# Higher order ( $n, m$ )-Drazin normal operators 

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#### Abstract

The purpose of this paper is to introduce and study the structure of $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal operators. This is a generalization of the class of $p$-tuple of $n$-normal operators. We consider a generalization of these single variable $n$ - $\mathcal{D}$-normal and $(n, m)$-D -normal operators and explore some of their basic properties.


Keywords: $\mathcal{D}$-normal; $n$ - $\mathcal{D}$-normal operators; ( $n, m$ )-D -normal; Joint normal operators; Joint $n$-normal tuples

## 1 Introduction

Let $\mathbf{K}$ be a complex Hilbert space, $\mathbf{B}[\mathbf{K}]$ be the algebra of all bounded linear operators defined in $\mathbf{K}$. For every $N$ in $\mathbf{B}[\mathbf{K}]$, denote $\operatorname{ker}(N)$ as the null space and $N^{*}$ as the adjoint of $R$, respectively.

The Drazin inverse of bounded linear operators on complex Banach spaces was introduced by Caradus [14] and King [26]. For more detailed study and applications of the concepts of Drazin invertibility, we invite the interested readers to refer to ([11, 12, 32]). It is well known that the Drazin inverse of the operator $N \in \mathbf{B}[\mathbf{K}]$ is the unique operator $N^{\mathcal{D}} \in \mathbf{B}[\mathbf{K}]$ if it exists and satisfies the following conditions

$$
\left\{\begin{array}{l}
N^{\mathcal{D}} N=N N^{\mathcal{D}}, \\
\left(N^{\mathcal{D}}\right)^{2} N=N^{\mathcal{D}}, \\
N^{v+1} N^{\mathcal{D}}=N^{v} \quad \text { for some integer } v \geq 0 .
\end{array}\right.
$$

We denote by $\mathbf{B}_{d}[\mathbf{K}]$ the set of all Drazin invertible elements of $\mathbf{B}[\mathbf{K}]$.
For $N \in \mathbf{B}_{d}[\mathbf{K}]$, it was observed that the Drazin inverse $N^{\mathcal{D}}$ of $N$ satisfies the following conditions

$$
\left\{\begin{array}{l}
\left(N^{*}\right)^{\mathcal{D}}=\left(N^{\mathcal{D}}\right)^{*} \\
\left(N^{k}\right)^{\mathcal{D}}=\left(N^{\mathcal{D}}\right)^{k}, \quad \forall k \in \mathbb{N} .
\end{array}\right.
$$

Moreover, it was observed that if $N \in \mathbf{B}_{d}[\mathbf{K}]$ and $T \in \mathbf{B}[\mathbf{K}]$ is an invertible operator, then $T^{-1} N T \in \mathbf{B}_{d}[\mathbf{K}]$ and $\left(T^{-1} N T\right)^{\mathcal{D}}=T^{-1} N^{\mathcal{D}} T$.

[^0]Lemma $1.1([14,34])$ Let $N, T \in \mathbf{B}_{d}[\mathbf{K}]$. Then the following properties hold.
(1) NT is Drazin invertible if and only if TN is Drazin invertible. Moreover

$$
(N T)^{\mathcal{D}}=N\left[(T N)^{\mathcal{D}}\right]^{\mathcal{D}} T \quad \text { and } \quad \operatorname{ind}(N T) \leq \operatorname{ind}(T N)+1
$$

(2) If $N T=T N$, then $(N T)^{\mathcal{D}}=T^{\mathcal{D}} N^{\mathcal{D}}=N^{\mathcal{D}} T^{\mathcal{D}}, N^{\mathcal{D}} T=T N^{\mathcal{D}}$ and $N T^{\mathcal{D}}=T^{\mathcal{D}} N$.
(3) If $N T=T N=0$, then $(N+T)^{\mathcal{D}}=N^{\mathcal{D}}+T^{\mathcal{D}}$.

The success of the theory of normal operators on Hilbert spaces has led to several attempts to generalize it to large classes of operators, including normal operators.

For $N, T \in \mathbf{B}[\mathbf{K}]$, we set $[N, T]=N T-T N$. An operator $N \in \mathbf{B}[\mathbf{K}]$ is called
(i) normal if $\left[N, N^{*}\right]=0([17,22,30])$,
(ii) $n$-normal if $\left[N^{n}, N^{*}\right]=0([2,24,25])$,
(iii) $(n, m)$-normal if $\left[N^{n},\left(N^{m}\right)^{*}\right]=0$, where $n, m$ are two nonnegative integers $([1,3,4])$.

These concepts of normality, studied for $N \in \mathbf{B}[\mathbf{K}]$, have been extended to the class of Drazin inverse of bounded linear operators on $\mathbf{K}$ as follows: For $R \in \mathbf{B}_{d}[\mathbf{K}], R$ is said to be
(i) $\mathcal{D}$-normal if $\left[N^{\mathcal{D}}, N^{*}\right]=0$ ([19]),
(ii) $n$-power $\mathcal{D}$-normal if $\left[\left(N^{\mathcal{D}}\right)^{n}, N^{*}\right]=0$ ([19]),
(iii) $(n, m)$-power $\mathcal{D}$-normal if $\left[\left(N^{\mathcal{D}}\right)^{n},\left(N^{*}\right)^{m}\right]=0$ for some positive integers $n$ and $m$ ([28]),
The study of $p$-tuples of operators has received great interest from many authors in recent years. Some developments in this field have been presented in [7, 9, 10, 13, 15, 16, 23, 27, 29], and further references can be found therein.

Given a $p$-tuple $\mathbf{N}:=\left(N_{1}, \ldots, N_{p}\right) \in \mathbf{B}[\mathbf{K}]^{p}$, we define $\left[\mathbf{N}^{*}, \mathbf{N}\right] \in \mathbf{B}[\mathbf{K} \oplus \cdots \oplus \mathbf{K}]$ as the self-commutator of $\mathbf{N}$, which is given by

$$
\left[\mathbf{N}^{*}, \mathbf{N}\right]_{k, l}:=\left[N_{l}^{*}, N_{k}\right] \quad \forall(k, l) \in\{1, \ldots, p\}^{2}
$$

where $\mathbf{N}^{*}:=\left(N_{1}^{*}, \ldots, N_{p}^{*}\right)$.
We shall say, following ([7, 18]), that $\mathbf{N}$ is jointly hyponormal if

$$
\left[\mathbf{N}^{*}, \mathbf{N}\right]=\left(\begin{array}{cccc}
{\left[N_{1}^{*}, N_{1}\right]} & {\left[N_{2}^{*}, N_{1}\right]} & \cdots & {\left[N_{p}^{*}, N_{1}\right]} \\
{\left[N_{1}^{*}, N_{2}\right]} & {\left[N_{2}^{*}, N_{2}\right]} & \cdots & {\left[N_{p}^{*}, N_{2}\right]} \\
\vdots & \vdots & \vdots & \vdots \\
{\left[N_{1}^{*}, N_{p}\right]} & {\left[N_{2}^{*}, N_{p}\right]} & \cdots & {\left[N_{p}^{*}, N_{p}\right]}
\end{array}\right)
$$

is a positive operator on $\mathbf{K} \oplus \cdots \oplus \mathbf{K}$, or equivalently

$$
\sum_{1 \leq i, k \leq p}\left\langle\left.\left[\begin{array}{ll}
N_{i}^{*} & N_{k}
\end{array}\right] x \right\rvert\, x\right\rangle \geq 0 \quad \forall x \in \mathbf{K} .
$$

