\theta s_{n-1}}},[\theta s_n]_{\alpha_{\theta s_n}}\big), \quad \forall n\ge 1. \tag{3.7}$$

Since $\frac{1}{\sigma+1}d(s_{n-1},[\theta s_{n-1}]_{\alpha_{\theta s_{n-1}}})\le\frac{1}{\sigma+1}d(s_{n-1},s_n)\le\varpi(s_{n-1},s_n)$ and by hypothesis, we obtain

$$\begin{aligned} H\big([\theta s_{n-1}]_{\alpha_{\theta s_{n-1}}},[\theta s_n]_{\alpha_{\theta s_n}}\big) &\le x\big\{d\big(s_{n-1},[\theta s_{n-1}]_{\alpha_{\theta s_{n-1}}}\big)+d\big(s_n,[\theta s_n]_{\alpha_{\theta s_n}}\big)\big\} \\ &\le x\big\{d(s_{n-1},s_n)+d(s_n,s_{n+1})\big\}. \end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we have

$$\varpi(s_n,s_{n+1}) < g_n x\big\{\varpi(s_{n-1},s_n)+\varpi(s_n,s_{n+1})\big\}.$$

We can choose $g_n = \frac{k}{x} > 1$ with $x \in (0,k)$ and $0 < k < \frac{1}{2}$. Then, we obtain $\varpi_n < \frac{k}{1-k}\varpi_{n-1}$, where $\frac{k}{1-k} < 1$ and $\varpi_n = \varpi(s_n, s_{n+1})$. Thus, $\varpi_n < (\frac{k}{1-k})^n \varpi_0$ for all $n \geq 1$. Hence,

$$\sum_{n=1}^{\infty} \varpi_n \leq \varpi_0 \sum_{n=1}^{\infty} \left(\frac{k}{1-k}\right)^n < +\infty.$$

By Proposition 2.5, $\{s_n\}$ is a CS in S. Since S is complete, $\exists\, r \in S$ such that $\lim_{n\to\infty} s_n = r$. Now, we show that for any $n \geq 0$, either

$$\frac{1}{\sigma+1}\varpi\big(s_n, [\theta s_n]_{\alpha_{\theta s_n}}\big) \leq \varpi(s_n, r) \quad \text{or} \quad \frac{1}{\sigma+1}\varpi\big(s_{n+1}, [\theta s_{n+1}]_{\alpha_{\theta s_{n+1}}}\big) \leq \varpi(s_{n+1}, r). \tag{3.9}$$

Arguing by contradiction, we suppose that for some $n \geq 0$,

$$\varpi(s_n, r) < \frac{1}{\sigma+1} d\big(s_n, [\theta s_n]_{\alpha_{\theta s_n}}\big) \quad \text{or} \quad \varpi(s_{n+1}, r) < \frac{1}{\sigma+1} d\big(s_{n+1}, [\theta s_{n+1}]_{\alpha_{\theta s_{n+1}}}\big).$$

Then, by the triangular inequality, we obtain

$$\begin{aligned}
\varpi_n &= \varpi(s_n, s_{n+1}) \leq \varpi(s_n, r) + \sigma\varpi(s_{n+1}, r) \\
&< \frac{1}{\sigma+1} d\big(s_n, [\theta s_n]_{\alpha_{\theta s_n}}\big) + \frac{\sigma}{\sigma+1} d\big(s_{n+1}, [\theta s_{n+1}]_{\alpha_{\theta s_{n+1}}}\big) \\
&\leq \frac{1}{\sigma+1}\varpi(s_n, s_{n+1}) + \frac{\sigma}{\sigma+1}\varpi(s_{n+1}, s_{n+2}) \\
&\leq \varpi_n.
\end{aligned}$$

This is a contradiction. Hence, by hypothesis for each $n \geq 0$ and from (3.9), either

$$H\big([\theta s_n]_{\alpha_{\theta s_n}}, [\theta r]_{\alpha_{\theta r}}\big) \leq x\big\{d\big(s_n, [\theta s_n]_{\alpha_{\theta s_n}}\big) + d\big(r, [\theta r]_{\alpha_{\theta r}}\big)\big\}, \tag{3.10}$$

or

$$H\big([\theta s_{n+1}]_{\alpha_{\theta s_{n+1}}}, [\theta r]_{\alpha_{\theta r}}\big) \leq x\big\{d\big(s_{n+1}, [\theta s_{n+1}]_{\alpha_{\theta s_{n+1}}}\big) + d\big(r, [\theta r]_{\alpha_{\theta r}}\big)\big\}. \tag{3.11}$$

