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Approximation by bivariate Bernstein–Kantorovich–Stancu operators that reproduce exponential functions

Lian-Ta Su^{1†} , Kadir Kanat^{2*} , Melek Sofyalioğlu Aksoy^{2†} and Merve Kisakol^{2†}

*Correspondence:
kadir.kanat@hbv.edu.tr
²Department of Mathematics,
Polatlı Faculty of Science and
Letters, Ankara Haci Bayram Veli
University, 06900, Ankara, Turkey
Full list of author information is
available at the end of the article
†Equal contributors

Abstract

In this study, we construct a Stancu-type generalization of bivariate Bernstein–Kantorovich operators that reproduce exponential functions. Then, we investigate some approximation results for these operators. We use test functions to prove a Korovkin-type convergence theorem. Then, we show the rate of convergence by the modulus of continuity and give a Voronovskaya-type theorem. We give a convergence comparison about bivariate Bernstein–Kantorovich–Stancu operators and their exponential form.

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Bivariate operators; Voronovskaya-type theorem

1 Introduction

The goal of approximation theory is to approximate a target function using straightforward, computable, and more useful functions. In 1912, Bernstein [1] defined the Bernstein operators for every function on the interval $[0, 1]$. Later, the various generalizations of Bernstein polynomials were investigated in [2–4].

In addition to classical Bernstein polynomials, there are many studies on two-dimensional Bernstein polynomials and generalizations, such as [5]. Different types of Bernstein–Kantorovich operators have been studied in [6–9]. In [10], for $n \in \mathbb{N}$, $f \in L_1([0, 1] \times [0, 1])$ Pop and Farcas constructed two variable Bernstein–Kantorovich-type operators $K_n : L_1(S) \rightarrow C([0, 1] \times [0, 1])$. For any $(x, y) \in S$, these operators are defined as:

$$K_n(f; x, y) = (n+1)^2 \sum_{k=0}^n \sum_{j=0}^{n-k} p_{n,k,j}(x, y) \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t, s) dt ds,$$

where $k, j \geq 0$.

In 2020, the Stancu variant of Bernstein–Kantorovich operators based on the shape parameter α was introduced [11]. Also, the Stancu variant of well-known operators such

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as Bernstein, Baskakov, and Szász was introduced in [12–19]. The bivariate form of Bernstein operators has been also studied in the literature (see, for instance, [20–27] references therein). One of these extensions is bivariate Bernstein–Kantorovich–Stancu operators. These operators are defined [28] for the functions $f \in C(A)$, $A = [0, 1] \times [0, 1]$ as

$$\begin{aligned} K_{m,n}^{\alpha,\beta}(f; x, y) \\ = (m + \beta_1 + 1)(n + \beta_2 + 1) \sum_{k=0}^m \sum_{l=0}^n p_{m,n,k}(x, y) \int_{\frac{l+\alpha_2+1}{n+\beta_2+1}}^{\frac{l+\alpha_2+1}{n+\beta_2+1}} \int_{\frac{k+\alpha_1+1}{m+\beta_1+1}}^{\frac{k+\alpha_1+1}{m+\beta_1+1}} f(t, s) dt ds \end{aligned} \quad (1)$$

and

$$p_{m,n,k}(x, y) = \binom{m}{k} \binom{n}{l} x^k (1-x)^{m-k} y^l (1-y)^{n-l},$$

where $s, t \in [0, 1]$ and $m, n \in \mathbb{N}$, $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$, $0 \leq \alpha_1 \leq \beta_1$, $0 \leq \alpha_2 \leq \beta_2$. In [22], Aral et al. gave the modification of exponential forms of Bernstein operators as follows

$$G_n f(x) = G_n(f; x) = \sum_{k=0}^n e^{-\alpha k/n} e^{-\alpha x} p_{n,k}(\alpha_n(x)) f\left(\frac{k}{n}\right), \quad x \in [0, 1], n \in \mathbb{N},$$

where

$$p_{n,k}(\alpha_n(x)) = \binom{n}{k} (\alpha_n(x))^k (1 - \alpha_n(x))^{n-k}$$

and

$$\alpha_n(x) = \frac{e^{\alpha x/n} - 1}{e^{\alpha/n} - 1}. \quad (2)$$

They defined the relation of their operators between the classical Bernstein operators as

$$G_n f(x) = \exp_\alpha(x) B_n\left(\frac{f}{\exp_\alpha}; \alpha_n(x)\right). \quad (3)$$

Here, the exponential function is symbolized as $\exp_\alpha(x) = e^{\alpha x}$, for a real parameter $\alpha > 0$. The generalization of Bernstein operators given by Aral et al. [22] is a particular case of the modification introduced by Morigi and Neamtu in [29].

In 2019, Aral et al. [30] gave the Bernstein–Kantorovich operators that reproduce exponential functions for $n \in \mathbb{N}$ and $\alpha, \beta, \mu > 0$ and $x \in [0, 1]$. They considered the operator $\tilde{K}_n : C[0, 1] \rightarrow [0, 1]$ for the functions $f \in C[0, 1]$ as

$$\tilde{K}_n(f; x) = \alpha'_{n+1}(n+1) e^{\mu x} \sum_{k=0}^n \binom{n}{k} (\alpha_{n+1}(x))^k (1 - \alpha_{n+1}(x))^{n-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} e^{\mu t} f(t) dt.$$

This paper consists of 6 sections. In Sect. 2, we give the definition of generalized bivariate Bernstein–Kantorovich–Stancu operators and we obtain some auxiliary results. In Sect. 3, we mention the rate of convergence with the help of the modulus of continuity. In Sect. 4, we present Voronovskaya-type results. In Sect. 5, we illustrate numerical examples with graphics. In Sect. 6, we give the conclusions.

2 Preliminaries

In this article, we construct bivariate Bernstein–Kantorovich–Stancu operators that reproduce exponential functions.