$\mathbf{N}$ is said to be jointly normal if $\mathbf{N}$ ([8]) satisfying

$$
\begin{cases}{\left[N_{k}, N_{l}\right]=0,} & k, l \in\{1, \ldots, p\} \\ {\left[N_{k}^{*}, N_{k}\right]=0,} & k=1, \ldots, p\end{cases}
$$

Recently, in [5], the author has introduced the concept of jointly $n$-normal tuple as follows: $\mathbf{N}=\left(N_{1}, \ldots, N_{p}\right) \in \mathbf{B}[\mathbf{K}]^{p}$ is said to be joint $n$-normal operators if $\mathbf{R}$ satisfying

$$
\begin{cases}{\left[N_{k}, N_{l}\right]=0,} & k, l \in\{1, \ldots, p\} \\ {\left[N_{k}^{n}, N_{k}^{*}\right]=0,} & k=1, \ldots, p,\end{cases}
$$

for some positive integer $n$.
Let $\mathbf{N}=\left(N_{1}, \ldots, N_{p}\right) \in \mathbf{B}_{d}[\mathbf{K}]^{p}$. We set $\mathbf{N}^{\mathcal{D}}:=\left(N_{1}^{\mathcal{D}}, \ldots, N_{p}^{\mathcal{D}}\right)$.
The present paper proposes and studies the concept of $p$-tuples of $(n, m)$ - $\mathcal{D}$-normal operators. These are natural generalizations of $\mathcal{D}$-normal, $n$-power $\mathcal{D}$-normal, and $(n, m)$ power $\mathcal{D}$-normal single operators as done in [19, 28]. For more details on some classes of Drazin inverse operators, the reader is invited to consult [20, 21, 33].
This paper has been organized into two sections. In section two, we introduce the class of $p$-tuples of $(n, m)$ - $\mathcal{D}$-normal operators associated with Drazin invertible operators using their Drazin inverses. Some properties of this class are studied along with examples. In the third section, the tensor product of some members of this class is discussed.

## 2 p-tuple of ( $n, m$ )-Drazin normal oreators

In this section, we introduce and study the class of jointly ( $n, m$ )-power $D$-normal multioperators.

Definition 2.1 Let $\mathbf{N}:=\left(N_{1}, \ldots, N_{p}\right) \in \mathbf{B}_{d}[\mathbf{K}]^{p}$. We said that $\mathbf{N}$ is $p$-tuple of $(n, m)$-Drazin normal operators for some positive integers $n$ and $m$ if $\mathbf{N}$ satisfies the following conditions

$$
\left\{\begin{array}{l}
{\left[N_{k}, N_{l}\right]=0 ; \quad \forall(k, l) \in\{1, \ldots, p\}^{2}} \\
{\left[\left(N_{k}^{\mathcal{D}}\right)^{n}, N_{k}^{* m}\right]=0 \quad \forall k=1, \ldots, p}
\end{array}\right.
$$

When $n=m=1$, we said that $\mathbf{N}$ is $p$-tuple of $\mathcal{D}$-normal operators and if $m=1, \mathbf{N}$ is $p$-tuple of $n$ - $\mathcal{D}$-normal operators.

Example 2.1 Let $N \in \mathbf{B}_{d}[\mathbf{K}]$ be an ( $n, m$ )-D -normal operator, then $\mathbf{N}=(N, \ldots, N) \in$ $\mathbf{B}_{d}[\mathbf{K}]^{p}$ is $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal operators.

Example 2.2 Let $\mathbf{N}:=\left(N_{1}, \ldots, N_{p}\right) \in \mathbf{B}_{d}[\mathbf{K}]^{p}$ be commuting operators. If each $N_{k}$ is ( $n, m$ )-D -normal single operator, then $\mathbf{N}$ is $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal operators.

The following example shows that there exists a $p$-tuple of operators $\mathbf{N}=\left(N_{1}, \ldots, N_{p}\right) \in$ $\mathbf{B}_{d}(\mathbf{K})^{p}$ such that each $N_{k}$ is $(n, m)$ - $\mathcal{D}$-normal for $k=1, \ldots, p$, however $\mathbf{N}$ is not $p$-tuple of ( $n, m$ )-D-normal operators. This means that the study of these concepts is not trivial.

Example 2.3 Let $\mathbf{N}=\left(N_{1}, N_{2}\right) \in \mathbf{B}\left[\mathbb{C}^{4}\right]$ where

$$
N_{1}=\left(\begin{array}{llll}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{array}\right) \quad \text { and } \quad N_{2}=\left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

It was observed in [19] that $N_{1}$ and $N_{2}$ are in $\mathbf{B}_{d}\left[\mathbb{C}^{4}\right]$ and

$$
N_{1}^{\mathcal{D}}=\left(\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right) \text { and } N_{2}^{\mathcal{D}}=\left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

It is easy to check that $\left[N_{1}, N_{2}\right] \neq 0$ and $\left[\left(N_{j}^{\mathcal{D}}\right)^{n}, N_{j}^{* m}\right]=0$ for $j=1,2$. This means that, each $N_{j}$ is $(n, m)$-power $\mathcal{D}$-normal, while that $\mathbf{N}$ is not $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal operators.

In the the following theorem we collect some properties of $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal operators.

Theorem 2.1 Let $\mathbf{N}=\left(N_{1}, \ldots, N_{p}\right) \in \mathbf{B}_{d}[\mathbf{K}]^{p}$ be p-tuple of $(n, m)$-D-normal operators, then the following properties hold.
(1) $\mathbf{N}$ is p-tuple of $(r n, s m)$-D-normal operators for some positive integers $r$ and $s$.
(2) $\mathbf{N}^{q}:=\left(N_{1}^{q_{1}}, \ldots, N_{p}^{q_{p}}\right)$ is $p$-tuple of $(n, m)$-D-normal operators for $q=\left(q_{1}, \ldots, q_{p}\right) \in \mathbb{N}^{p}$.
(3) $\mathbf{N}^{*}=\left(N_{1}^{*}, \ldots, N_{p}^{*}\right)$ is p-tuple of $(n, m)$-D-normal operators.
(4) If $V$ is an unitary operator, then $V^{*} \mathbf{N} V:=\left(V^{*} N_{1} V, \ldots, V^{*} N_{p} V\right)$ is $p$-tuple of $(n, m)$ - $\mathcal{D}$ normal operators.

