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Multiple positive solutions for Schrödinger-Poisson system with singularity on the Heisenberg group

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Abstract

In this work, we study the following Schrödinger-Poisson system

$$\begin{cases} -\Delta_H u + \mu\phi u = \lambda u^{-\gamma}, & \text{in } \Omega, \\ -\Delta_H \phi = u^2, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where Δ_H is the Kohn-Laplacian on the first Heisenberg group $\mathbb{H}^1$, and $\Omega \subset \mathbb{H}^1$ is a smooth bounded domain, $\mu = \pm 1$, $0 < \gamma < 1$, and $\lambda > 0$ are some real parameters. For the above system, we prove the existence and uniqueness of positive solution for $\mu = 1$ and each $\lambda > 0$. Multiple solutions of the system are also considered for $\mu = -1$ and $\lambda > 0$ small enough using the critical point theory for nonsmooth functional.

Mathematics Subject Classification: 35A15; 35R03

Keywords: Schrödinger-Poisson system; Singularity; Positive solutions; Heisenberg group

1 Introduction and main results

This paper consider the following singular Schrödinger-Poisson system

$$\begin{cases} -\Delta_H u + \mu\phi u = \lambda u^{-\gamma}, & \text{in } \Omega, \\ -\Delta_H \phi = u^2, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ_H is the Kohn-Laplacian on the first Heisenberg group $\mathbb{H}^1$, Ω is a smooth bounded domain of $\mathbb{H}^1$, $\mu = \pm 1$, $0 < \gamma < 1$, and $\lambda > 0$ are some real parameters.

Over the years, many scholars have been widely studied the Heisenberg group due to its crucial role in several branches of mathematics, such as quantum mechanics, complex variables, and harmonic analysis, so one can refer to [9, 12, 19] and the references therein.

In 2022, Liu et al. [15] investigated the following Schrödinger-Poisson system on the Heisenberg group

$$\begin{cases} -(a-b\int_\Omega |\nabla_H u|^2\,d\xi)\Delta_H u + \mu\phi u = \lambda|u|^{q-2}u + |u|^2u, & \text{in } \Omega, \\ -\Delta_H\phi = u^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{H}^1$ is a smooth bounded domain, $a, b > 0$, $1 < q < 2$ or $2 < q < 4$, $\lambda > 0$, and $\mu \in \mathbb{R}$ are some real parameters. They obtained the existence and multiplicity of solutions. In particular, when $a = 1$, $b = 0$, An and Liu in [1] established the existence and multiplicity of solutions of problem (1.2). Using the Green representation formula, the concentration compactness, and the critical point theory, they proved that the above system has at least two positive solutions for $\mu < S \times \text{meas}(\Omega)^{-\frac{1}{2}}$ and λ small enough. In addition, they also established that there is a positive ground-state solution for (1.2).

Lei and Liao [13] considered the following system

$$\begin{cases} -\Delta u + \lambda\phi u = \frac{\lambda}{|x|^\beta u^\gamma} + |u|^4u, & \text{in } \Omega, \\ -\Delta\phi = u^2, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $0 < \gamma < 1$, $0 \le \beta < \frac{5+\gamma}{2}$ and $\lambda > 0$ is parameter, they obtained two positive solutions using the variational method and the Nehari manifold method.

In [20], Pucci and Ye studied the logarithmic and critical nonlinearities for the Kirchhoff-Poisson system

$$\begin{cases} -M(\int_\Omega |\nabla_H u|^2\,d\xi)\Delta_H u + \mu\phi u = |u|^2u + \lambda|u|^{q-2}u\ln|u|^2, & \text{in } \Omega, \\ -\Delta_H\phi = u^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{H}^1$, $q \in (2\theta, 4)$, $\mu \in \mathbb{R}$, and $\lambda > 0$ are some real parameters. Under suitable assumptions on the Kirchhoff function M, covering the degenerate case, they proved the existence of nontrivial solutions for the above system when $\lambda > 0$ is sufficiently large. For more on the results of the Heisenberg group, we refer the reader to [2, 8, 10, 14, 16–18, 21] and the references therein.

Furthermore, for the system (1.1) in the Heisenberg group, there is no result that this paper answers positively. Before giving the theorem, we define the solutions of (1.1) if u satisfies

$$\int_\Omega \nabla_H u\nabla_H v\,d\xi + \mu\int_\Omega \phi_u uv\,d\xi - \lambda\int_\Omega \frac{v}{u^\gamma}\,d\xi = 0, \quad \forall v \in S_0^1(\Omega),$$

we say that u is a solution of problem (1.1).

Theorem 1.1 *Assume that* $0 < \gamma < 1$, $\mu = 1$ *and* $\lambda > 0$, *then system* (1.1) *has a unique solution.*

Theorem 1.2 *Assume that* $0 < \gamma < 1$ *and* $\mu = -1$, *then there exists* $\Lambda_0 > 0$ *such that for every* $\lambda \in (0, \Lambda_0)$, *system* (1.1) *has at least two positive solutions.*

Remark 1.3 Our approach is novel, unlike the Euclidean case, since the presence of singular terms gives us great difficulties; the critical point theory for nonsmooth functional is used to overcome the difficulties, generalizing the results of literature [22].

2 Some preliminary results

In this section, we review the Heisenberg group. For more results, see [7, 11]. Let $\mathbb{H}^1$ be the Heisenberg group of topological dimension 3, that is, the Lie group where underlying manifold is $\mathbb{R}^3$, endowed with the non-Abelian law

$$\tau : \mathbb{H}^1 \to \mathbb{H}^1, \qquad \tau_\xi\left(\xi'\right) = \xi \circ \xi',$$

where

$$\xi \circ \xi' = \left(x + x', y + y', t + t' + 2\left(x'y - xy'\right)\right),$$

for $\forall \xi, \xi' \in \mathbb{H}^1$, with $\xi = (x, y, t)$ and $\xi' = (x', y', t')$, satisfy the inverse operation. Consider the family of dilations on $\mathbb{H}^1$ defined by

$$\delta_s(\xi) = \left(sx, sy, s^2 t\right), \quad \forall \xi \in \mathbb{H}^1,$$

so $\delta_s(\xi \circ \xi') = \delta_s(\xi) \circ \delta_s(\xi')$ (see [19]). The number $Q = 4$ is the homogeneous dimension of $\mathbb{H}^1$, definition

$$\left|B_H(\xi_0, r)\right| = \omega_Q r^Q,$$

where $B_H(\xi_0, r)$ is the Heisenberg ball of radius r centered at ξ_0, i.e.,

$$B_H(\xi_0, r) = \left\{\xi \in \mathbb{H}^1 : d_H(\xi_0, \xi) < r\right\},$$

$d_H(\xi_0, \xi) = |\xi^{-1} \circ \xi_0|_H$ and $\omega_Q = |B_H(0, 1)|$.