Definition 2.1 Let $S_{\mu,\nu} = \{(x,y) \in \mathbb{R}^2; x, y \geq 0, r_\mu + r_\nu \leq 1\} \subset S$ for each $m, n \in \mathbb{N}$ and $\mu, \nu > 0$. We define Bernstein–Kantorovich–Stancu operators for the functions $f \in C(S_{\mu,\nu})$ as

$$\begin{aligned} \tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(f; x, y) &= r'_\mu(x)r'_\nu(y)(m+\beta_1+1)(n+\beta_2+1)e^{\mu x+\nu y}\sum_{k=0}^m\sum_{l=0}^n p_{m,n,k,l}(r_\mu(x), r_\nu(y)) \\ &\times \int_{\frac{l+\alpha_2+1}{n+\beta_2+1}}^{\frac{l+\alpha_2+1}{n+\beta_2+1}} \int_{\frac{k+\alpha_1+1}{m+\beta_1+1}}^{k+\alpha_1+1} e^{-\mu t-\nu s} f(t, s) dt ds, \\ p_{m,n,k,l}(r_\mu(x), r_\nu(y)) &= \binom{m}{k} \binom{n}{l} r_\mu(x)^k (1-r_\mu(x))^{m-k} r_\nu(y)^l (1-r_\nu(y))^{n-l}, \end{aligned} \quad (4)$$

where $s, t \in [0, 1]$ and $m, n \in \mathbb{N}$, and $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$, $0 \leq \alpha_1 \leq \beta_1$, $0 \leq \alpha_2 \leq \beta_2$. Here,

$$r_\mu(x) = \frac{e^{\mu x/m+\beta_1+1}-1}{e^{\mu/m+\beta_1+1}-1}, \quad r_\nu(y) = \frac{e^{\nu y/n+\beta_2+1}-1}{e^{\nu/n+\beta_2+1}-1} \quad (5)$$

and so

$$r'_\mu(x) = \frac{\mu e^{\mu x/m+\beta_1+1}}{(e^{\mu/m+\beta_1+1}-1)(m+\beta_1+1)}, \quad (6)$$

$$r'_\nu(y) = \frac{\nu e^{\nu y/n+\beta_2+1}}{(e^{\nu/n+\beta_2+1}-1)(n+\beta_2+1)}, \quad (7)$$

$\mu, \nu > 0$ are real parameters and $\exp_{i,j}^{\mu,\nu}$ represents the exponential function defined by $\exp_{i,j}^{\mu,\nu}(t, s) := e^{i\mu t+j\nu s}$ for $0 \leq i, j \leq 4$.

Lemma 2.1 Let $m, n \in \mathbb{N}$ and $(x, y) \in S_{\mu,\nu}$. The following equalities hold:

$$\begin{aligned} \tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(1; x, y) &= e^{\frac{\mu x-\mu(\alpha_1+1)}{m+\beta_1+1} + \frac{\nu y-\nu(\alpha_2+1)}{n+\beta_2+1} + \mu x + \nu y} (e^{\frac{-\mu}{m+\beta_1+1}} - e^{\frac{\mu(x-1)}{m+\beta_1+1}} + 1)^m \\ &\times (e^{\frac{-\nu}{n+\beta_2+1}} - e^{\frac{\nu(y-1)}{n+\beta_2+1}} + 1)^n, \end{aligned}$$

$$\begin{aligned} \tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(e^{\mu t}; x, y) &= \frac{\mu}{(m+\beta_1+1)(e^{\frac{\mu}{m+\beta_1+1}}-1)} e^{\frac{\mu x}{m+\beta_1+1} + \frac{\nu y-\nu(\alpha_2+1)}{n+\beta_2+1} + \mu x + \nu y} \\ &\times (e^{\frac{-\nu}{n+\beta_2+1}} - e^{\frac{\nu(y-1)}{n+\beta_2+1}} + 1)^n, \end{aligned}$$

$$\begin{aligned} \tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(e^{\nu s}; x, y) &= \frac{\nu}{(n+\beta_2+1)(e^{\frac{\nu}{n+\beta_2+1}}-1)} e^{\frac{\nu y}{n+\beta_2+1} + \frac{\mu x-\mu(\alpha_1+1)}{m+\beta_1+1} + \mu x + \nu y} \\ &= \frac{\nu}{(n+\beta_2+1)(e^{\frac{\nu}{n+\beta_2+1}}-1)} e^{\frac{\nu y}{n+\beta_2+1} + \frac{\mu x-\mu(\alpha_1+1)}{m+\beta_1+1} + \mu x + \nu y} \end{aligned}$$

$$\begin{aligned}
& \times \left(e^{\frac{-\mu}{m+\beta_1+1}} - e^{\frac{\mu(x-1)}{m+\beta_1+1}} + 1 \right)^m, \\
\tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(e^{2\mu t};x,y) &= e^{\frac{\mu x+\mu\alpha_1+\mu xm}{m+\beta_1+1} + \frac{vy-v(\alpha_2+1)}{n+\beta_2+1} + \mu x + vy} \left(e^{\frac{-v}{n+\beta_2+1}} - e^{\frac{v(y-1)}{n+\beta_2+1}} + 1 \right)^n, \\
\tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(e^{2\nu s};x,y) &= e^{\frac{\mu x-\mu(\alpha_1+1)}{m+\beta_1+1} + \frac{vy+v\alpha_2+vyn}{n+\beta_2+1} + \mu x + vy} \left(e^{\frac{-\mu}{m+\beta_1+1}} - e^{\frac{\mu(x-1)}{m+\beta_1+1}} + 1 \right)^m, \\
\tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(e^{3\mu t};x,y) &= \frac{e^{\frac{\mu}{m+\beta_1+1}} + 1}{2} e^{\frac{\mu x+2\mu\alpha_1}{m+\beta_1+1} + \frac{vy-v(\alpha_2+1)}{n+\beta_2+1} + \mu x + vy} \left(e^{\frac{-v}{n+\beta_2+1}} - e^{\frac{v(y-1)}{n+\beta_2+1}} + 1 \right)^n \\
&\quad \times \left(e^{\frac{\mu(x+1)}{m+\beta_1+1}} - e^{\frac{\mu}{m+\beta_1+1}} + e^{\frac{\mu x}{m+\beta_1+1}} \right)^m, \\
\tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(e^{4\mu t};x,y) &= \frac{e^{\frac{2\mu}{m+\beta_1+1}} + e^{\frac{\mu}{m+\beta_1+1}} + 1}{3} e^{\frac{\mu x+3\mu\alpha_1}{m+\beta_1+1} + \frac{vy-v(\alpha_2+1)}{n+\beta_2+1} + \mu x + vy} \left(e^{\frac{-v}{n+\beta_2+1}} - e^{\frac{v(y-1)}{n+\beta_2+1}} + 1 \right)^n \\
&\quad \times \left(e^{\frac{\mu(x+2)}{m+\beta_1+1}} + e^{\frac{\mu(x+1)}{m+\beta_1+1}} - e^{\frac{\mu}{m+\beta_1+1}} + e^{\frac{\mu x}{m+\beta_1+1}} - e^{\frac{2\mu}{m+\beta_1+1}} \right)^m, \\
\tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(e^{\mu t+\nu s};x,y) &= \frac{\mu\nu e^{\frac{\mu x}{m+\beta_1+1} + \frac{vy}{n+\beta_2+1} + \mu x + vy}}{(n+\beta_2+1)(m+\beta_1+1)(e^{\frac{\mu}{m+\beta_1+1}} - 1)(e^{\frac{v}{n+\beta_2+1}} - 1)}.
\end{aligned}$$