Proof (1) Since $\mathbf{N}$ is a $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal operators, it follows that $\left[N_{k}, N_{l}\right]=0$ for $k, l=1, \ldots, p$. However,

$$
\begin{aligned}
{\left[\left(N_{k}^{\mathcal{D}}\right)^{r n}, N_{k}^{*(s m)}\right] } & \left.=\left(N_{k}^{\mathcal{D}}\right)^{r n} N_{k}^{*(s m)}-N_{k}^{*(s m)} N_{k}^{\mathcal{D}}\right)^{r n} \\
& =\underbrace{\left(N_{k}^{\mathcal{D}}\right)^{n} \cdots\left(N_{k}^{\mathcal{D}}\right)^{n}}_{r \text {-times }} \cdot \underbrace{N_{k}^{* m} \cdots N_{k}^{* m}}_{s-\text { times }}-\underbrace{N_{k}^{* m} \cdots N_{k}^{* m}}_{s \text {-times }} \underbrace{\left(N_{k}^{\mathcal{D}}\right)^{n} \cdots\left(N_{k}^{\mathcal{D}}\right)^{n}}_{r \text {-times }} \\
& =\underbrace{N_{k}^{* m} \cdots N_{k}^{* m}}_{s \text {-times }} \underbrace{\left(N_{k}^{\mathcal{D}}\right)^{n} \cdots\left(N_{k}^{\mathcal{D}}\right)^{n}}_{r \text {-times }}-\underbrace{N_{k}^{* m} \cdots N_{k}^{* m}}_{s-\text { times }} \underbrace{\left(N_{k}^{\mathcal{D}}\right)^{n} \cdots\left(N_{k}^{\mathcal{D}}\right)^{n}}_{r \text {-times }} \\
& =0 .
\end{aligned}
$$

(2) If $q_{k}=1$ for all $k \in\{1, \ldots, q\}$, then $\left[N_{k}^{q_{k}}, N_{l}^{q_{l}}\right]=0$.

If $q_{k}>1$ for all $k \in\{1, \ldots, p\}$, by taking into account [29, Lemma 2.1], we have

$$
\left[N_{k}^{q_{k}}, N_{l}^{q_{l}}\right]=\sum_{\substack{\alpha+\alpha^{\prime}=q_{k}-1 \\ \beta+\beta^{\prime}=q_{l}-1}} N_{k}^{\alpha} N_{l}^{\beta}\left[N_{k}, N_{l}\right] N_{l}^{\alpha^{\prime}} N_{k}^{\beta^{\prime}}
$$

Now, under the assumption that $\mathbf{N}$ is a $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal operators, it follows that

$$
\left[N_{k}^{q_{k}}, N_{l}^{q_{l}}\right]=\sum_{\substack{\alpha+\alpha^{\prime}=q_{k}-1 \\ \beta+\beta^{\prime}=q_{l}-1}} N_{k}^{\alpha} N_{l}^{\beta}\left[N_{k}, N_{l}\right] N_{l}^{\alpha^{\prime}} N_{k}^{\beta^{\prime}}=0, \quad \forall(k, l) \in\{1, \ldots, p\}^{2}
$$

By looking that $N_{k}$ is an $(n, m)$ - $\mathcal{D}$-normal, then from [28, Proposition 2.10], we obtain that $N_{k}^{q_{k}}$ is an $(n, m)$ - $\mathcal{D}$-normal for all $k \in\{1, \ldots, q\}$. This means that $\left(N_{1}^{q_{1}}, \ldots, N_{p}^{q_{p}}\right)$ is a $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal operators.
(3) From Definition 2.1, we have under the condition that $\mathbf{N}$ is a $p$-tuple of $(n, m)$-power $\mathcal{D}$-normal operators that is

$$
\left\{\begin{array}{l}
{\left[N_{k}, N_{l}\right]=0 \text { for all }(k, l) \in\{1, \ldots, p\}^{2}} \\
{\left[\left(N_{k}^{\mathcal{D}}\right)^{n}, N_{k}^{* m}\right]=0 \text { for } k=1, \ldots, p}
\end{array}\right.
$$

and therefore,

$$
\left\{\begin{array}{l}
{\left[N_{k}^{*}, N_{l}^{*}\right]=0 \quad \text { for all }(k, l) \in\{1, \ldots, p\}^{2}} \\
{\left[\left(N_{k}^{\mathcal{D}}\right)^{* n}, N_{k}^{m}\right]=0 \quad \text { for } k=1, \ldots, q}
\end{array}\right.
$$

Hence, $\mathbf{N}^{*}$ is a $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal operators.
(4) We observe that

$$
\begin{aligned}
{\left[V^{*} N_{k} V, V^{*} N_{l} V\right] } & =\left(V^{*} N_{k} V\right)\left(V^{*} N_{l} V\right)-\left(V^{*} N_{l} V\right)\left(V^{*} N_{k} V\right) \\
& =V^{*} N_{k} N_{l} V-V^{*} N_{l} N_{k} V \\
& =V^{*}\left[N_{k}, N_{l}\right] V \\
& =0
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
{\left[\left(V^{*} N_{k} V\right)^{\mathcal{D} n},\left(V^{*} N_{k} V\right)^{* m}\right] } & =V^{*}\left(N_{k}^{\mathcal{D}}\right)^{n} V V^{*} N_{k}^{* m} V-V^{*} N_{k}^{* m} V V^{*}\left(N_{k}^{\mathcal{D}}\right)^{n} V \\
& =V^{*}\left(N_{k}^{\mathcal{D}}\right)^{n} N_{k}^{* m} V-V^{*} N_{k}^{* m}\left(N_{k}^{\mathcal{D}}\right)^{n} V \\
& =V^{*}\left[\left(N_{k}^{\mathcal{D}}\right)^{n}, N_{k}^{* m}\right] V \\
& =0
\end{aligned}
$$

Hence, $V^{*} \mathbf{N} V$ is a $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal operators.

Proposition 2.1 Let $\mathbf{N}=\left(N_{1}, \ldots, N_{p}\right) \in \mathbf{B}_{d}[\mathbf{K}]^{p}$. The following statements are true.
(1) If $\mathbf{N}$ is a p-tuple of $(n, n)$-D-normal operators then $\left(\mathbf{N}^{\mathcal{D}}\right)^{n}:=\left(\left(N_{1}^{\mathcal{D}}\right)^{n}, \ldots,\left(N_{p}^{\mathcal{D}}\right)^{n}\right)$ is a p-tuple of normal operators.
(2) If $\left(\mathbf{N}^{\mathcal{D}}\right)^{n}$ is a p-tuple of normal operators and $N_{k} N_{l}-N_{l} N_{k}=0$ for all $k, l=1, \ldots, p$, then $\mathbf{N}$ is a p-tuple of $(n, m)$-D-normal operators.