The Kohn-Laplacian Δ_H on $\mathbb{H}^1$ is defined as

$$\Delta_H u = \operatorname{div}_H(\nabla_H u),$$

where $\nabla_H u = (Xu, Yu)$. Indeed, the vector fields

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \quad \text{and} \quad T = \frac{\partial}{\partial t},$$

are a basis of the Lie algebra of $\mathbb{H}^1$ thus constituting a set of left invariant vector fields on $\mathbb{H}^1$. Widely known that Δ_H is a degenerate elliptic operator, and the Bony maximum principle is satisfied (see [4]). In the present section, the existence and multiplicity of solutions of system (1.1), when $\mu = -1$, are studied. We prove that system (1.1) has two positive solutions using the critical point theory for nonsmooth functional and the variational method for $\lambda > 0$ small enough.

Let us review critical points of nonsmooth functions related concepts. Let (X, d) be a complete metric space with metric d and $f : X \to \mathbb{R}$ be a continuous functional in X. Denote by $|df|(u)$ the supremum of δ in $[0, \infty)$ such that there exist $r > 0$ and a continuous map $\sigma : U \times [0, r] \to X$, satisfying

$$\begin{cases} f(\sigma(v,t)) \leq f(v) - \delta t, & (v,t) \in U \times [0,r], \\ d(\sigma(v,t), v) \leq t, & (v,t) \in U \times [0,r]. \end{cases} \tag{2.1}$$

The number $|df|(u)$ is called the weak slope of f at u. Thus, $u \in X$ is a critical point of f if $|df|(u) = 0$, and $c \in \mathbb{R}$ is a critical value of f if there exists a critical point $u \in X$ of f with $f(u) = c$.

Since we are solving for the positive solution of system (1.1), so consider the functional I_λ defined on the closed positive cone U^+ of $S_0^1(\Omega)$, which is defined as

$$U^+ = \left\{ u \in S_0^1(\Omega), u(x) \geq 0, \text{a.e. } x \in \Omega \right\}.$$

The Hilbert space $S_0^1(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ under the inner product $\langle u, v \rangle = \int_\Omega \nabla_H u \nabla_H v \, d\xi$. Accordingly, the norm is denoted by $\|u\| = \|u\|_{S_0^1(\Omega)} = (\int_\Omega |\nabla_H u|^2 \, d\xi)^{\frac{1}{2}}$. The norm in $L^p(\Omega)$ is denoted by $\|u\|_p = (\int_\Omega |u|^p \, d\xi)^{\frac{1}{p}}$. The embedding $S_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for $p \in [1, Q^*]$, where $Q^* = \frac{2Q}{Q-2} = 4$ is the critical exponent in $\mathbb{H}^1$. Let us denote by B_ρ and S_ρ a closed ball and a sphere, respectively, of a center of zero and radius ρ. Let S be the best Sobolev constant, namely

$$S = \inf_{u \in S_0^1(\mathbb{H}^1) \setminus \{0\}} \frac{\int_{\mathbb{H}^1} |\nabla_H u|^2 \, d\xi}{(\int_{\mathbb{H}^1} |u|^4 \, d\xi)^{\frac{1}{2}}}. \tag{2.2}$$

First, using the Lax-Milgram theorem, for each $u \in S_0^1(\Omega)$, there exists a unique solution $\phi_u \in S_0^1(\Omega)$, which satisfies the second equation of system (1.1). Then, system (1.1) is transformed into the following problem

$$\begin{cases} -\Delta_H u + \mu \phi_u u = \lambda u^{-\gamma}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

For problem (2.3), we define the functional

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{\mu}{4} \int_\Omega \phi_u u^2 \, d\xi - \frac{\lambda}{1-\gamma} \int_\Omega |u|^{1-\gamma} \, d\xi. \tag{2.4}$$

We know that the functional I_λ is well defined and $I_\lambda \in C^1(S_0^1(\Omega), \mathbb{R})$. Besides, we say that u is a weak solution of problem (2.3) if u satisfies

$$\left\langle I_\lambda'(u), v \right\rangle = \int_\Omega \nabla_H u \nabla_H v \, d\xi + \mu \int_\Omega \phi_u u v d\xi - \lambda \int_\Omega \frac{v}{u^\gamma} \, d\xi = 0, \quad \forall v \in S_0^1(\Omega). \tag{2.5}$$

By the Hölder inequality and (2.2), we obtain

$$\int_\Omega |u|^{1-\gamma}\,d\xi \le S^{-\frac{1-\gamma}{2}}|\Omega|^{\frac{3+\gamma}{4}}\|u\|^{1-\gamma}. \tag{2.6}$$

Lemma 2.1 (See [1]) *For all $u \in S_0^1(\Omega)$, there exists a unique solution $\phi_u \in S_0^1(\Omega)$ of*

$$\begin{cases} -\Delta_H \phi = u^2, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

and

(1) $\phi_u \ge 0$ *and* $\phi_{tu} = t^2\phi_u$ *for each* $t > 0$;

(2) *If* $u_n \rightharpoonup u$ *in* $S_0^1(\Omega)$, *then* $\phi_{u_n} \to \phi_u$ *in* $S_0^1(\Omega)$ *and*

$$\lim_{n\to\infty}\int_\Omega \phi_{u_n}u_n v\,d\xi = \int_\Omega \phi_u u v\,d\xi, \quad \forall v \in S_0^1(\Omega);$$

(3) *For all* $u \in S_0^1(\Omega)$, *there holds that*

$$\int_\Omega |\nabla_H \phi_u|^2\,d\xi = \int_\Omega \phi_u u^2\,d\xi \le S^{-1}\|u\|_{8/3}^4 \le S^{-3}|\Omega|^{\frac{1}{2}}\|u\|^4;$$

(4) *For* $u, v \in S_0^1(\Omega)$, $\int_\Omega (\phi_u u - \phi_v v)(u - v)\,d\xi \ge \frac{1}{2}\|\phi_u - \phi_v\|^2$.