Proof By taking $f(t,s) = 1$ in (4), we obtain

$$\begin{aligned}
& \tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(1;x,y) \\
&= r'_\mu(x)r'_v(y)(m+\beta_1+1)(n+\beta_2+1)e^{\mu x+vy} \sum_{k=0}^m \sum_{l=0}^n p_{m,n,k,l}(r_\mu(x), r_v(y)) \\
&\quad \times \int_{\frac{l+\alpha_2}{n+\beta_2+1}}^{\frac{l+\alpha_2+1}{n+\beta_2+1}} \int_{\frac{k+\alpha_1}{m+\beta_1+1}}^{\frac{k+\alpha_1+1}{m+\beta_1+1}} e^{-\mu t-\nu s} dt ds, \\
&= e^{\frac{\mu x-\mu(\alpha_1+1)}{m+\beta_1+1} + \frac{vy-v(\alpha_2+1)}{n+\beta_2+1} + \mu x + vy} \left(e^{\frac{-\mu}{m+\beta_1+1}} - e^{\frac{\mu(x-1)}{m+\beta_1+1}} + 1 \right)^m \left(e^{\frac{-v}{n+\beta_2+1}} - e^{\frac{v(y-1)}{n+\beta_2+1}} + 1 \right)^n.
\end{aligned}$$

By taking $f(t,s) = e^{\mu t}$ in (4), we achieve

$$\begin{aligned}
& \tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(e^{\mu t};x,y) \\
&= r'_\mu(x)r'_v(y)(m+\beta_1+1)(n+\beta_2+1)e^{\mu x+vy} \sum_{k=0}^m \sum_{l=0}^n p_{m,n,k,l}(r_\mu(x), r_v(y)) \\
&\quad \times \int_{\frac{l+\alpha_2}{n+\beta_2+1}}^{\frac{l+\alpha_2+1}{n+\beta_2+1}} \int_{\frac{k+\alpha_1}{m+\beta_1+1}}^{\frac{k+\alpha_1+1}{m+\beta_1+1}} e^{-\nu s} dt ds, \\
&= \frac{\mu}{(m+\beta_1+1)(e^{\frac{\mu}{m+\beta_1+1}} - 1)} e^{\frac{\mu x}{(m+\beta_1+1)} + \frac{vy-v(\alpha_2+1)}{(n+\beta_2+1)} + \mu x + vy} \left(e^{\frac{-v}{(n+\beta_2+1)}} - e^{\frac{v(y-1)}{(n+\beta_2+1)}} + 1 \right)^n.
\end{aligned}$$

By taking $f(t, s) = e^{2\mu t}$ in (4), we obtain

$$\begin{aligned} & \tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(e^{2\mu t}; x, y) \\ &= r'_\mu(x)r'_v(y)(m + \beta_1 + 1)(n + \beta_2 + 1)e^{\mu x + \nu y} \sum_{k=0}^m \sum_{l=0}^n p_{m,n,k,l}(r_\mu(x), r_v(y)) \\ &\quad \times \int_{\frac{l+\alpha_2}{n+\beta_2+1}}^{\frac{l+\alpha_2+1}{n+\beta_2+1}} \int_{\frac{k+\alpha_1}{m+\beta_1+1}}^{\frac{k+\alpha_1+1}{m+\beta_1+1}} e^{\mu t - \nu s} dt ds, \\ &= e^{\frac{\mu x + \mu \alpha_1 + \mu x m}{m + \beta_1 + 1} + \frac{\nu y - \nu(\alpha_2 + 1)}{n + \beta_2 + 1} + \mu x + \nu y} \left(e^{\frac{-\nu}{n + \beta_2 + 1}} - e^{\frac{\nu(y-1)}{n + \beta_2 + 1}} + 1 \right)^n. \end{aligned}$$

By taking $f(t, s) = e^{3\mu t}$ in (4), we have

$$\begin{aligned} & \tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(e^{3\mu t}; x, y) \\ &= r'_\mu(x)r'_v(y)(m + \beta_1 + 1)(n + \beta_2 + 1)e^{\mu x + \nu y} \sum_{k=0}^m \sum_{l=0}^n p_{m,n,k,l}(r_\mu(x), r_v(y)) \\ &\quad \times \int_{\frac{l+\alpha_2}{n+\beta_2+1}}^{\frac{l+\alpha_2+1}{n+\beta_2+1}} \int_{\frac{k+\alpha_1}{m+\beta_1+1}}^{\frac{k+\alpha_1+1}{m+\beta_1+1}} e^{2\mu t - \nu s} dt ds, \\ &= \frac{e^{\frac{\mu}{m+\beta_1+1}} + 1}{2} e^{\frac{\mu x}{m+\beta_1+1} + \frac{\nu y}{n+\beta_2+1} + \frac{2\mu\alpha_1}{m+\beta_1+1} - \frac{\nu(\alpha_2+1)}{n+\beta_2+1}} \left(e^{\frac{-\nu}{n+\beta_2+1}} - e^{\frac{\nu(y-1)}{n+\beta_2+1}} + 1 \right)^n \\ &\quad \times \left(e^{\frac{\mu(x+1)}{m+\beta_1+1}} - e^{\frac{\mu}{m+\beta_1+1}} + e^{\frac{\mu x}{m+\beta_1+1}} \right)^m. \end{aligned}$$