Then, either (3.10) holds for infinity natural numbers n or (3.11) holds for infinity natural numbers n. Suppose (3.10) holds for infinity natural numbers n. We can choose that in that infinity set the sequence $\{n_k\}$ is a monotone strictly increasing sequence of natural numbers. Therefore, sequence $\{s_{n_k}\}$ is a subsequence of $\{s_n\}$ and

$$\begin{aligned}
d\big(r, [\theta r]_{\alpha_{\theta r}}\big) &\leq d\big([\theta s_{n_k}]_{\alpha_{\theta s_{n_k}}}, r\big) + \sigma H\big([\theta s_{n_k}]_{\alpha_{\theta s_{n_k}}}, [\theta r]_{\alpha_{\theta r}}\big) \\
&\qquad \leq \varpi(s_{n_k+1}, r) + Kx\big\{d\big(s_{n_k+1}, [\theta s_{n_k+1}]_{\alpha_{\theta s_{n_k+1}}}\big) + d\big(r, [\theta r]_{\alpha_{\theta r}}\big)\big\},
\end{aligned}$$

which is equivalent to

$$d\big(r, [\theta r]_{\alpha_{\theta r}}\big) \leq \frac{1+\sigma x}{1-\sigma x}\varpi(s_{n_k+1}, r) + \frac{\sigma^2 x}{1-\sigma x}\varpi(s_{n_k+2}, r).$$

By taking limits on both sides of the above inequality, we obtain $d(r, [\theta r]_{\alpha_{\theta r}}) = 0$. This means that $r \in [\theta r]_{\alpha_{\theta r}}$. If (3.11) holds for infinity natural numbers n, by using an argument similar to that of above we have r is a FP of θ. Suppose $\bar{r}$ is another FP of θ, then $0 =$

$\frac{1}{\sigma+1}d(r,[\theta r]_{\alpha_{\theta r}}) \leq \varpi(r,\bar{r})$ and by hypothesis,

$$\begin{aligned}H\big([\theta r]_{\alpha_{\theta r}},[\theta\bar{r}]_{\alpha_{\theta\bar{r}}}\big) &\leq x\big\{d\big(r,[\theta r]_{\alpha_{\theta r}}\big)+d\big(\bar{r},[\theta\bar{r}]_{\alpha_{\theta\bar{r}}}\big)\big\}\\ &\leq x\big\{d(r,r)+d(\bar{r},\bar{r}\big\}=0\end{aligned}$$

and so $H([\theta r]_{\alpha_{\theta r}},[\theta\bar{r}]_{\alpha_{\theta\bar{r}}})=0$ implies $[\theta r]_{\alpha_{\theta r}}=[\theta\bar{r}]_{\alpha_{\theta\bar{r}}}$ means $r=\bar{r}$. Hence, θ has a unique FP $r\in S$. □

Example 3.11 Consider a set $S=\{2,3,4\}$. A mapping $\varpi : S\times S\to[0,\infty)$ defined by

$$\begin{aligned}&\varpi(2,3)=1=\varpi(3,2),\\ &\varpi(2,4)=4=\varpi(4,2),\\ &\varpi(3,4)=1=\varpi(4,3),\\ &\varpi(2,2)=\varpi(3,3)=\varpi(4,4)=0\end{aligned}$$

is a strong b-metric. The triplet $(S,\varpi,\sigma=4)$ is a complete strong b-MS.

For any $\alpha\in(0,1]$, define a mapping $\theta : S\to F(S)$ and $\theta(s) : S\to[0,1]$ by

$$\theta(2)(t)=\begin{cases}\frac{\alpha}{3}, & t=2;\\ \alpha, & t=3;\\ \frac{\alpha}{4}, & t=4,\end{cases}$$

$$\theta(3)(t)=\begin{cases}\frac{\alpha}{2}, & t=2,4;\\ \alpha, & t=3,\end{cases}$$

$$\theta(4)(t)=\begin{cases}\alpha, & t=3;\\ \frac{\alpha}{3}, & t=2,4\end{cases}$$

and

$$\begin{aligned}&[\theta 2]_{\alpha_{\theta 2}}=\big\{t\in S:\theta(2)(t)\geq\alpha\big\}=\{3\},\\ &[\theta 3]_{\alpha_{\theta 3}}=\big\{t\in S:\theta(3)(t)\geq\alpha\big\}=\{3\},\\ &[\theta 4]_{\alpha_{\theta 4}}=\big\{t\in S:\theta(4)(t)\geq\alpha\big\}=\{3\}.\end{aligned}$$

Then,

$$\begin{aligned}&H\big([\theta 2]_{\alpha_{\theta 2}},[\theta 3]_{\alpha_{\theta 3}}\big)=H\big(\{3\},\{3\}\big)=0,\\ &H\big([\theta 3]_{\alpha_{\theta 3}},[\theta 4]_{\alpha_{\theta 4}}\big)=H\big(\{3\},\{3\}\big)=0,\\ &H\big([\theta 2]_{\alpha_{\theta 2}},[\theta 4]_{\alpha_{\theta 4}}\big)=H\big(\{3\},\{3\}\big)=0.\end{aligned}$$