Lemma 2.2 *Assume that* $u \in U^+$ *and* $|dI_\lambda|(u) < +\infty$. *Then, for all* $v \in U^+$, *one obtains*

$$\lambda\int_\Omega \frac{v-u}{u^\gamma}\,d\xi \le \int_\Omega \nabla_H u \nabla_H (v-u)\,d\xi - \int_\Omega \phi_u u(v-u)\,d\xi + |dI_\lambda|(u)\|v-u\|. \tag{2.7}$$

Proof Let $u \ne v \in U^+$ and $\|v-u\| > 2\delta$. Define $\sigma : U \times [0,\delta] \to U^+$ by

$$\sigma(z,t) = z + t\frac{v-z}{\|v-z\|},$$

where U is a neighborhood of u, then $\|\sigma(z,t) - z\| = t$. By (2.1), there exists $(z,t) \in U \times [0,\delta]$ such that $I_\lambda(\sigma(z,t)) > I_\lambda(z) - ct$. Hence, we assume that there exist sequences $\{u_n\} \subset U^+$ and $\{t_n\} \subset [0,+\infty)$, such that $u_n \to u$, $t_n \to 0^+$, and

$$I_\lambda\left(u_n + t_n\frac{v-u_n}{\|v-u_n\|}\right) \ge I_\lambda(u_n) - ct_n.$$

That is say

$$I_\lambda\big(u_n + s_n(v-u_n)\big) \ge I_\lambda(u_n) - cs_n\|v-u_n\|, \tag{2.8}$$

where $s_n = \frac{t_n}{\|v-u_n\|} \to 0^+$ as $n \to \infty$. Dividing (2.8) by s_n, we deduce that

$$\begin{aligned}
&\frac{\lambda}{1-\gamma}\int_\Omega \frac{[u_n + s_n(v-u_n)]^{1-\gamma} - u_n^{1-\gamma}}{s_n}\,d\xi \\
&\quad\le \frac{1}{2}\frac{\|u_n + s_n(v-u_n)\|^2 - \|u_n\|^2}{s_n} \\
&\qquad - \frac{1}{4}\int_\Omega \frac{\phi_{u_n+s_n(v-u_n)}(u_n + s_n(v-u_n))^2 - \phi_{u_n}u_n^2}{s_n}\,d\xi + c\|v-u_n\|.
\end{aligned} \tag{2.9}$$

Further, we can infer that

$$\begin{aligned}
\int_\Omega \frac{[u_n + s_n(v-u_n)]^{1-\gamma} - u_n^{1-\gamma}}{s_n(1-\gamma)}\,d\xi &= \int_\Omega \frac{[u_n + s_n(v-u_n)]^{1-\gamma} - [(1-s_n)u_n]^{1-\gamma}}{s_n(1-\gamma)}\,d\xi \\
&\quad + \int_\Omega \frac{[(1-s_n)u_n]^{1-\gamma} - u_n^{1-\gamma}}{s_n(1-\gamma)}\,d\xi \\
&= \int_\Omega \frac{[u_n + s_n(v-u_n)]^{1-\gamma} - [(1-s_n)u_n]^{1-\gamma}}{s_n(1-\gamma)}\,d\xi \\
&\quad + \frac{(1-s_n)^{1-\gamma} - 1}{s_n(1-\gamma)}\int_\Omega |u_n|^{1-\gamma}\,d\xi \\
&= I_{1n} + I_{2n}.
\end{aligned} \tag{2.10}$$

By mean value theorem, one has

$$I_{1n} = \int_\Omega \frac{\zeta_n^{-\gamma} s_n v}{s_n}\,d\xi = \int_\Omega \frac{v}{\zeta_n^\gamma}\,d\xi,$$

where $\zeta_n \in (u_n - s_n u_n, u_n + s_n(v-u_n))$, that is $\zeta_n \to u(u_n \to u)$ as $s_n \to 0^+$, since $I_{1n} \ge 0$ for all n. Applying Fatou's Lemma to I_{1n}, one gets

$$\liminf_{n\to\infty} I_{1n} \ge \int_\Omega \frac{v}{u^\gamma}\,d\xi, \quad \forall v \in U^+.$$

For I_{2n}, by the dominated convergence theorem, it holds that

$$\lim_{n\to\infty} I_{2n} = -\int_\Omega |u|^{1-\gamma}\,d\xi.$$

For every $v \in U^+$, and the above information, we have

$$\begin{aligned}
\lambda\int_\Omega \frac{v-u}{u^\gamma}\,d\xi &\le \liminf_{n\to\infty}(I_{1n} + I_{2n}) \\
&\le \int_\Omega \nabla_H u \nabla_H (v-u)\,d\xi - \int_\Omega \phi_u u(v-u)\,d\xi + c\|v-u\|,
\end{aligned}$$

where $|dI_\lambda|(u) < c$ is arbitrary. □

Lemma 2.3 *I_λ satisfies the (P.S.) condition.*

Proof Let $\{u_n\} \subset U^+$ be $(P.S.)$ sequence of I_λ, that is

$$|dI_\lambda|(u_n) \to 0, \qquad I_\lambda(u_n) \to c \in \mathbb{R} \quad \text{as } n \to \infty.$$

By Lemma 2.2, $\forall v \in U^+$, we can infer that

$$\lambda \int_\Omega \frac{v - u_n}{u_n^\gamma}\, d\xi \le \int_\Omega \nabla_H u_n \nabla_H (v - u_n)\, d\xi - \int_\Omega \phi_{u_n} u_n (v - u_n)\, d\xi + o(1)\|v - u_n\|, \quad (2.11)$$

taking $v = 2u_n \in U^+$ in (2.11), we have that

$$\lambda \int_\Omega |u_n|^{1-\gamma}\, d\xi \le \int_\Omega |\nabla_H u_n|^2\, d\xi - \int_\Omega \phi_{u_n} u_n^2\, d\xi + o(1)\|u_n\|. \quad (2.12)$$

Since $I_\lambda(u_n) \to c$,

$$\frac{1}{2}\int_\Omega |\nabla_H u_n|^2\, d\xi - \frac{1}{4}\int_\Omega \phi_{u_n} u_n^2\, d\xi - \frac{\lambda}{1-\gamma}\int_\Omega |u_n|^{1-\gamma}\, d\xi = c + o(1). \quad (2.13)$$