By taking $f(t, s) = e^{4\mu t}$ in (4), we obtain

$$\begin{aligned} & \tilde{K}_n^{\alpha,\beta,\mu,\nu}(e^{4\mu t}; x, y) \\ &= r'_\mu(x)r'_v(y)(m + \beta_1 + 1)(n + \beta_2 + 1)e^{\mu x + \nu y} \sum_{k=0}^m \sum_{l=0}^n p_{m,n,k,l}(r_\mu(x), r_v(y)) \\ &\quad \times \int_{\frac{l+\alpha_2}{n+\beta_2+1}}^{\frac{l+\alpha_2+1}{n+\beta_2+1}} \int_{\frac{k+\alpha_1}{m+\beta_1+1}}^{\frac{k+\alpha_1+1}{m+\beta_1+1}} e^{3\mu t - \nu s} dt ds, \\ &= \frac{e^{\frac{2\mu}{m+\beta_1+1}} + e^{\frac{\mu}{m+\beta_1+1}} + 1}{3} e^{\frac{\mu x}{m+\beta_1+1} + \frac{\nu y}{n+\beta_2+1} + \frac{3\mu\alpha_1}{m+\beta_1+1} - \frac{\nu(\alpha_2+1)}{n+\beta_2+1}} \left(e^{\frac{-\nu}{n+\beta_2+1}} - e^{\frac{\nu(y-1)}{n+\beta_2+1}} + 1 \right)^n \\ &\quad \times \left(e^{\frac{\mu(x+2)}{m+\beta_1+1}} + e^{\frac{\mu(x+1)}{m+\beta_1+1}} - e^{\frac{\mu}{m+\beta_1+1}} + e^{\frac{\mu x}{m+\beta_1+1}} - e^{\frac{2\mu}{m+\beta_1+1}} \right)^m. \end{aligned}$$

By taking $f(t, s) = e^{\mu t + \nu s}$ in (4), we obtain

$$\begin{aligned} & \tilde{K}_n^{\alpha,\beta,\mu,\nu}(e^{\mu t + \nu s}; x, y) \\ &= r'_\mu(x)r'_v(y)(m + \beta_1 + 1)(n + \beta_2 + 1)e^{\mu x + \nu y} \sum_{k=0}^m \sum_{l=0}^n p_{m,n,k,l}(r_\mu(x), r_v(y)) \\ &\quad \times \int_{\frac{l+\alpha_2}{n+\beta_2+1}}^{\frac{l+\alpha_2+1}{n+\beta_2+1}} \int_{\frac{k+\alpha_1}{m+\beta_1+1}}^{\frac{k+\alpha_1+1}{m+\beta_1+1}} dt ds, \end{aligned}$$

$$= \frac{\mu \nu e^{\frac{\mu x}{m+\beta_1+1} + \frac{\nu y}{n+\beta_2+1} + \mu x + \nu y}}{(n+\beta_2+1)(m+\beta_1+1)(e^{\frac{\mu}{m+\beta_1+1}-1})(e^{\frac{\nu}{n+\beta_2+1}-1}).}$$

Other results can be obtained in a similar way. \square

Theorem 2.1 Let $\alpha, \beta \in (0, \infty)$. Then, we have

$$\lim_{n \rightarrow \infty} \tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(\exp_{i,j}^{\mu,\nu}; x, y) = \exp_{i,j}^{\mu,\nu}(x, y)$$

for $(i, j) \in \{(0, 0), (1, 0), (0, 1), (2, 0), (0, 2)\}$.

Proof Hereby, by choosing the test functions $\exp_{i,j}^{\mu,\nu}(t, s) := e^{i\mu t + j\nu s}$ for $(i, j) \in \{(0, 0), (1, 0), (0, 1)\}$, we obtain that

$$\lim_{m,n \rightarrow \infty} \tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(1; x, y) = 1, \quad (8)$$

$$\lim_{m,n \rightarrow \infty} \tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(e^{\mu t}; x, y) = e^{\mu x}, \quad (9)$$

$$\lim_{m,n \rightarrow \infty} \tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(e^{\nu s}; x, y) = e^{\nu y}. \quad (10)$$

By choosing $(i, j) = (2, 0)$ and $(i, j) = (0, 2)$, in (4), respectively, we obtain

$$\lim_{m,n \rightarrow \infty} \tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(e^{2\mu t} + e^{2\nu s}; x, y) = e^{2\mu x} + e^{2\nu y}. \quad (11)$$

Theorem 2.2 Let $\mu, \nu \in (0, \infty)$ and $f \in C(S_{\mu,\nu})$, then $\tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(f; x, y)$ converges to f uniformly.

Proof Applying the Korovkin theorem, and from (8), (9), (10), and (11),

$$\lim_{m,n \rightarrow \infty} \|\tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(f; x, y) - f(x, y)\|_{C(S_{\mu,\nu})} = 0,$$

where $(i, j) \in \{(0, 0), (1, 0), (0, 1), (2, 0), (0, 2)\}$, we obtain the desired result. \square

Lemma 2.2 For any $(x, y) \in S_{\mu,\nu}$, we obtain the limits of the central moments as follows:

$$\begin{aligned} & \lim_{m \rightarrow \infty} m(\tilde{K}_{m,m}^{\alpha,\beta,\mu,\nu}(1; x, y) - 1) \\ &= -\mu^2 x(x-1) + \mu(2x + x\beta_1 - \alpha_1 - 1) \\ &+ \nu(2y + \nu y + \beta_2 y - \alpha_2 - 1 - \nu y^2), \end{aligned} \quad (12)$$

$$\lim_{m \rightarrow \infty} m(\tilde{K}_{m,m}^{\alpha,\beta,\mu,\nu}((e^{\mu t} - e^{\mu x}); x, y)) = \frac{\mu}{2} e^{\mu x} (1 + 2\alpha_1 - 2x(1 + \mu + \beta_1) + 2\mu x^2), \quad (13)$$

$$\lim_{m \rightarrow \infty} m(\tilde{K}_{m,m}^{\alpha,\beta,\mu,\nu}((e^{\mu t} - e^{\mu x})^2; x, y)) = -\mu^2 e^{2\mu x} x(x-1), \quad (14)$$

$$\lim_{m \rightarrow \infty} m(\tilde{K}_{m,m}^{\alpha,\beta,\mu,\nu}((e^{\mu t} - e^{\mu x})(e^{\nu s} - e^{\nu y}); x, y)) = 0, \quad (15)$$

$$\lim_{m \rightarrow \infty} m^2(\tilde{K}_{m,m}^{\alpha,\beta,\mu,\nu}((e^{\mu t} - e^{\mu x})^4); x, y) = 0, \quad (16)$$

$$\lim_{m \rightarrow \infty} m^2(\tilde{K}_{m,m}^{\alpha,\beta,\mu,\nu}((e^{\nu s} - e^{\nu y})^4); x, y) = 0. \quad (17)$$