Proof (1) If $\mathbf{N}$ is a $p$-tuple of $(n, n)$ - $\mathcal{D}$-normal operators. Then we get

$$
\left[N_{k}, N_{l}\right]=0 \quad \Longrightarrow \quad\left[N_{k}^{\mathcal{D}}, N_{l}^{\mathcal{D}}\right]=0 \quad \forall k, l=1, \ldots, p .
$$

However,

$$
\left[\left(N_{k}^{\mathcal{D}}\right)^{n}, N_{k}^{* n}\right]=0 \quad \Longrightarrow \quad\left[\left(N_{k}^{\mathcal{D}}\right)^{n},\left(N_{k}^{\mathcal{D}}\right)^{* n}\right]=0, \quad \forall k \in 1, \ldots, p
$$

Therefore $\left(\mathbf{N}^{\mathcal{D}}\right)^{n}$ is a $p$-tuple of normal operators.
(2) Since $\left(\mathbf{N}^{\mathcal{D}}\right)^{n}$ is a $p$-tuple of normal operators, we have that

$$
\left[\left(N_{k}^{\mathcal{D}}\right)^{n},\left(N_{k}^{\mathcal{D}}\right)^{* n}\right]=0, \quad \text { for each } k=1, \ldots, p
$$

Moreover, it is well known that $\left[N_{k}^{\mathcal{D}}, N_{k}\right]=0$ for each $k=1, \ldots, p$ and hence

$$
\left[\left(N_{k}^{\mathcal{D}}\right)^{n}, N_{k}\right]=0, \quad k=1, \ldots, p
$$

By taking into account the Fugled-Putnam theorem ([31]), it follows that $\left[\left(N_{k}^{\mathcal{D}}\right)^{n}, N_{k}^{* m}\right]=0$ for each $k=1, \ldots, p$. Therefore $\mathbf{N}$ is a $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal operators.

Proposition 2.2 Let $\mathbf{N}=\left(N_{1}, \ldots, N_{p}\right) \in \mathbf{B}_{d}[\mathbf{K}]^{p}$. The following assertions hold.
(1) If $\mathbf{N}$ is a p-tuple of $(n, m)$-D-normal and a p-tuple of $(n+1, m)$-D-normal operators, then $\mathbf{N}$ is a p-tuple of $(n+2, m)$-D -normal operators.
(2) If $\mathbf{N}$ is a p-tuple of $(n, m)$-D-normal and a p-tuple of $(n, m+1)$-D-normal operators, then $\mathbf{N}$ is a p-tuple of $(n, m+2)$ - $\mathcal{D}$-normal operators.

Proof Since $\mathbf{N}$ is a $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal and a $p$-tuple of $(n+1, m)$ - $\mathcal{D}$-normal operators, we have

$$
\left\{\begin{array}{l}
{\left[N_{k}, N_{l}\right]=0 \quad \forall k, l=1, \ldots, p} \\
{\left[\left(N_{k}^{\mathcal{D}}\right)^{n}, N_{k}^{* m}\right]=0 \quad k=1, \ldots, p} \\
{\left[\left(N_{k}^{\mathcal{D}}\right)^{n+1}, N_{k}^{* m}\right]=0, \quad k=1, \ldots, p .}
\end{array}\right.
$$

This implies that

$$
\left\{\begin{array}{l}
{\left[N_{k}, N_{l}\right]=0, \quad \forall k, l=1, \ldots, p} \\
\left(N_{k}^{\mathcal{D}}\right)^{n}\left[N_{k}^{\mathcal{D}} N_{k}^{* m}-N_{k}^{* m} N_{k}^{\mathcal{D}}\right]=0, \quad k=1, \ldots, p,
\end{array}\right.
$$

and therefore,

$$
\left\{\begin{array}{l}
{\left[N_{k}, N_{l}\right]=0 \quad \forall k, l=1, \ldots, p} \\
{\left[\left(N_{k}^{\mathcal{D}}\right)^{n+2}, N_{k}^{* m}\right]=0, \quad k=1, \ldots, p .}
\end{array}\right.
$$

So, $\mathbf{N}$ is a $p$-tuple of $(n+2, m)$ - $\mathcal{D}$-normal operators.
(2) The proof of the statement (2) is similar to the proof of statement (1), so we omit it.

Proposition 2.3 Let $\mathbf{N}=\left(N_{1}, \ldots, N_{p}\right) \in \mathbf{B}_{d}[\mathbf{K}]^{p}$, the following statements hold:
(1) If $\mathbf{N}$ is a p-tuple of $\left(n_{1}, m\right)$ - $\mathcal{D}$-normal and a p-tuple of $\left(n_{2}, m\right)$ - $\mathcal{D}$-normal operators, then $\mathbf{N}$ is a p-tuple of $\left(n_{1}+n_{2}, m\right)$-D -normal operators.
(2) If $\mathbf{N}$ is a p-tuple of $\left(n, m_{1}\right)$ - $\mathcal{D}$-normal operators and a p-tuple of $\left(n, m_{2}\right)$ - $\mathcal{D}$-normal operators, then $\mathbf{N}$ is a p-tuple of $\left(n, m_{1}+m_{2},\right)$-D-normal operators.
(3) If $\mathbf{N}$ is a p-tuple of $\left(n_{1}, m\right)$ - $\mathcal{D}$-normal and a $p$-tuple of $\left(n_{2}, m\right)$ - $\mathcal{D}$-normal operators, then $\mathbf{N}$ is a p-tuple of $\left(r n_{1}+s n_{2}, m\right)$-D-normal operators for $r, s \in \mathbf{N}$.
(4) If $\mathbf{N}$ is a p-tuple of $\left(n, m_{1}\right)$-D-normal and a p-tuple of $\left(n, m_{2}\right)$ - $\mathcal{D}$-normal operators, then $\mathbf{N}$ is a p-tuple of $\left(n, r m_{1}+s m_{2},\right)$-D-normal operators for $r, s \in \mathbb{N}$.