From (2.12) and (2.13), we have

$$\begin{aligned} \frac{1}{4}\int_\Omega |\nabla_H u_n|^2\, d\xi &\le \lambda\left(\frac{1}{1-\gamma} - \frac{1}{4}\right)\int_\Omega |u_n|^{1-\gamma}\, d\xi + c + o(1) + o(1)\|u_n\| \\ &\le \lambda \frac{3+\gamma}{4(1-\gamma)}\|u_n\|^{1-\gamma} + C + o(1)\|u_n\|, \end{aligned} \quad (2.14)$$

which implies that $\{u_n\}$ is bounded in $S_0^1(\Omega)$. Thus, there exists a subsequence, still denoted by itself, and a function $u \in S_0^1(\Omega)$, such that $u_n \rightharpoonup u$ in $S_0^1(\Omega)$, $u_n(x) \to u(x)$ a.e. in Ω as $n \to \infty$. Choosing $v = u_m$ as the test function in (2.11), we have

$$\lambda \int_\Omega \frac{u_m - u_n}{u_n^\gamma}\, d\xi \le \int_\Omega \nabla_H u_n \nabla_H (u_m - u_n)\, d\xi - \int_\Omega \phi_{u_n} u_n (u_m - u_n)\, d\xi + o(1)\|u_m - u_n\|.$$

Exchanging u_m and u_n gives a similar inequality, and adding two inequalities together and Lemma 2.1(4), it holds that

$$\begin{aligned} \|u_n - u_m\|^2 &\le \lambda \int_\Omega (u_n - u_m)\left(\frac{1}{u_n^\gamma} - \frac{1}{u_m^\gamma}\right) d\xi \\ &\quad - \int_\Omega (\phi_{u_m} u_m - \phi_{u_n} u_n)(u_n - u_m)\, d\xi + o(1)\|u_m - u_n\| \\ &\le - \int_\Omega (\phi_{u_m} u_m - \phi_{u_n} u_n)(u_m - u_n)\, d\xi + o(1)\|u_m - u_n\| \\ &\le -\frac{1}{2}\|\phi_{u_m} - \phi_{u_n}\|^2 + o(1)\|u_m - u_n\| \\ &\le o(1)\|u_m - u_n\|. \end{aligned}$$

We have $\lim_{n\to\infty}\|u_n - u_m\| = 0$. Therefore, $u_n \to u$ in $S_0^1(\Omega)$ as $n \to \infty$. □

Lemma 2.4 *Suppose that* $|dI_\lambda|(u) = 0$, *then* u *is a weak solution of the problem* (2.3). *Namely*, $u^{-\gamma}\varphi \in L^1(\Omega)$ *for all* $\varphi \in S_0^1(\Omega)$, *there holds*

$$\int_\Omega \nabla_H u \nabla_H \varphi \, d\xi - \int_\Omega \phi_u u \varphi \, d\xi = \lambda \int_\Omega \frac{\varphi}{u^\gamma} \, d\xi. \tag{2.15}$$

Proof By Lemma 2.2, we deduce that

$$\lambda \int_\Omega \frac{v-u}{u^\gamma} \, d\xi \le \int_\Omega \nabla_H u \nabla_H (v-u) \, d\xi - \int_\Omega \phi_u u (v-u) \, d\xi,$$

for every $v \in U^+$. Letting $s \in \mathbb{R}$, $\varphi \in S_0^1(\Omega)$, taking $v = (u + s\varphi)^+$ and $v \in U^+$ as a test function in (2.7), one gets

$$\begin{aligned}
0 &\le \int_\Omega \nabla_H u \nabla_H \big((u+s\varphi)^+ - u\big) \, d\xi - \int_\Omega \phi_u u \big((u+s\varphi)^+ - u\big) \, d\xi - \lambda \int_\Omega \frac{(u+s\varphi)^+ - u}{u^\gamma} \, d\xi \\
&= s\left(\int_\Omega \nabla_H u \nabla_H \varphi \, d\xi - \int_\Omega \phi_u u \varphi \, d\xi - \lambda \int_\Omega \frac{\varphi}{u^\gamma} \, d\xi\right) - \int_{\{u+s\varphi<0\}} \nabla_H u \nabla_H (u+s\varphi) \, d\xi \\
&\quad + \int_{\{u+s\varphi<0\}} \phi_u u (u+s\varphi) \, d\xi + \int_{\{u+s\varphi<0\}} \frac{u+s\varphi}{u^\gamma} \, d\xi \\
&\le s\left(\int_\Omega \nabla_H u \nabla_H \varphi \, d\xi - \int_\Omega \phi_u u \varphi \, d\xi - \lambda \int_\Omega \frac{\varphi}{u^\gamma} \, d\xi\right) \\
&\quad - s \int_{\{u+s\varphi<0\}} (\nabla_H u \nabla_H \varphi - \phi_u u \varphi) \, d\xi,
\end{aligned}$$

since $\nabla_H u(x) = 0$ for a.e. $x \in \Omega$ with $u(x) = 0$, and $\mathrm{Meas}\{x \in \Omega : u(x) + s\varphi(x) < 0, u(x) > 0\} \to 0$ as $s \to 0$, one obtains

$$\int_{\{u+s\varphi<0\}} (\nabla_H u \nabla_H \varphi - \phi_u u \varphi) \, d\xi = \int_{\{u+s\varphi<0, u>0\}} (\nabla_H u \nabla_H \varphi - \phi_u u \varphi) \, d\xi \to 0.$$

Therefore

$$0 \le s\left(\int_\Omega \nabla_H u \nabla_H \varphi \, d\xi - \int_\Omega \phi_u u \varphi \, d\xi - \lambda \int_\Omega \frac{\varphi}{u^\gamma} \, d\xi\right) + o(s),$$

as $s \to 0$, we obtain that

$$\int_\Omega \nabla_H u \nabla_H \varphi \, d\xi - \int_\Omega \phi_u u \varphi \, d\xi - \lambda \int_\Omega \frac{\varphi}{u^\gamma} \, d\xi \ge 0.$$

By the arbitrariness of φ, also holds for $-\varphi$

$$\int_\Omega \nabla_H u \nabla_H \varphi \, d\xi - \int_\Omega \phi_u u \varphi \, d\xi - \lambda \int_\Omega \frac{\varphi}{u^\gamma} \, d\xi = 0.$$

Hence, we can deduce that (2.15) holds. □

Lemma 2.5 *Given* $0 < \gamma < 1$, *there exist constants* $r, \rho, \Lambda_0 > 0$, *such that the functional* I_λ *satisfies the following conditions for* $0 < \lambda < \Lambda_0$:

(i) $I_\lambda(u)|_{u\in S_\rho} \geq r > 0$, $\inf_{u\in B_\rho} I_\lambda(u) < 0$;
(ii) *There exists* $e \in S_0^1(\Omega)$ *with* $\|e\| > \rho$ *such that* $I_\lambda(e) < 0$.