On the other hand, since

$$\begin{aligned}&\frac{1}{\sigma+1}\varpi\big(s,[\theta s]_{\alpha_{\theta s}}\big)\leq\varpi(s,u),\\ &\frac{1}{5}=\frac{1}{5}\varpi\big(2,[\theta 2]_{\alpha_{\theta 2}}\big)\leq\varpi(2,u),\end{aligned}$$

for any $u \in S$ and

$$0 = H\big([\theta 2]_{\alpha_{\theta 2}}, [\theta 3]_{\alpha_{\theta 3}}\big) \leq x\big\{\varpi\big(2, [\theta 2]_{\alpha_{\theta 2}}\big) + \varpi\big(3, [\theta 3]_{\alpha_{\theta 3}}\big)\big\} = x,$$

$$0 = H\big([\theta 2]_{\alpha_{\theta 2}}, [\theta 4]_{\alpha_{\theta 4}}\big) \leq x\big\{\varpi\big(2, [\theta 2]_{\alpha_{\theta 2}}\big) + \varpi\big(4, [\theta 4]_{\alpha_{\theta 4}}\big)\big\} = 2x,$$

then $\frac{1}{5}\varpi(2, [\theta 2]_{\alpha_{\theta 2}}) \leq \varpi(2, u)$ implies $H([\theta 2]_{\alpha_{\theta 2}}, [\theta u]_{\alpha_{\theta u}}) \leq x\{\varpi(2, [\theta 2]_{\alpha_{\theta 2}}) + \varpi(u, [\theta u]_{\alpha_{\theta u}})\}$, for all $u \in S$. Again, since $0 = \frac{1}{5}\varpi(3, [\theta 3]_{\alpha_{\theta 3}}) \leq \varpi(3, u)$ holds for all $u \in S$ and

$$0 = H\big([\theta 3]_{\alpha_{\theta 3}}, [\theta 2]_{\alpha_{\theta 2}}\big) \leq x\big\{\varpi\big(3, [\theta 3]_{\alpha_{\theta 3}}\big) + \varpi\big(2, [\theta 2]_{\alpha_{\theta 2}}\big)\big\} = x,$$

$$0 = H\big([\theta 3]_{\alpha_{\theta 3}}, [\theta 4]_{\alpha_{\theta 4}}\big) \leq x\big\{\varpi\big(3, [\theta 3]_{\alpha_{\theta 3}}\big) + \varpi\big(4, [\theta 4]_{\alpha_{\theta 4}}\big)\big\} = x,$$

then $\frac{1}{5}\varpi(3, [\theta 3]_{\alpha_{\theta 3}}) \leq \varpi(3, u)$ implies $H([\theta 3]_{\alpha_{\theta 3}}, [\theta u]_{\alpha_{\theta u}}) \leq x\{\varpi(3, [\theta 3]_{\alpha_{\theta 3}}) + \varpi(u, [\theta u]_{\alpha_{\theta u}})\}$, for all $u \in S$. Finally, by $\frac{1}{5} = \frac{1}{5}\varpi(4, [\theta 4]_{\alpha_{\theta 4}}) \leq \varpi(4, u)$ for all $u \in S$ and

$$0 = H\big([\theta 4]_{\alpha_{\theta 4}}, [\theta 3]_{\alpha_{\theta 3}}\big) \leq x\big\{\varpi\big(4, [\theta 4]_{\alpha_{\theta 4}}\big) + \varpi\big(3, [\theta 3]_{\alpha_{\theta 3}}\big)\big\} = x,$$

$$0 = H\big([\theta 4]_{\alpha_{\theta 4}}, [\theta 2]_{\alpha_{\theta 2}}\big) \leq x\big\{\varpi\big(4, [\theta 4]_{\alpha_{\theta 4}}\big) + \varpi\big(2, [\theta 2]_{\alpha_{\theta 2}}\big)\big\} = 2x,$$

then $\frac{1}{5}\varpi(4, [\theta 4]_{\alpha_{\theta 4}}) \leq \varpi(4, u)$ implies $H([\theta 4]_{\alpha_{\theta 4}}, [\theta u]_{\alpha_{\theta u}}) \leq x\{\varpi(4, [\theta 4]_{\alpha_{\theta 4}}) + \varpi(u, [\theta u]_{\alpha_{\theta u}})\}$, for all $u \in S$. Thus, all hypotheses of Theorem 3.10 are satisfied and $r = 3$ is a unique FP of θ.

4 Applications

Here, we find FPs for multivalued mappings with the help of our results obtained in Theorems 3.4, 3.7, and 3.10.