Proof (1) We have $\left[N_{k}, N_{l}\right]=0$ for $k, l=1, \ldots, p$ and moreover for $k=1, \ldots, p$,

$$
\begin{aligned}
{\left[\left(N_{k}^{\mathcal{D}}\right)^{n_{1}+n_{2}}, N_{k}^{* m}\right] } & =\left(N_{k}^{\mathcal{D}}\right)^{n_{1}+n_{2}} N_{k}^{* m}-N_{k}^{* m}\left(N_{k}^{\mathcal{D}}\right)^{n_{1}+n_{2}} \\
& =\left(N_{k}^{\mathcal{D}}\right)^{n_{1}}\left[\left(N_{k}^{\mathcal{D}}\right)^{n_{2}}, N_{k}^{* m}\right] \\
& =0 .
\end{aligned}
$$

(2) We have $\left[N_{k}, N_{l}\right]=0$ for $k, l=1, \ldots, p$ and moreover for $k=1, \ldots, p$,

$$
\begin{aligned}
{\left[\left(N_{k}^{\mathcal{D}}\right)^{n}, N_{k}^{*\left(m_{1}+m_{2}\right)}\right] } & =\left(N_{k}^{\mathcal{D}}\right)^{n} N_{k}^{*\left(m_{1}+m_{2}\right)}-N_{k}^{*\left(m_{1}+m_{2}\right)}\left(N_{k}^{\mathcal{D}}\right)^{n_{1}+n_{2}} \\
& =\left[\left(N_{k}^{\mathcal{D}}\right)^{n}, N_{k}^{* m_{1}}\right] N_{k}^{* m_{2}} \\
& =0 .
\end{aligned}
$$

Therefore, the required results are satisfied.

Theorem 2.2 Let $\mathbf{N}=\left(N_{1}, \ldots, N_{p}\right) \in \mathbf{B}_{d}[\mathbf{K}]^{p}$ such that

$$
\operatorname{ker}\left(\mathbf{N}^{\mathcal{D}}\right):=\bigcap_{1 \leq k \leq p} \operatorname{ker} N_{k}^{\mathcal{D}}=\{0\} .
$$

If $\mathbf{N}$ is a p-tuple of $\left(n_{1}, m\right)$-D-normal and a $p$-tuple of $\left(n_{2}, m\right)$-D-normal operators for some positive integer $n_{1}, n_{2}$ and $m$, then, $\mathbf{N}$ is a $p$-tuple of $\left(\max \left\{n_{1}, n_{2}\right\}-\min \left\{n_{1}, n_{2}\right\}, m\right)$ - $\mathcal{D}$ normal operators. In particular, if $\mathbf{N}$ is jointly $(n, 1)$ - $\mathcal{D}$-normal and a p-tuple of $(n+1,1)$ - $\mathcal{D}$ normal operators, then $\mathbf{N}$ is p-tuple of $\mathcal{D}$-normal operators.

Proof We have $\left[N_{k}, N_{l}\right]=0$ for all $(k, l) \in\{1, \ldots, p\}^{2}$. Moreover, for each $k=1, \ldots, p$, we have

$$
\left\{\begin{array}{l}
{\left[\left(N_{k}^{\mathcal{D}}\right)^{n_{1}} N_{k}^{* m}\right]=0} \\
{\left[\left(N_{k}^{\mathcal{D}}\right)^{n_{2}}, N_{k}^{* m}\right]=0}
\end{array}\right.
$$

Considering the case where $n_{1} \geq n_{2}$, we get

$$
\begin{aligned}
{\left[\left(N_{k}^{\mathcal{D}}\right)^{n_{1}}, N_{k}^{* m}\right]=0 } & \left.\Longrightarrow\left(N_{k}^{\mathcal{D}}\right)^{n_{2}}\left[\left(N_{k}\right)^{\mathcal{D}}\right)^{n_{1}-n_{2}}, N_{k}^{* m}\right]=0 \\
& \Longrightarrow\left[\left(N_{k}^{\mathcal{D}}\right)^{n_{1}-n_{2}}, N_{k}^{* m}\right]=0
\end{aligned}
$$

and hence $\mathbf{N}$ is a $p$-tuple of $\left(n_{1}-n_{2}, m\right)$ - $\mathcal{D}$-normal operators.
Proposition 2.4 Let $\mathbf{N}=\left(N_{1}, \ldots, N_{p}\right) \in \mathbf{B}[\mathbf{K}]^{p}$ be commuting tuple of Drazin invertible operators. For $n, m \in \mathbb{N}$, set

$$
\mathbf{N}^{\prime}=\left(N_{1}^{\prime}, \ldots, N_{p}^{\prime}\right)=\left(\left(N_{1}^{\mathcal{D}}\right)^{n}+N 1^{* m}, \ldots,\left(N_{p}^{\mathcal{D}}\right)^{n}+N_{p}^{* m}\right)
$$

and

$$
\mathbf{N}^{\prime \prime}=\left(N_{1}^{\prime}, \ldots, N_{p}^{\prime \prime}\right)=\left(\left(N_{1}^{\mathcal{D}}\right)^{n}-N_{1}^{* m}, \ldots,\left(N_{p}^{\mathcal{D}}\right)^{n}-N_{p}^{* m}\right)
$$

Then the following axioms hold.
(1) $\mathbf{N}$ is a p-tuple of $n$ - $\mathcal{D}$-normal operators if and only if $\left[N_{k}^{\prime}, N_{k}^{\prime \prime}\right]=0$ for each $k=1, \ldots, p$.
(2) If $\mathbf{N}$ is a p-tuple of $(n, m)$ - $\mathcal{D}$-normal operators, then $\mathbf{Z}=\left(\left(N_{1}^{D}\right)^{n} N_{1}^{* m}, \ldots,\left(N_{p}^{\mathcal{D}}\right)^{n} N_{p}^{* m}\right)$ commutes with $\mathbf{N}^{\prime}$ and $\mathbf{N}^{\prime \prime}$.
(3) $\mathbf{N}$ is a p-tuple of $(n, m)$-D-normal operators, if and only if $\left(\mathbf{N}^{\mathcal{D}}\right)^{n}$ commutes with $\mathbf{N}^{\prime}$.
(4) $\mathbf{N}$ is a p-tuple of $(n, m)$-D -normal operators if and only if $\left(\mathbf{N}^{\mathcal{D}}\right)^{n}$ commutes with $\mathbf{N}^{\prime \prime}$.

Proof Obviously, $\left[N_{k}, N_{l}\right]=0 \forall(k, l) \in\{1, \ldots, p\}^{2}$. On the other hand,

$$
\begin{aligned}
{\left[N_{k}^{\prime}, N_{k}^{\prime \prime}\right] } & =0 \\
& \Longleftrightarrow \quad N_{k}^{\prime} N_{k}^{\prime \prime}-N_{k}^{\prime \prime} N_{k}^{\prime}=0 \\
& \Longleftrightarrow \quad\left(\left(N_{k}^{\mathcal{D}}\right)^{n}+N_{k}^{* m}\right)\left(\left(N_{k}^{\mathcal{D}}\right)^{n}-N_{k}^{* m}\right)-\left(\left(N_{k}^{\mathcal{D}}\right)^{n}-N_{k}^{* m}\right)\left(\left(N_{k}^{\mathcal{D}}\right)^{n}+N_{k}^{* m}\right)=0 \\
\Longleftrightarrow & \left(N_{k}^{\mathcal{D}}\right)^{2 n}-\left(N_{k}^{\mathcal{D}}\right)^{n} N_{k}^{* m}+N_{k}^{* m}\left(N_{k}^{\mathcal{D}}\right)^{n}-N_{k}^{* 2 m} \\
& \quad-\left(\left(N_{k}^{\mathcal{D}}\right)^{2 n}+\left(N_{k}^{\mathcal{D}}\right)^{n} N_{k}^{* m}-N_{k}^{* m}\left(N_{k}^{\mathcal{D}}\right)^{n}-N_{k}^{* 2 m}\right)=0 \\
& \quad\left(N_{k}^{\mathcal{D}}\right)^{n} N_{k}^{* m}-N_{k}^{* m}\left(N_{k}^{\mathcal{D}}\right)^{n}=0, \quad \forall k \in\{1, \ldots, p\} .
\end{aligned}
$$

This completes the proof.