3 Rate of convergence

The modulus of continuity $\omega(f, \delta)$ for two-dimensional functions is given as follows:

$$\omega(f, \delta) = \sup \{ |f(x, y) - f(t, s)| : (t, s) \in S, \sqrt{(t-x)^2 + (s-y)^2} \leq \delta \}.$$

Theorem 3.1 Let $f \in C(S_{\mu, v})$. The following inequality holds

$$|\tilde{K}_{m,n}^{\alpha, \beta, \mu, v}(f; x, y) - f(x, y)| \leq \left(1 + \frac{1}{\mu^2} + \frac{1}{v^2} \right) \omega(f, \delta),$$

where

$$\begin{aligned} \delta^2 &= \left(e^{\frac{\mu x}{m+\beta_1+1} + \frac{v y}{n+\beta_2+1} + \mu x + v y} \right) \left[\left(e^{\frac{\mu(\alpha_1+xm)}{m+\beta_1+1} - \frac{v(\alpha_2+1)}{n+\beta_2+1}} - 2 \frac{e^{\mu x - \frac{v(\alpha_2+1)}{n+\beta_2+1}} \mu}{(m+\beta_1+1)(e^{\frac{\mu}{m+\beta_1+1}} - 1)} \right. \right. \\ &\quad + \left(e^{\mu x} + e^{vy} \right) e^{\frac{-\mu(\alpha_1+1)}{m+\beta_1+1} - \frac{v(\alpha_2+1)}{n+\beta_2+1}} \left(e^{\frac{-\mu}{(m+\beta_1+1)}} - e^{\frac{\mu(x-1)}{(m+\beta_1+1)}} + 1 \right)^m \left(e^{\frac{-v}{(n+\beta_2+1)}} - e^{\frac{v(y-1)}{(n+\beta_2+1)}} + 1 \right)^n \\ &\quad \left. \left. + \left(e^{\frac{v(\alpha_2+yn)}{n+\beta_2+1} - \frac{\mu(\alpha_1+1)}{m+\beta_1+1}} - 2 \frac{e^{vy - \frac{\mu(\alpha_1+1)}{m+\beta_1+1}} v}{(n+\beta_2+1)(e^{\frac{\mu}{n+\beta_2+1}} - 1)} \right) \left(e^{\frac{-\mu}{(m+\beta_1+1)}} - e^{\frac{\mu(x-1)}{(m+\beta_1+1)}} + 1 \right)^m \right] \right]. \end{aligned}$$

Proof From the definition of the modulus of continuity, we have

$$|\tilde{K}_{m,n}^{\alpha, \beta, \mu, v}(f; x, y) - f(x, y)| \leq \left(1 + \frac{\tilde{K}_{m,n}^{\alpha, \beta, \mu, v}((t-x)^2 + (s-y)^2; x, y)}{\delta^2} \right) \omega(f, \delta).$$

By using the Mean Value Theorem, we obtain

$$\begin{aligned} &|\tilde{K}_{m,n}^{\alpha, \beta, \mu, v}(f; x, y) - f(x, y)| \\ &\leq \left\{ 1 + \left(\frac{1}{\mu^2} + \frac{1}{v^2} \right) \frac{\tilde{K}_{m,n}^{\alpha, \beta, \mu, v}((e^{\mu t} - e^{\mu x})^2 + (e^{vs} - e^{vy})^2; x, y)}{\delta^2} \right\} \omega(f, \delta). \end{aligned}$$

Here, if we choose

$$\delta^2 = \tilde{K}_{m,n}^{\alpha, \beta, \mu, v}((e^{\mu t} - e^{\mu x})^2 + (e^{vs} - e^{vy})^2; x, y),$$

we have

$$|\tilde{K}_{m,n}^{\alpha, \beta, \mu, v}(f; x, y) - f(x, y)| \leq \left(1 + \frac{1}{\mu^2} + \frac{1}{v^2} \right) \omega(f, \delta). \quad \square$$

4 Voronovskaya-type theorem

In this section, we mention a Voronovskaya-type theorem for the $\tilde{K}_{m,n}^{\alpha, \beta, \mu, v}(f; x, y)$. Let the inverse of the exponential function for the first variable t be denoted by \log_μ^v and the inverse of the exponential function for the second variable s be shown as \log_v^μ .

Theorem 4.1 Let $f \in C(S_{\mu, v})$. We have

$$\lim_{m \rightarrow \infty} m(\tilde{K}_{m,n}^{\alpha, \beta, \mu, v}(f; x, y) - f(x, y))$$

$$\begin{aligned}
&= f(x, y)(-\mu^2(x-1)x + \mu(2x + x\beta_1 - \alpha_1 - 1) \\
&\quad + 2\nu(2y + \nu y + \beta_2 y) + 2\nu(-\alpha_2 - 1 - \nu y^2)) \\
&\quad + \frac{\partial f(x, y)}{\partial x}(1 + \alpha_1 - 2(1 + \mu + \beta_1)x + 2\mu x^2) \\
&\quad + \frac{\partial f(x, y)}{\partial y}(1 + 2\alpha_2 - 2(1 + \nu + \beta_2)y + 2\nu y^2) \\
&\quad + \frac{1}{2} \left\{ \left(-\frac{\partial^2 f(x, y)}{\partial x^2} + \mu \frac{\partial f(x, y)}{\partial x} \right) 2x(x-1) + \left(-\frac{\partial^2 f(x, y)}{\partial y^2} + \nu \frac{\partial f(x, y)}{\partial y} \right) 2y(y-1) \right\}
\end{aligned}$$

uniformly in $(x, y) \in S_{\mu, \nu}$.