In the following, $CB(S)$ denotes the collection of all closed and bounded subsets of S.

Theorem 4.1 *Suppose (S, ϖ, σ) is a complete strong b-MS with $\sigma \geq 1$ and $A : S \to CB(S)$ is a multivalued mapping such that*

$$H\big(A(s), A(u)\big) \leq \beta\varpi(s, u),$$

for all $s, u \in S$ and $\beta \in [0, 1)$. Then, there exists r such that $r \in A(r)$.

Proof Consider an arbitrary mapping $B : S \to (0, 1]$. Define a FM $\theta : S \to F(S)$ as follows:

$$\theta(s)(g) = \begin{cases} B(s), & \text{if } g \in A(s) \\ 0, & \text{if } g \notin A(s). \end{cases}$$

Then, for $s \in S$,

$$\big[\theta(s)\big]_{\alpha_{\theta(s)}} = \big\{g \in S : \theta(s)(g) \geq \alpha_{\theta(s)} = B(s)\big\} = A(s).$$

Now, since $H([\theta(s)]_{\alpha_{\theta(s)}}, [\theta(u)]_{\alpha_{\theta(u)}}) = H(A(s), A(u))$, Theorem 3.4 can be applied to obtain required FP of A in S. □

Theorem 4.2 *Suppose (S,ϖ,σ) is a complete strong b-MS with $\sigma \geq 1$ and $P: S \to CB(S)$ is a multivalued mapping such that*

$$H\big(P(s),P(u)\big) \leq \beta\varpi\big(s,P(u)\big) + \varpi\big(u,P(s)\big), \tag{4.1}$$

for all $s,u \in S$ and $\beta \in [0,1)$. Then, there exist r in S such that $r \in P(r)$.

Proof Consider an arbitrary mapping $Q: S \to (0,1]$. Define a FM $\theta: S \to F(S)$ as follows:

$$\theta(s)(g) = \begin{cases} Q(s), & \text{if } g \in P(s) \\ 0, & \text{if } g \notin P(s). \end{cases}$$

Then, for $s \in S$,

$$\big[\theta(s)\big]_{\alpha_{\theta(s)}} = \big\{g \in S : \theta(s)(g) \geq \alpha_{\theta(s)} = Q(s)\big\} = P(s).$$

Now, since $H([\theta(s)]_{\alpha_{\theta(s)}},[\theta(u)]_{\alpha_{\theta(u)}}) = H(P(s),P(u))$, Theorem 3.7 can be applied to obtain the required FP of P in S. □

Theorem 4.3 *Suppose (S,ϖ,σ) is a complete strong b-MS and $A: S \to CB(S)$ is a multivalued mapping. Suppose $x \in (0,k)$ with $0 < k < \frac{1}{2}$ satisfying $\frac{1}{\sigma+1}\varpi(s,As) \leq \varpi(s,u)$ implies $H(A(s),A(u)) \leq x\{\varpi(s,A(s)) + \varpi(u,A(u))\}$, for all $s,u \in S$. Then, A has a unique FP $r \in S$. Moreover, for each $s \in S$ the sequence of iterates $\{A^n s\}$ converges to r.*

Proof Consider an arbitrary mapping $P: S \to (0,1]$. Define a FM $\theta: S \to F(S)$ as follows:

$$\theta(s)(g) = \begin{cases} P(s), & \text{if } g \in A(s) \\ 0, & \text{if } g \notin A(s). \end{cases}$$

Then, for $s \in S$,

$$\big[\theta(s)\big]_{\alpha_{\theta(s)}} = \big\{g \in S : \theta(s)(g) \geq \alpha_{\theta(s)} = P(s)\big\} = A(s).$$

Now, since $H([\theta(s)]_{\alpha_{\theta(s)}},[\theta(u)]_{\alpha_{\theta(u)}}) = H(A(s),A(u))$, Theorem 3.10 can be applied to obtain the required FP of A in S. □

5 Conclusion

FP theory is a useful theoretical tool in diverse fields, such as logic programming, functional analysis, artificial intelligence, and many others. In 2021, Doan [9] extended the results in [12] for a class of contractive mappings in strong b-MSs. He proved new versions of FP theorems for single-valued and multivalued mappings by combining the results in [15] and [29]. In this article, we obtained the idea from [9] and extended it to [6] and [29]. We have established FP theorems for fuzzy and nonfuzzy mappings in complete strong b-MS by combining results [6] and [9] and the obtained results are furnished with interesting and nontrivial examples. Moreover, some other contractions are also applied to find fuzzy and nonfuzzy fixed points. Some results for FMs and multivalued mappings are incorporated as corollaries and as applications. Moreover, other direct consequences are obtained as well. We hope these existence results will provide an appropriate environment to approximate further operator equations in applied science.

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