Theorem 2.3 Let $\mathbf{N}=\left(N_{1}, \ldots, N_{p}\right) \in \mathbf{B}_{d}[\mathbf{K}]^{p}$ be $p$-tuple of $(n, m)$-D-normal operators for $n \geq m$. If each $N_{k}^{m}$ is a partial isometry for $k=1, \ldots, m$, then $\mathbf{N}$ is a p-tuple of $(n+m, m)-\mathcal{D}$ normal operators.

Proof Suppose $\mathbf{N}$ is $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal operators for $n \geq m$. It is easy to see that each $N_{k}$ is $(n, m)$ - $D$-normal for $1 \leq k \leq d$. Under the hypothesis that $N_{k}^{m}$ is a partial isometry, it follows from [28, Theorem 2.4] that $N_{k}$ is $(n+m, m)$ - $\mathcal{D}$-normal operator for $k=1, \ldots, p$. Consequently, $\mathbf{N}$ is a $p$-tuple of $(n+m, m)$ - $\mathcal{D}$-normal operators.

The following proposition shows that the class of $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal operators is closed subset of $\mathbf{B}_{d}[\mathbf{K}]^{p}$.

Proposition 2.5 The class of p-tuple of $(n, m)$-D-normal operators is a closed subset of $\mathbf{B}_{d}[\mathbf{K}]^{p}$.

Proof Suppose that $\left(\mathbf{N}_{k}=\left(N_{1}(k), \ldots, N_{p}(k)\right)\right)_{k} \in \mathbf{B}_{d}[\mathbf{K}]^{p}$ is a sequence of $p$-tuple of $(n, m)-$ power $\mathcal{D}$-normal operators for which

$$
\left\|\mathbf{N}_{k}-\mathbf{N}\right\|=\sup _{1 \leq j \leq p}\left(\left\|N_{j}(k)-N_{j}\right\|\right) \longrightarrow 0, \quad \text { as } k \longrightarrow \infty,
$$

where $\mathbf{N}=\left(N_{1}, \ldots, N_{p}\right) \in \mathbf{B}_{d}[\mathbf{K}]^{p}$. Obviously, for each $j \in\{1, \ldots, p\}$, we have

$$
\begin{equation*}
\lim _{k \rightarrow+\infty}\left\|N_{j}(k)-N_{j}\right\|=0 \tag{2.1}
\end{equation*}
$$

Since $\left(N_{j}(k)^{\mathcal{D}}\right)^{n} N_{j}(k)^{* m}=N_{j}(k)^{* m}\left(N_{j}(k)^{\mathcal{D}}\right)^{n}$ for each $j=1, \ldots, p$, it follows from [28, Theorem 2.4] that

$$
\left(N_{j}^{\mathcal{D}}\right)^{n} N_{j}^{* m}=N_{j}^{* m}\left(N_{j}^{\mathcal{D}}\right)^{n}, \quad \forall j \in\{1, \ldots, p\} .
$$

Moreover, for all $i, j \in\{1, \ldots, p\}$ and $k \in \mathbb{N}$, we can see that

$$
\begin{aligned}
\left\|N_{i}(k) N_{j}(k)-N_{i} N_{j}\right\| & =\left\|N_{i}(k)\left(N_{j}(k)-N_{j}\right)+\left(N_{i}(k)-N_{i}\right) N_{j}\right\| \\
& \leq\left\|N_{i}(k)\right\|\left\|N_{j}(k)-N_{j}\right\|+\left\|N_{i}(k)-N_{i}\right\|\left\|N_{j}\right\| \\
& \leq\left(\left\|N_{i}(k)-N_{i}\right\|+\left\|N_{i}\right\|\right)\left\|N_{j}(k)-N_{j}\right\|+\left\|N_{i}(k)-N_{i}\right\|\left\|N_{j}\right\| .
\end{aligned}
$$

Hence, in view of (2.1), we obtain

$$
\left\|N_{i}(k) N_{j}(k)-N_{i} N_{j}\right\| \longrightarrow 0, \quad \text { as } k \rightarrow+\infty, \forall(i, j) \in\{1, \ldots, q\}^{2}
$$

On the other hand, since $\left\{\mathbf{N}_{k}\right\}_{k}=\left\{\left(N_{1}(k), \ldots, N_{p}(k)\right)\right\}_{k}$ is a sequence of $p$-tuple of $(n, m)-\mathcal{D}-$ normal operators, then

$$
\left[N_{i}(k), N_{j}(k)\right]=0 \quad \forall(i, j) \in\{1, \ldots, p\}^{2} ; \text { and } k \in \mathbb{N} .
$$

Therefore, we immediately get

$$
\left[N_{i}, N_{j}\right]=0 \quad \forall(i, j) \in\{1,2, \ldots, p\}^{2}
$$

Therefore, $\mathbf{N}$ is a $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal operators.

Proposition 2.6 Let $\mathbf{N}=\left(N_{1}, \ldots, N_{p}\right) \in \mathbf{B}_{d}[\mathbf{K}]^{p}$ and $\mathbf{S}=\left(S_{1}, \ldots, S_{p}\right) \in \mathbf{B}_{d}[\mathbf{K}]^{p}$ be two $p-$ tuple of $(n, m)$-D-normal operators. The following statements hold.
(1) If $\left[N_{k}, S_{l}\right]=0, \forall k, l \in\{1, \ldots, p\}^{2}$ and $\left[N_{k}, S_{k}^{*}\right]=0$ for all $k \in\{1, \ldots, p\}$, then $\mathbf{N S}=$ $\left(N_{1} S_{1}, \ldots, N_{p} S_{p}\right)$ and $\mathbf{S N}=\left(S_{1} N_{1}, \ldots, S_{p} N_{p}\right)$ are p-tuple of $(n, m)$-D-normal operators.
(2) If $\left[N_{i}, S_{j}\right]=0, \forall i, j \in\{1, \ldots, p\}$ and $N_{k} S_{k}=N_{k} S_{k}^{*}=0$ for all $k \in\{1, \ldots, p\}$, then $\mathbf{N}+\mathbf{S}=$ $\left(N_{1}+S_{1}, \ldots, N_{p}+S_{p}\right)$ is a p-tuple of $(n, m)$-D-normal operators.