Proof From Taylor's expansion for $(x, y) \in S_{\mu, \nu}$, we have

$$\begin{aligned}
f(t, s) &= f(x, y) + (e^{\mu t} - e^{\mu x}) \left[\frac{\partial}{\partial x} f(\log_\mu^v, \cdot) \right] \Big|_{(e^{\mu x}, e^{\nu y})} \\
&\quad + (e^{\nu s} - e^{\nu y}) \left[\frac{\partial}{\partial y} f(\cdot, \log_\nu^\mu) \right] \Big|_{(e^{\mu x}, e^{\nu y})} + \frac{1}{2} \left\{ (e^{\mu t} - e^{\mu x})^2 \left[\frac{\partial^2}{\partial x^2} f(\log_\mu^v, \cdot) \right] \Big|_{(e^{\mu x}, e^{\nu y})} \right. \\
&\quad \left. + (e^{\nu s} - e^{\nu y})^2 \left[\frac{\partial^2}{\partial y^2} f(\cdot, \log_\nu^\mu) \right] \Big|_{(e^{\mu x}, e^{\nu y})} \right. \\
&\quad \left. + 2(e^{\mu t} - e^{\mu x})(e^{\nu s} - e^{\nu y}) \left[\frac{\partial^2}{\partial y \partial x} f(\log_\mu^v, \log_\nu^\mu) \right] \Big|_{(e^{\mu x}, e^{\nu y})} \right\} \\
&\quad + R(f, t, s; x, y) ((e^{\mu t} - e^{\mu x})^2 + (e^{\nu s} - e^{\nu y})^2), \tag{18}
\end{aligned}$$

where $R(f, t, s; x, y) \rightarrow 0$ as $(t, s) \rightarrow (x, y)$.

By applying the operator $\tilde{K}_{m,n}^{\alpha, \beta, \mu, \nu}(\cdot; x, y)$ to both sides of (18), we write

$$\begin{aligned}
&\tilde{K}_{m,n}^{\alpha, \beta, \mu, \nu}(f; x, y) - f(x, y) \\
&= f(x, y) (\tilde{K}_{m,n}^{\alpha, \beta, \mu, \nu}(1; x, y) - 1) \\
&\quad + \tilde{K}_{m,n}^{\alpha, \beta, \mu, \nu}((e^{\mu t} - e^{\mu x}); x, y) \left[\frac{\partial}{\partial x} f(\log_\mu^v, \cdot) \right] \Big|_{(e^{\mu x}, e^{\nu y})} \\
&\quad + \tilde{K}_{m,n}^{\alpha, \beta, \mu, \nu}((e^{\nu s} - e^{\nu y}); x, y) \left[\frac{\partial}{\partial y} f(\cdot, \log_\nu^\mu) \right] \Big|_{(e^{\mu x}, e^{\nu y})} \\
&\quad + \frac{1}{2} \left\{ \tilde{K}_{m,n}^{\alpha, \beta, \mu, \nu}((e^{\mu t} - e^{\mu x})^2; x, y) \left[\frac{\partial^2}{\partial x^2} f(\log_\mu^v, \cdot) \right] \Big|_{(e^{\mu x}, e^{\nu y})} \right. \\
&\quad \left. + 2\tilde{K}_{m,n}^{\alpha, \beta, \mu, \nu}((e^{\mu t} - e^{\mu x})(e^{\nu s} - e^{\nu y}); x, y) \left[\frac{\partial^2}{\partial y \partial x} f(\log_\mu^v, \log_\nu^\mu) \right] \Big|_{(e^{\mu x}, e^{\nu y})} \right. \\
&\quad \left. + \tilde{K}_{m,n}^{\alpha, \beta, \mu, \nu}((e^{\nu s} - e^{\nu y})^2; x, y) \left[\frac{\partial^2}{\partial y^2} f(\cdot, \log_\nu^\mu) \right] \Big|_{(e^{\mu x}, e^{\nu y})} \right\} \\
&\quad + \tilde{K}_{m,n}^{\alpha, \beta, \mu, \nu}(R(f, t, s; x, y)((e^{\mu t} - e^{\mu x})^2 + (e^{\nu s} - e^{\nu y})^2); x, y). \tag{19}
\end{aligned}$$

Hence, we have the following derivatives:

$$\begin{aligned} \left[\frac{\partial f(\log_\mu^v, \cdot)}{\partial x} \right] \Big|_{(e^{\mu x}, e^{vy})} &= \frac{e^{-\mu x}}{\mu} \frac{\partial f(x, y)}{\partial x}, \\ \left[\frac{\partial^2 f(\log_\mu^v, \cdot)}{\partial x^2} \right] \Big|_{(e^{\mu x}, e^{vy})} &= e^{-2\mu x} \left(\frac{1}{\mu^2} \frac{\partial^2 f(x, y)}{\partial x^2} - \frac{1}{\mu} \frac{\partial f(x, y)}{\partial x} \right), \\ \left[\frac{\partial^2 f(\log_\mu^v, \log_v^\mu)}{\partial y \partial x} \right] \Big|_{(e^{\mu x}, e^{vy})} &= \frac{e^{-(\mu x+vy)}}{\mu v} \frac{\partial f(x, y)}{\partial y \partial x} \end{aligned} \quad (20)$$