Proof (i) It follows from (2.6) and Lemma 2.1(3) that

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_\Omega \phi_u u^2\, d\xi - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}\, d\xi \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{4}S^{-3}|\Omega|^{\frac{1}{2}}\|u\|^4 - \frac{\lambda}{1-\gamma}S^{-\frac{1-\gamma}{2}}|\Omega|^{\frac{3+\gamma}{4}}\|u\|^{1-\gamma}, \end{aligned}$$

which implies that there exist constants $r, \rho, \Lambda_0 > 0$, such that $I_\lambda|_{u\in S_\rho} \geq r > 0$ for every $\lambda \in (0, \Lambda_0)$. Moreover, for $u \in S_0^1(\Omega)\backslash\{0\}$, it holds that

$$\lim_{t\to 0^+} \frac{I_\lambda(tu)}{t^{1-\gamma}} = -\frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}\, d\xi < 0.$$

So, we obtain that $I_\lambda(tu) < 0$ for all $u \neq 0$ and t small enough. Therefore, for $\|u\|$ small enough, one has

$$m = \inf_{u\in B_\rho} I_\lambda(u) < 0.$$

(ii) For every $u^+ \in S_0^1(\Omega)$, $u^+ \neq 0$ and $t > 0$, we get

$$I_\lambda(tu) = \frac{t^2}{2}\|u\|^2 - \frac{t^4}{4}\int_\Omega \phi_u u^2\, d\xi - \frac{\lambda t^{1-\gamma}}{1-\gamma}\int_\Omega |u|^{1-\gamma}\, d\xi \to -\infty,$$

as $t \to +\infty$. Therefore, we can find $e \in S_0^1(\Omega)$ such that $\|e\| > \rho$ and $I_\lambda(e) < 0$. □

3 Proof of main results

In this section, we show that for each $\lambda > 0$, the functional I_λ attains the global minimizer in $S_0^1(\Omega)$, which is the unique solution of system (1.1) for $\mu = 1$ and multiple solutions of the system for $\mu = -1$, $\lambda > 0$ small enough.

Proof of Theorem 1.1 We prove Theorem 1.1 in three steps.

Step 1. For every $\lambda > 0$ and $\mu = 1$, the functional I_λ attains the global minimizer in $S_0^1(\Omega)$, in other words, there exists $u_* \in S_0^1(\Omega)$ such that

$$I_\lambda(u_*) = m_\lambda = \inf_{S_0^1(\Omega)} I_\lambda < 0.$$

In fact, for all $u \in S_0^1(\Omega)$, combining with Lemma 2.1(1) and (2.6), we infer that

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_\Omega \phi_u u^2\, d\xi - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}\, d\xi \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{1-\gamma}S^{-\frac{1-\gamma}{2}}|\Omega|^{\frac{3+\gamma}{4}}\|u\|^{1-\gamma}, \end{aligned} \tag{3.1}$$

this implies that I_λ is coercive and bounded from below on $S_0^1(\Omega)$ for each $\lambda > 0$. Thus, $m_\lambda = \inf_{S_0^1(\Omega)} I_\lambda$. For $t > 0$ and given $u \in S_0^1(\Omega)\backslash\{0\}$,

$$I_\lambda(tu) = \frac{t^2}{2}\|u\|^2 + \frac{t^4}{4}\int_\Omega \phi_u u^2\,d\xi - \frac{\lambda t^{1-\gamma}}{1-\gamma}\int_\Omega |u|^{1-\gamma}\,d\xi.$$

We deduce from that for $t > 0$ small enough, $I_\lambda(tu) < 0$. Therefore, $m_\lambda = \inf_{S_0^1(\Omega)} I_\lambda < 0$.

From the definition of m_λ, existence of minimizing sequence $\{u_n\} \subset S_0^1(\Omega)$ such that $\lim_{n\to\infty} I_\lambda(u_n) = m_\lambda < 0$. Since $I_\lambda(u_n) = I_\lambda(|u_n|)$, we can assume that $u_n \geq 0$. By (3.1), we know that $\{u_n\}$ is bounded in $S_0^1(\Omega)$. Suppose there exists a subsequence, still denoted by $\{u_n\}$, and $u_* \in S_0^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u_*, & \text{weakly in } S_0^1(\Omega), \\ u_n \to u_*, & \text{strongly in } L^p(\Omega)\ (1 \leq p < 4), \\ u_n(x) \to u_*(x), & \text{a.e. in } \Omega. \end{cases}$$

Then, combining with the weakly lower semi-continuity of the norm and Lemma 2.1 (2), one has

$$\begin{aligned} I_\lambda(u_*) &= \frac{1}{2}\|u_*\|^2 + \frac{1}{4}\int_\Omega \phi_{u_*} u_*^2\,d\xi - \frac{\lambda}{1-\gamma}\int_\Omega |u_*|^{1-\gamma}\,d\xi \\ &\leq \liminf_{n\to\infty} I_\lambda(u_n) = m_\lambda. \end{aligned}$$

Furthermore, $I_\lambda(u_*) \geq m_\lambda$, thus $I_\lambda(u_*) = m_\lambda < 0$.

In addition, we show $u_* > 0$ in Ω. From the information above, $u_* \geq 0$ and $u_* \not\equiv 0$. Fix $\eta \in S_0^1(\Omega)$, $\eta > 0$ and $t \geq 0$, we obtain that

$$\begin{aligned} 0 &\leq \liminf_{t\to 0} \frac{I_\lambda(u_* + t\eta) - I_\lambda(u_*)}{t} \\ &= \int_\Omega (\nabla_H u_* \nabla_H \eta + \phi_{u_*} u_* \eta)\,d\xi - \frac{\lambda}{1-\gamma}\limsup_{t\to 0}\int_\Omega \frac{(u_* + t\eta)^{1-\gamma} - u_*^{1-\gamma}}{t}\,d\xi, \end{aligned}$$

that is

$$\frac{\lambda}{1-\gamma}\limsup_{t\to 0}\int_\Omega \frac{(u_* + t\eta)^{1-\gamma} - u_*^{1-\gamma}}{t}\,d\xi \leq \int_\Omega [\nabla_H u_* \nabla_H \eta + \phi_{u_*} u_* \eta]\,d\xi. \tag{3.2}$$