Proof For the statement (1), we have for all $k, l \in\{1, \ldots, q\}$,

$$
\begin{aligned}
{\left[N_{k} S_{k}, N_{l} S_{l}\right] } & =N_{k} S_{k} N_{l} S_{l}-N_{l} S_{l} N_{k} S_{k} \\
& =N_{k} N_{l} S_{k} S_{l}-N_{l} N_{k} S_{l} S_{k} \\
& =N_{k} N_{l} S_{k} S_{l}-N_{k} N_{l} S_{l} S_{k} \\
& =N_{k} N_{l}\left(S_{k} S_{l}-S_{l} S_{k}\right) \\
& =N_{k} N_{l}\left[S_{k}, S_{l}\right]=0 .
\end{aligned}
$$

On the other hand, let $k \in\{1, \ldots, p\}$, we have

$$
\begin{aligned}
\left(N_{k} S_{k}\right)^{*}\left(N_{k} S_{k}\right)^{\mathcal{D} n} & =S_{k}^{*} N_{k}^{*}\left(N_{k}^{\mathcal{D}}\right)^{n}\left(S_{k}^{\mathcal{D}}\right)^{n} \\
& =S_{k}^{*}\left(N_{k}^{\mathcal{D}}\right)^{n} N_{k}^{*}\left(S_{k}^{\mathcal{D}}\right)^{n}=S_{k}^{*}\left(N_{k}^{\mathcal{D}}\right)^{n} S_{k}^{n} N_{k}^{*} \\
& =\left(N_{k}^{\mathcal{D}}\right)^{n}\left(S_{k}^{\mathcal{D}}\right)^{n} S_{k}^{*} N_{k}^{*} \\
& =\left(N_{k} S_{k}\right)^{n \mathcal{D}}\left(N_{k} S_{k}\right)^{*} .
\end{aligned}
$$

This implies that NS is a $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal operators. In same way, we show that $\mathbf{S N}$ is a $p$-tuple of $(n . m)-\mathcal{D}$-normal operators.
(2) For all $(i, j) \in\{1, \ldots, p\}^{2}$, we have

$$
\begin{aligned}
{\left[N_{k}+S_{k}, N_{l}+S_{l}\right] } & =\left(N_{k}+S_{k}\right)\left(N_{l}+S_{l}\right)-\left(N_{l}+S_{l}\right)\left(N_{k}+S_{k}\right) \\
& =\left[N_{k}, N_{l}\right]+\left[S_{k}, S_{l}\right]+\left[N_{k}, S_{l}\right]+\left[S_{k}, N_{l}\right]=0 .
\end{aligned}
$$

Besides, for $k \in\{1,2, \ldots, p\}$, we get

$$
\begin{aligned}
\left(N_{k}\right. & \left.+S_{k}\right)^{* m}\left(\left(N_{k}+S_{k}\right)^{\mathcal{D}}\right)^{n} \\
& =\left(N_{k}+S_{k}\right)^{* m}\left(N_{k}^{\mathcal{D}}+S_{k}^{\mathcal{D}}\right)^{n} \\
& =\left(\sum_{j=0}^{m}\binom{m}{j} N_{k}^{* j} S_{k}^{* m-j}\right)\left(\sum_{j=0}^{n}\binom{n}{k}\left(N_{k}^{\mathcal{D}}\right)^{j}\left(S_{k}^{\mathcal{D}}\right)^{n-j}\right) \\
& =\left(N_{k}^{* m}+S_{k}^{* m}\right)\left(\left(N_{k}^{\mathcal{D}}\right)^{n}+\left(S_{k}^{\mathcal{D}}\right)^{n}\right) \\
& =\left(N_{k}^{* m}\left(N_{k}^{\mathcal{D}}\right)^{n}+N_{k}^{* m}\left(S_{k}^{\mathcal{D}}\right)^{n}+S_{k}^{* m}\left(N_{k}^{\mathcal{D}}\right)^{n}+S_{k}^{* m}\left(S_{k}^{\mathcal{D}}\right)^{n}\right. \\
& =\left(N_{k}^{\mathcal{D}}\right)^{n} N_{k}^{* m}+\left(S_{k}^{\mathcal{D}}\right)^{n} S_{k}^{* m} \\
& =\left(\left(N_{k}^{\mathcal{D}}\right)^{n}+\left(S_{k}^{\mathcal{D}}\right)^{n}\right)\left(N_{k}+S_{k}\right)^{* m} \\
& =\left(\sum_{j=0}^{n}\binom{n}{k}\left(N_{k}^{\mathcal{D}}\right)^{j}\left(S_{k}^{\mathcal{D}}\right)^{n-j}\right)\left(N_{k}+S_{k}\right)^{* m} \\
& =\left(\left(N_{k}+S_{k}\right)^{\mathcal{D}}\right)^{n}\left(N_{k}+S_{k}\right)^{* m} .
\end{aligned}
$$

So, $\mathbf{N}+\mathbf{S}$ is a $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal operators.

## 3 Tensor product

Let $\mathbf{N}=\left(N_{1}, \ldots, N_{p}\right) \in \mathbf{B}[\mathbf{K}]^{p}$ and $\mathbf{S}=\left(S_{1}, \ldots, S_{p}\right) \in \mathbf{B}[\mathbf{K}]^{p}$. We denote by

$$
\mathbf{N} \otimes \mathbf{S}=\left(N_{1} \otimes S_{1}, \ldots, N_{p} \otimes S_{p}\right)
$$

If $N, S \in \mathbf{B}[\mathbf{K}]$, then $N \otimes S$ is $n$-normal if and only if $N$ and $S$ are $n$-normal (see [6]) However, If $N, S \in \mathbf{B}_{d}[\mathbf{K}]$ such that $N$ and $S$ are ( $n, m$ )-D -normal operators, then $N \otimes S$ is ( $n, m$ )-D-normal (see in [28]). The following theorem studied the tensor product of two $p$-tuples of $(n, m)$ - $\mathcal{D}$-normal operators.

Theorem 3.1 Let $\mathbf{N}=\left(N_{1}, \ldots, N_{p}\right) \in \mathbf{B}_{d}[\mathbf{K}]^{p}$ and $\mathbf{S}=\left(S_{1}, \ldots, S_{p}\right) \in \mathbf{B}_{d}[\mathbf{K}]^{p}$ are two p-tuples of $(n, m)$-D-normal operators, then $\mathbf{N} \otimes \mathbf{S}$ is a p-tuple of $(n, m)$-D-normal operators.