and by substituting (20) into (19) and then by taking the limit we obtain

$$\begin{aligned} &\lim_{m \rightarrow \infty} m(\tilde{K}_{m,n}^{\alpha,\beta,\mu,v}(f; x, y) - f(x, y)) \\ &= f(x, y) \lim_{m \rightarrow \infty} m(\tilde{K}_{m,m}^{\alpha,\beta,\mu,v}(1; x, y) - 1) \\ &\quad + \frac{e^{-\mu x}}{\mu} \frac{\partial f(x, y)}{\partial x} \lim_{m \rightarrow \infty} m(\tilde{K}_{m,m}^{\alpha,\beta,\mu,v}((e^{\mu t} - e^{\mu x}); x, y)) \\ &\quad + \frac{e^{-vy}}{v} \frac{\partial f(x, y)}{\partial y} \lim_{m \rightarrow \infty} m(\tilde{K}_{m,m}^{\alpha,\beta,\mu,v}((e^{vs} - e^{vy}); x, y)) \\ &\quad + \frac{1}{2} \left\{ e^{-2\mu x} \left(\frac{1}{\mu^2} \frac{\partial^2 f(x, y)}{\partial x^2} - \frac{1}{\mu} \frac{\partial f(x, y)}{\partial x} \right) \lim_{m \rightarrow \infty} m(\tilde{K}_{m,m}^{\alpha,\beta,\mu,v}((e^{\mu t} - e^{\mu x})^2; x, y)) \right. \\ &\quad + \frac{2}{\mu v} e^{-(\mu x+vy)} \frac{\partial f(x, y)}{\partial y \partial x} \lim_{m \rightarrow \infty} m(\tilde{K}_{m,m}^{\alpha,\beta,\mu,v}((e^{\mu t} - e^{\mu x})(e^{vs} - e^{vy}); x, y)) \\ &\quad \left. + e^{-2vy} \left(\frac{1}{v^2} \frac{\partial^2 f(x, y)}{\partial y^2} - \frac{1}{v} \frac{\partial f(x, y)}{\partial y} \right) \lim_{m \rightarrow \infty} m(\tilde{K}_{m,m}^{\alpha,\beta,\mu,v}((e^{vs} - e^{vy})^2; x, y)) \right\} \\ &\quad + \lim_{m \rightarrow \infty} m(\tilde{K}_{m,m}^{\alpha,\beta,\mu,v}(R(f, t, s; x, y)((e^{\mu t} - e^{\mu x})^2 + (e^{vs} - e^{vy})^2); x, y)). \end{aligned}$$

By using equalities (12)–(17), we obtain

$$\begin{aligned} &\lim_{m \rightarrow \infty} m(\tilde{K}_{m,n}^{\alpha,\beta,\mu,v}(f; x, y) - f(x, y)) \\ &= f(x, y)(-\mu^2(x-1)x + \mu(2x + x\beta_1 - \alpha_1 - 1) \\ &\quad + v(2y + vy + \beta_2 y - \alpha_2 - 1 - vy^2)) + \frac{\partial f(x, y)}{\partial x} \left(\frac{1}{2} + \alpha_1 - (1 + \mu + \beta_1)x + \mu x^2 \right) \\ &\quad + \frac{\partial f(x, y)}{\partial y} \left(\frac{1}{2} + \alpha_2 - (1 + v + \beta_2)y + vy^2 \right) + \frac{1}{2} \left\{ \left(-\frac{\partial^2 f(x, y)}{\partial x^2} + \mu \frac{\partial f(x, y)}{\partial x} \right) x(x-1) \right. \\ &\quad \left. + \left(-\frac{\partial^2 f(x, y)}{\partial y^2} + v \frac{\partial f(x, y)}{\partial y} \right) y(y-1) \right\} \\ &\quad + \lim_{m \rightarrow \infty} m(\tilde{K}_{m,m}^{\alpha,\beta,\mu,v}(R(f, t, s; x, y)(e^{\mu t} - e^{\mu x})^2; x, y)) \\ &\quad + \lim_{m \rightarrow \infty} m(\tilde{K}_{m,m}^{\alpha,\beta,\mu,v}(R(f, t, s; x, y)(e^{vs} - e^{vy})^2; x, y)). \end{aligned} \quad (21)$$

When we apply the Cauchy–Schwarz inequality to (21), we obtain

$$m\tilde{K}_{m,m}^{\alpha,\beta,\mu,v}(R(t, s; x, y)((e^{\mu t} - e^{\mu x})^2; x, y) + m\tilde{K}_{m,m}^{\alpha,\beta,\mu,v}(R(t, s; x, y)(e^{vs} - e^{vy})^2; x, y))$$

$$\begin{aligned} &\leq \sqrt{\tilde{K}_{m,m}^{\alpha,\beta,\mu,\nu}(R^2(t,s;x,y))} \left(\sqrt{m^2 \tilde{K}_{m,m}^{\alpha,\beta,\mu,\nu}((e^{\mu t} - e^{\mu x})^4; x, y)} \right. \\ &\quad \left. + \sqrt{m^2 \tilde{K}_{m,m}^{\alpha,\beta,\mu,\nu}((e^{vs} - e^{vy})^4; x, y)} \right). \end{aligned}$$

Since $R(t,s;x,y) \rightarrow 0$ as $(t,s) \rightarrow (x,y)$,

$$\lim_{m \rightarrow \infty} \tilde{K}_{m,m}^{\alpha,\beta,\mu,\nu}(R(t,s;x,y); x, y) = 0$$

is verified uniformly in $C(S_{\mu,\nu})$. By using (16) and (17), we achieve the desired result. \square

5 Graphical and numerical analysis

In this section, we give a graphical and numerical analysis of $\tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(f; x, y)$ operators that illustrate the modeling of the approximation for the function f .

Example 5.1 Let $f(x, y) = \frac{\cos(x+1)\cos(y+1)}{e^{x+y+5}}$ for $x, y \in [0.1, 0.9]$. In Fig. 1, we show the graphs of $\tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(f; x, y)$ operators for fixed $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, the various values of $\mu = \nu \in \{1, 2, 3\}$ and $m = n \in \{70, 80, 90\}$.