Notice that

$$\int_\Omega \frac{(u_* + t\eta)^{1-\gamma} - u_*^{1-\gamma}}{t}\,d\xi = (1-\gamma)\int_\Omega (u_* + t\eta\zeta)^{-\gamma}\eta\,d\xi,$$

where $\zeta(x) \in (0,1)$ and

$$\big(u_*(x) + t\eta(x)\zeta(x)\big)^{-\gamma}\eta(x) \to u_*(x)^{-\gamma}\eta(x), \quad \text{a.e. } x \in \Omega, t \to 0.$$

Since $(u_*(x) + t\eta(x)\zeta(x))^{-\gamma}\eta(x) \geq 0$, using Fatou's Lemma, from (3.2), it holds

$$\frac{\lambda}{1-\gamma}\int_\Omega u_*^{-\gamma}\eta\,d\xi \leq \int_\Omega [\nabla_H u_* \nabla_H \eta + \phi_{u_*} u_* \eta]\,d\xi.$$

Using a similar approach, the above equation also holds for $0 \le \eta \in S_0^1(\Omega)$, that is

$$\int_\Omega (\nabla_H u_* \nabla_H \eta + \phi_{u_*} u_* \eta)\, d\xi - \frac{\lambda}{1-\gamma} \int_\Omega u_*^{-\gamma} \eta\, d\xi \ge 0, \quad \eta \in S_0^1(\Omega), \eta \ge 0. \tag{3.3}$$

Thus,

$$-\Delta_H u_* + \phi_{u_*} u_* \ge 0.$$

Note that $\phi_{u_*}(\xi) > 0$ for any $\xi \in \Omega$, $u_* \ge 0$ and $u_* \not\equiv 0$. According to the maximum principle (see [3, 4]), $u_* > 0$ in Ω.

Step 2. We prove that u_* satisfies (2.5) for $\mu = 1$. Let $\delta > 0$ and define $h : [-\delta, \delta] \to \mathbb{R}$ by $h(t) = I_\lambda(u_* + tu_*)$, then h attains its minimum at $t = 0$, and it holds that

$$h'(0) = \|u_*\|^2 + \int_\Omega \phi_{u_*} u_*^2\, d\xi - \lambda \int_\Omega |u_*|^{1-\gamma}\, d\xi = 0. \tag{3.4}$$

We take $\eta \in S_0^1(\Omega)\backslash\{0\}$, $\varepsilon > 0$ and define $\Phi = (u_* + \varepsilon\eta)^+$. Let

$$\Omega_1 = \{x \in \Omega : u_*(x) + \varepsilon\eta(x) > 0\}, \qquad \Omega_2 = \{x \in \Omega : u_*(x) + \varepsilon\eta(x) \le 0\}.$$

Then, $\Phi|_{\Omega_1} = u_* + \varepsilon\eta$, $\Phi|_{\Omega_2} = 0$. Inserting Φ into (3.3) and using (3.4), we can get

$$\begin{aligned}
0 &\le \int_\Omega \big(\nabla_H u_* \nabla_H \Phi + \phi_{u_*} u_* \Phi - \lambda u_*^{-\gamma} \Phi\big)\, d\xi \\
&= \int_{\Omega_1} \big[\nabla_H u_* \nabla_H (u_* + \varepsilon\eta) + \phi_{u_*} u_* (u_* + \varepsilon\eta) - \lambda u_*^{-\gamma}(u_* + \varepsilon\eta)\big]\, d\xi \\
&= \int_{\Omega\backslash\Omega_2} \big[\nabla_H u_* \nabla_H (u_* + \varepsilon\eta) + \phi_{u_*} u_* (u_* + \varepsilon\phi) - \lambda u_*^{-\gamma}(u_* + \varepsilon\eta)\big]\, d\xi \\
&= \varepsilon \int_\Omega \big(\nabla_H u_* \nabla_H \eta + \phi_{u_*} u_* \eta - \lambda u_*^{-\gamma} \eta\big)\, d\xi - \int_{\Omega_2} \big[\nabla_H u_* \nabla_H (u_* + \varepsilon\eta) \\
&\quad + \phi_{u_*} u_* (u_* + \varepsilon\eta) - \lambda u_*^{-\gamma}(u_* + \varepsilon\eta)\big]\, d\xi \\
&\le \varepsilon \int_\Omega \big(\nabla_H u_* \nabla_H \eta + \phi_{u_*} u_* \eta - \lambda u_*^{-\gamma} \eta\big)\, d\xi - \varepsilon \int_{\Omega_2} (\nabla_H u_* \nabla_H \eta + \phi_{u_*} u_* \eta)\, d\xi.
\end{aligned} \tag{3.5}$$

Due to $u_* > 0$ and the measure of the domain $\Omega_2 = \{x \in \Omega : u_*(x) + \varepsilon\eta(x) \le 0\}$ tends to zero as $\varepsilon \to 0$, there holds

$$\int_{\Omega_2} (\nabla_H u_* \nabla_H \eta + \phi_{u_*} u_* \eta)\, d\xi \to 0.$$

Then, dividing by $\varepsilon > 0$ and letting $\varepsilon \to 0$ in (3.5), we have

$$\int_\Omega \big(\nabla_H u_* \nabla_H \eta + \phi_{u_*} u_* \eta - \lambda u_*^{-\gamma} \eta\big)\, d\xi \ge 0, \quad \eta \in S_0^1(\Omega).$$

The above inequality also holds for $-\eta$, and we can get

$$\int_\Omega \left(\nabla_H u_* \nabla_H \eta + \phi_{u_*} u_* \eta - \lambda u_*^{-\gamma} \eta\right) d\xi = 0, \quad \eta \in S_0^1(\Omega).$$

Then, $u_* \in S_0^1(\Omega)$ is a solution of system (1.1) for $\lambda > 0$ and $\mu = 1$.