Proof Since $\mathbf{N}=\left(N_{1}, \ldots, N_{p}\right)$ and $\mathbf{S}=\left(S_{1}, \ldots, S_{p}\right)$ are $p$-tuples of $(n, m)$ - $\mathcal{D}$-normal operators, we have all $(k, l) \in\{1, \ldots, p\}^{2}$

$$
\begin{aligned}
& {\left[\left(N_{k} \otimes S_{k}\right),\left(N_{l} \otimes S_{l}\right)\right]} \\
& \quad=\left[\left(N_{k} \otimes S_{k}\right)\left(N_{l} \otimes S_{l}\right)-\left(N_{l} \otimes S_{l}\right)\left(N_{k} \otimes S_{k}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
& =N_{k} N_{l} \otimes S_{k} S_{l}-N_{l} N_{k} \otimes S_{l} S_{k} \\
& =N_{l} N_{k} \otimes S_{l} S_{k}-N_{l} N_{k} \otimes S_{l} S_{k} \\
& =0
\end{aligned}
$$

Moreover, for all $k \in\{1, \ldots, p\}$, we have

$$
\begin{aligned}
\left(\left(N_{k} \otimes S_{k}\right)^{\mathcal{D}}\right)^{n}\left(N_{k} \otimes S_{k}\right)^{* m} & =\left(N_{k}^{\mathcal{D}}\right)^{n} N_{k}^{* m} \otimes\left(S_{k}^{\mathcal{D}}\right)^{n} S_{k}^{* m} \\
& =N_{k}^{* m}\left(N_{k}^{\mathcal{D}}\right)^{n} \otimes S_{k}^{* m}\left(S_{k}^{D}\right)^{n} \\
& =\left(N_{k} \otimes S_{k}\right)^{* m}\left(\left(N_{k} \otimes S_{k}\right)^{\mathcal{D}}\right)^{n}
\end{aligned}
$$

So, $\mathbf{N} \otimes \mathbf{S}$ is $p$-tuple of $(n, m)$ - $\mathcal{D}$-normal operators.
The converse of the above theorem need not hold in general, as shown in the following example.

Example 3.1 Let $N_{1}=\left(\begin{array}{ccc}-1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right) \in \mathcal{B}\left[\mathbb{C}^{3}\right]$ and $N_{2}=\left(\begin{array}{lll}0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0\end{array}\right) \in \mathbf{B}\left[\mathbb{C}^{3}\right]$. A direct calculation shows that

$$
N_{1} \otimes N_{1}=\left(\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

and

$$
N_{2} \otimes N_{2}=\left(\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) .
$$

Let $\mathbf{N}=\left(N_{1}, N_{2}\right)$ and $\left.\mathbf{N} \otimes \mathbf{N}=\left(N_{1} \otimes N_{1}, N_{2} \otimes N_{2}\right)\right)$. We observe that $\mathbf{N}$ is not 2-tuple of (2,3)- $\mathcal{D}$-normal operators since $N_{1} N_{2} \neq N_{2} N_{1}$. However

$$
\begin{aligned}
& \left(N_{1} \otimes N_{1}\right)^{\mathcal{D}}=N_{1} \otimes N_{1}, \quad\left(N_{2} \otimes N_{2}\right)^{\mathcal{D}}=N_{2} \otimes N_{2} \quad \text { and } \\
& \left(N_{k} \otimes N_{k}\right)^{* 3}\left(\left(N_{k} \otimes N_{k}\right)^{\mathcal{D}}\right)^{2}=\left(\left(N_{k} \otimes N_{k}\right)^{\mathcal{D}}\right)^{2}\left(N_{k} \otimes N_{k}\right)^{* 3}, \quad k \in\{1,2\} .
\end{aligned}
$$

Hence, $\mathbf{N} \otimes \mathbf{N}$ is 2-tuple of (2,3)-D-normal pairs.

In the following theorem we give the conditions under which the converse of Theorem 3.1 is true.

Theorem 3.2 Let $\mathbf{N}=\left(N_{1}, \ldots, N_{p}\right) \in \mathbf{B}_{d}[\mathbf{K}]^{p}$ and $\mathbf{S}=\left(S_{1}, \ldots, S_{p}\right) \in \mathbf{B}_{d}[\mathbf{K}]^{p}$ be a commuting p-tuple of operators. Then, if $\mathbf{N} \otimes \mathbf{S}$ is a p-tuple of $(n, n)$-D-normal operators, then and only then $\mathbf{N}$ and $\mathbf{S}$ are p-tuples of $(n, n)$-D-normal operators.

Proof Assume that $\mathbf{N} \otimes \mathbf{S}$ is a $p$-tuple of $(n, n)$ - $\mathcal{D}$-normal operators. By taking into account the statement (1) of Proposition 2.1 it follows that

$$
\begin{aligned}
\left((\mathbf{N} \otimes \mathbf{S})^{\mathcal{D}}\right)^{n} & =\left(\left(\left(N_{1} \otimes S_{1}\right)^{\mathcal{D}}\right)^{n}, \ldots,\left(\left(N_{p} \otimes S_{p}\right)^{\mathcal{D}}\right)^{n}\right) \\
& =\left(\left(N_{1}^{\mathcal{D}}\right)^{n} \otimes\left(S_{1}^{\mathcal{D}}\right)^{n}, \ldots\left(N_{p}^{\mathcal{D}}\right)^{n} \otimes\left(S_{p}^{\mathcal{D}}\right)^{n}\right)
\end{aligned}
$$

is a $p$-tuple of normal operators. From which we deduce that

$$
\left(\left(N_{k} \otimes S_{k}\right)^{\mathcal{D}}\right)^{n}=\left(R_{k}^{\mathcal{D}}\right)^{n} \otimes\left(S_{k}^{\mathcal{D}}\right)^{n}
$$

is normal for each $k=1, \ldots, p$. By [19, Propositon 3.2] it is well known that
$\left(N_{k}^{\mathcal{D}}\right)^{n} \otimes\left(S_{k}^{D}\right)^{n}$ is normal if and only if $\left(N_{k}^{\mathcal{D}}\right)^{n}$ and $\left(S_{k}^{\mathcal{D}}\right)^{n}$ are normal operators.

However, According to [19, Propositon 3.2] it is well known that $\left(N_{k}^{\mathcal{D}}\right)^{n}$ is normal if and only if that $N_{k}$ is $n$-power $\mathcal{D}$-normal and similarly, $\left(S_{k}^{\mathcal{D}}\right)^{n}$ is normal if and only if that $S_{k}$ is $n$ - $\mathcal{D}$-normal. Therefore, $\mathbf{N}$ and $\mathbf{S}$ are $p$-tuple of $(n, n)$ - $\mathbb{D}$-normal operators.
The converse follows from Theorem 3.1.

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No datasets were generated or analysed during the current study.

## Declarations

## Competing interests

The authors declare no competing interests.

## Author contributions

Hadi Obaid Alshammari conceived and designed the study, conducted the experiments, analyzed the data, and wrote the manuscript.

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