Step 3. We prove that u_* is the unique solution of (1.1) for $\mu = 1$. We may assume that $v_\star \in S_0^1(\Omega)$ is also a solution of system (1.1), and from (2.5), we get

$$\int_\Omega \left[\nabla_H u_* \nabla_H (u_* - v_\star) + \phi_{u_*} u_* (u_* - v_\star)\right] d\xi - \lambda \int_\Omega u_*^{-\gamma} (u_* - v_\star)\, d\xi = 0, \tag{3.6}$$

and

$$\int_\Omega \left[\nabla_H v_\star \nabla_H (u_* - v_\star) + \phi_{v_\star} v_\star (u_* - v_\star)\right] d\xi - \lambda \int_\Omega v_\star^{-\gamma} (u_* - v_\star)\, d\xi = 0. \tag{3.7}$$

Combining with (3.6) and (3.7), it holds that

$$\|u_* - v_\star\|^2 + \int_\Omega (\phi_{u_*} u_* - \phi_{v_\star} v_\star)(u_* - v_\star)\, d\xi = \lambda \int_\Omega \left(u_*^{-\gamma} - v_\star^{-\gamma}\right)(u_* - v_\star)\, d\xi. \tag{3.8}$$

For $\gamma \in (0,1)$, $u_*, v_\star > 0$ in Ω, and

$$\int_\Omega \left(u_*^{-\gamma} - v_\star^{-\gamma}\right)(u_* - v_\star)\, d\xi \le 0.$$

Hence, by (3.8) and Lemma 2.1(4), we get

$$\|u_* - v_\star\|^2 \le 0,$$

that is say

$$\|u_* - v_\star\|^2 = 0,$$

that is $u_* = v_\star$. Hence, $u_* \in S_0^1(\Omega)$ is the unique solution of system (1.1). □

Proof of Theorem 1.2 We prove Theorem 1.2 in two steps.

Step 1. Suppose that $0 < \lambda < \Lambda_0$, then system (1.1) admits a positive solution u_* such that $I_\lambda(u_\lambda) = m < 0$.

In fact, we claim that there exists $u_\lambda \in B_\rho$, such that $I_\lambda(u_\lambda) = m < 0$. By the definition of m, we know that there exists a minimizing sequence $\{u_n\} \subset B_\rho \subset U^+$ such that $\lim_{n\to\infty} I_\lambda(u_n) = m < 0$. Since $\{u_n\}$ is bounded in B_ρ, we may assume that, up to a subsequence still denoted by itself, there exists $u_\lambda \in S_0^1(\Omega)$, such that

$$\begin{cases} u_n \rightharpoonup u_\lambda, & \text{weakly in } S_0^1(\Omega), \\ u_n \to u_\lambda, & \text{strongly in } L^p(\Omega)\ (1 \le p < 4), \\ u_n(x) \to u_\lambda(x), & \text{a.e. in } \Omega, \end{cases}$$

as $n \to \infty$. Set $w_n = u_n - u_\lambda$, and using Brézis-Lieb's Lemma (see [5]), one has

$$\|u_n\|^2 = \|w_n\|^2 + \|u_\lambda\|^2 + o(1). \tag{3.9}$$

Hence, by Lemma 2.3, we can infer that

$$m = \lim_{n\to\infty} I_\lambda(u_n) = I_\lambda(u_\lambda) + \lim_{n\to\infty} \frac{1}{2}\|w_n\|^2 \geq I_\lambda(u_\lambda), \tag{3.10}$$

which implies that $m \geq I_\lambda(u_\lambda)$. Since B_ρ is closed and convex, one has that $u_\lambda \in B_\rho$. Thus, we obtain $I_\lambda(u_\lambda) = m < 0$ and $u_\lambda \not\equiv 0$ in Ω. From the above arguments, we know that u_λ is a local minimizer of I_λ.

Now, we prove that u_λ is a critical point of I_λ. Note that $u_\lambda \geq 0$ and $u_\lambda \not\equiv 0$. Then, for any $\psi \in U^+ \subset S_0^1(\Omega)$, let $t > 0$ such that $u_\lambda + t\psi \in S_0^1(\Omega)$, and one has

$$\begin{aligned} 0 &\leq I_\lambda(u_\lambda + t\psi) - I_\lambda(u) \\ &= \frac{1}{2}\|u_\lambda + t\psi\|^2 - \frac{1}{4}\int_\Omega \phi_{u_\lambda+t\psi}(u_\lambda + t\psi)^2\, d\xi - \frac{\lambda}{1-\gamma}\int_\Omega |u_\lambda + t\psi|^{1-\gamma}\, d\xi \\ &\quad - \frac{1}{2}\|u_\lambda\|^2 + \frac{1}{4}\int_\Omega \phi_{u_\lambda} u_\lambda^2\, d\xi + \frac{\lambda}{1-\gamma}\int_\Omega |u_\lambda|^{1-\gamma}\, d\xi. \end{aligned} \tag{3.11}$$

Actually, from (3.11), we can see that

$$\begin{aligned} &\frac{\lambda}{1-\gamma}\int_\Omega \left[(u_\lambda + t\psi)^{1-\gamma} - u_\lambda^{1-\gamma}\right] d\xi \\ &\quad \leq \frac{1}{2}\left(\|u_\lambda + t\psi\|^2 - \|u_\lambda\|^2\right) - \frac{1}{4}\int_\Omega \left[\phi_{u_\lambda+t\psi}(u_\lambda + t\psi)^2 - \phi_{u_\lambda} u_\lambda^2\right] d\xi. \end{aligned}$$

Dividing by $t > 0$ and passing to the limit as $t \to 0^+$, it holds that

$$\frac{\lambda}{1-\gamma}\liminf_{t\to 0^+}\int_\Omega \frac{(u_\lambda + t\psi)^{1-\gamma} - u_\lambda^{1-\gamma}}{t}\, d\xi \leq \int_\Omega \nabla_H u_\lambda \nabla_H \psi\, d\xi - \int_\Omega \phi_{u_\lambda} u_\lambda \psi\, d\xi. \tag{3.12}$$

Notice that

$$\frac{\lambda}{1-\gamma}\int_\Omega \frac{(u_\lambda + t\psi)^{1-\gamma} - u_\lambda^{1-\gamma}}{t}\, d\xi = \lambda\int_\Omega (u_\lambda + \zeta t\psi)^{-\gamma}\psi\, d\xi.$$

Where $\zeta \to 0^+$ and $(u_\lambda + \zeta t\psi)^{-\gamma}\psi \to u_\lambda^{-\gamma}\psi$ a.e. $x \in \Omega$ as $t \to 0^+$, since $(u_\lambda + \zeta t\psi)^{-\gamma}\psi \geq 0$. By Fatou's Lemma, one has

$$\lambda\int_\Omega u_\lambda^{-\gamma}\psi\, d\xi \leq \frac{\lambda}{1-\gamma}\liminf_{t\to 0^+}\int_\Omega \frac{(u_\lambda + t\psi)^{1-\gamma} - u_\lambda^{1-\gamma}}{t}\, d\xi.$$

Therefore, we deduce from (3.12) and the above estimate that

$$\int_\Omega \nabla_H u_\lambda \nabla_H \psi\, d\xi - \int_\Omega \phi_{u_\lambda} u_\lambda \psi\, d\xi - \lambda\int_\Omega u_\lambda^{-\gamma}\psi\, d\xi \geq 0, \quad \psi \geq 0. \tag{3.13}$$

Since $I_\lambda(u_\lambda) < 0$, this, together with Lemma 2.5, implies that $u_\lambda \notin S_\rho$; therefore, we obtain $\|u_\lambda\| < \rho$. For u_λ, there is $\delta_1 \in (0,1)$ such that $(1+t)u_\lambda \in B_\rho$ for $|t| \leq \delta_1$. Define $k : [-\delta_1, \delta_1]$ by $k(t) = I_\lambda((1+t)u_\lambda)$. Clearly, $k(t)$ achieves its minimum at $t = 0$, namely

$$k'(t)|_{t=0} = \|u_\lambda\|^2 - \int_\Omega \phi_{u_\lambda} u_\lambda^2 \, d\xi - \lambda \int_\Omega u_\lambda^{1-\gamma} \, d\xi = 0. \tag{3.14}$$

Suppose that for any $\nu \in S_0^1(\Omega)$, $\epsilon > 0$. Define $\Psi \in U^+$ by

$$\Psi = (u_\lambda + \epsilon\nu)^+.$$

Combining with (3.13) and (3.14), we get

$$\begin{aligned}
0 &\leq \int_\Omega \left(\nabla_H u_\lambda \nabla_H \Psi - \phi_{u_\lambda} u_\lambda \Psi - \lambda u_\lambda^{-\gamma} \Psi\right) d\xi \\
&= \int_{\{u_\lambda + \epsilon\nu > 0\}} \left[\nabla_H u_\lambda \nabla_H (u_\lambda + \epsilon\nu) - \phi_{u_\lambda} u_\lambda (u_\lambda + \epsilon\nu) - \lambda u_\lambda^{-\gamma}(u_\lambda + \epsilon\nu)\right] d\xi \\
&= \left(\int_\Omega - \int_{\{u_\lambda + \epsilon\nu \leq 0\}}\right) \left[\nabla_H u_\lambda \nabla_H (u_\lambda + \epsilon\nu) - \phi_{u_\lambda} u_\lambda (u_\lambda + \epsilon\nu) - \lambda u_\lambda^{-\gamma}(u_\lambda + \epsilon\nu)\right] d\xi \\
&= \|u_\lambda\|^2 - \int_\Omega \phi_{u_\lambda} u_\lambda^2 \, d\xi - \lambda \int_\Omega |u_\lambda|^{1-\gamma} \, d\xi \\
&\quad + \epsilon \int_\Omega \left(\nabla_H u_\lambda \nabla_H \nu - \phi_{u_\lambda} u_\lambda \nu - \lambda u_\lambda^{-\gamma} \nu\right) d\xi \\
&\quad - \int_{\{u_\lambda + \epsilon\nu \leq 0\}} \left[\nabla_H u_\lambda \nabla_H (u_\lambda + \epsilon\nu) - \phi_{u_\lambda} u_\lambda (u_\lambda + \epsilon\nu) - \lambda u_\lambda^{-\gamma}(u_\lambda + \epsilon\nu)\right] d\xi \\
&\leq \epsilon \int_\Omega \left(\nabla_H u_\lambda \nabla_H \nu - \phi_{u_\lambda} u_\lambda \nu - \lambda u_\lambda^{-\gamma} \nu\right) d\xi \\
&\quad - \epsilon \int_{\{u_\lambda + \epsilon\nu \leq 0\}} (\nabla_H u_\lambda \nabla_H \nu - \phi_{u_\lambda} u_\lambda \nu) \, d\xi.
\end{aligned} \tag{3.15}$$

Since the measure of the domain of integration $\{u_\lambda + \epsilon\nu \leq 0\} \to 0$ as $\epsilon \to 0$, it follows that

$$\lim_{\epsilon \to 0} \int_{u_\lambda + \epsilon\nu \leq 0} (\nabla_H u_\lambda \nabla_H \nu - \phi_{u_\lambda} u_\lambda \nu) \, d\xi = 0.$$

Therefore, dividing by ϵ and setting $\epsilon \to 0$ in (3.15), one gets

$$\int_\Omega \nabla_H u_\lambda \nabla_H \nu \, d\xi - \int_\Omega \phi_{u_\lambda} u_\lambda \nu \, d\xi - \lambda \int_\Omega u_\lambda^{-\gamma} \nu \, d\xi \geq 0. \tag{3.16}$$

By the arbitrariness of ν, the inequality also holds for $-\nu$

$$\int_\Omega \nabla_H u_\lambda \nabla_H \nu \, d\xi - \int_\Omega \phi_{u_\lambda} u_\lambda \nu \, d\xi - \lambda \int_\Omega u_\lambda^{-\gamma} \nu \, d\xi = 0. \tag{3.17}$$

Since $u_\lambda \geq 0$ and $u_\lambda \not\equiv 0$, from (3.17), there holds

$$-\Delta_H u_\lambda \geq \phi_{u_\lambda} u_\lambda \geq 0.$$

Note that $u_\lambda \geq 0$ and $u_\lambda \not\equiv 0$, then by the maximum principle (see [3, 4]), it suggests that $u_\lambda > 0$ in Ω. From the above arguments, we obtain that u_λ is a positive solution of system (1.1) with $I_\lambda(u_\lambda) = m < 0$.

Step 2. Assume that $0 < \lambda < \Lambda_0$, then system (1.1) has a positive solution v_* such that $I_\lambda(v_*) > 0$.

In fact, by Lemma 2.5, I_λ satisfies the geometric structure of the mountain pass Lemma. By Lemma 2.3, there exists a sequence $\{v_n\}$ such that

$$|dI_\lambda|(v_n) \to 0, \qquad I_\lambda(v_n) \to c, \quad \text{as } n \to \infty.$$

We know that $\{v_n\} \subset S_0^1(\Omega)$ has a convergent subsequence, still denoted by $\{v_n\}$, we may assume that $v_n \to v_*$ in $S_0^1(\Omega)$, and

$$I_\lambda(v_*) = \lim_{n\to\infty} I_\lambda(v_n) = c, \qquad |dI_\lambda|(v_n) \to 0.$$

Applying Theorem 1.3.1 in [6], similar to step 1, v_* satisfies problem (2.3) with $I_\lambda(v_*) = c > 0$. Thus, v_* is the second positive solution of system (1.1). □

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Competing interests
The authors declare no competing interests.

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