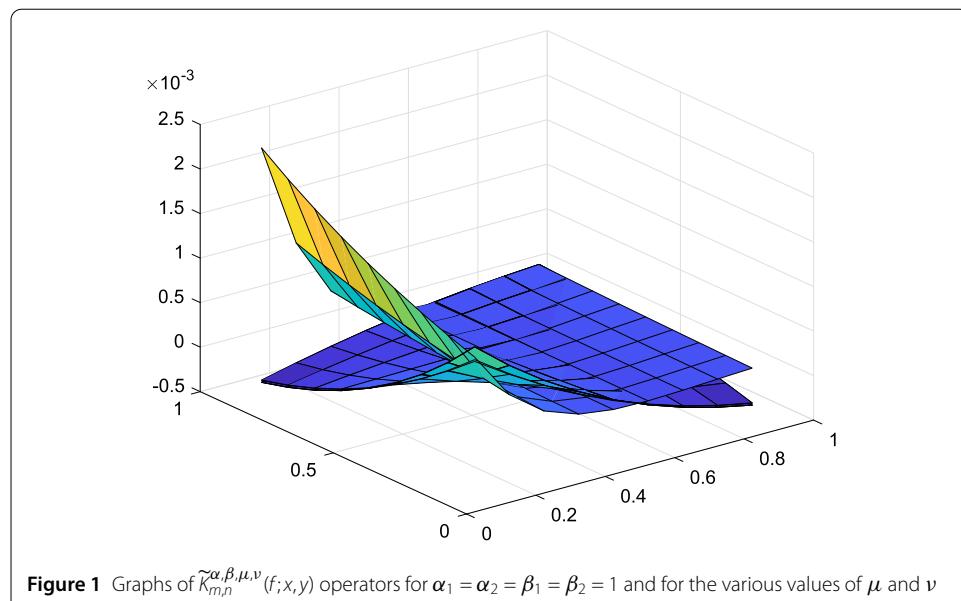


Figure 1 Graphs of $\tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(f; x, y)$ operators for $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$ and for the various values of μ and ν

Table 1 Error table for fixed $\mu = \nu \in \{1, 2, 3\}$ with different values of α and β

$\mu = \nu$	$\alpha_1 = \alpha_2$	$\beta_1 = \beta_2$	$\ \tilde{K}_{70,70}^{\alpha,\beta,\mu,\nu}(f) - f\ $	$\ \tilde{K}_{80,80}^{\alpha,\beta,\mu,\nu}(f) - f\ $	$\ \tilde{K}_{90,90}^{\alpha,\beta,\mu,\nu}(f) - f\ $
1	1	1	0.00015218	0.00013448	0.00012046
2	1	1	0.00014441	0.00016100	0.00018188
3	1	1	0.00020645	0.00018297	0.00016428
1	10	10	0.00016428	0.00071263	0.00066243
1	100	100	0.00046861	0.00042362	0.00038641

We calculate the maximum errors of $\|\tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(f) - f\|$ for the function $f(x,y) = \frac{\cos(x+1)\cos(y+1)}{e^{x+y+5}}$ by choosing $x = y \in [0.1, 0.9]$ and step size $h = 0.1$ in Table 1 for $m = n \in \{70, 80, 90\}$.

Example 5.2 Let $f(x,y) = e^{x+y}$. We give in Fig. 2 the graphs for $\tilde{K}_{70,70}^{5,10,0.9,0.9}(f; x, y)$, $\tilde{K}_{70,70}^{10,20,0.9,0.9}(f; x, y)$, and $\tilde{K}_{70,70}^{25,50,0.9,0.9}(f; x, y)$ for $x = y \in [0.1, 0.9]$.

We calculate the maximum errors of $\|\tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(f) - f\|$ for the function $f(x,y) = e^{x+y}$ by choosing $x = y \in [0.1, 0.9]$, $\mu = \nu = 1$ and step size $h = 0.1$ in Table 2 for $m = n \in \{70, 80, 90\}$.

In Table 3, by choosing $f(x,y) = e^{x+y}$, we give the comparison of $\tilde{K}_{m,n}^{\alpha,\beta}(f; x, y)$ and our new bivariate Bernstein–Kantorovich–Stancu operators $\tilde{K}_{m,n}^{\alpha,\beta,\mu,\nu}(f; x, y)$.

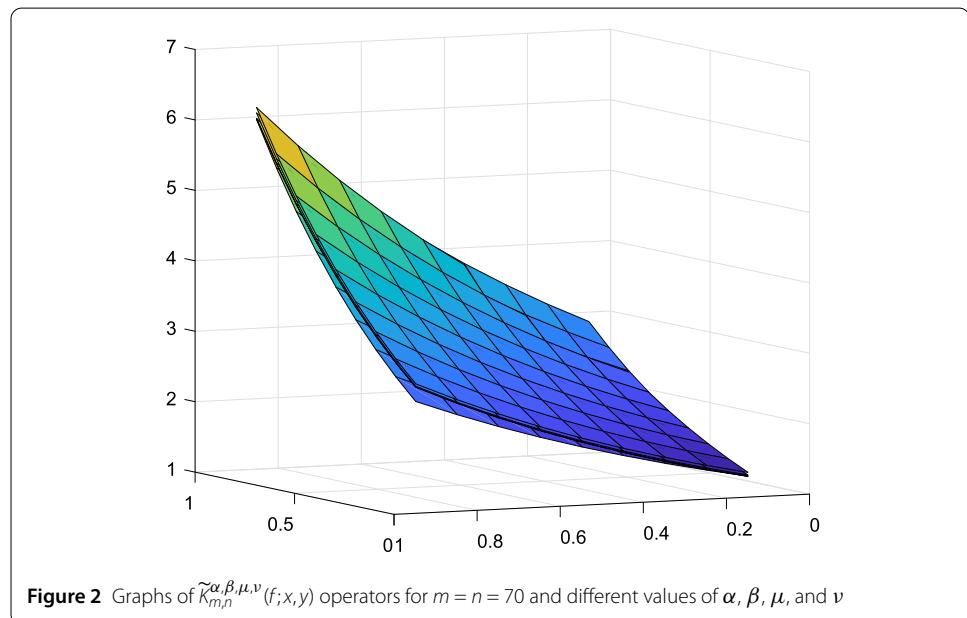


Table 2 Error table for fixed $\mu = \nu = 1$ with different values of α and β

$\alpha_1 = \alpha_2 = \beta_1 = \beta_2$	$\ \tilde{K}_{70,70}^{\alpha,\beta,1,1}(f) - f\ $	$\ \tilde{K}_{80,80}^{\alpha,\beta,1,1}(f) - f\ $	$\ \tilde{K}_{90,90}^{\alpha,\beta,1,1}(f) - f\ $
1	0.0342	0.0300	0.0267
10	0.0642	0.0567	0.0507
100	0.0484	0.0434	0.0394
1000	0.0364	0.0322	0.0288
10000	0.0344	0.0302	0.0269

Table 3 Comparison of $\tilde{K}_{90,90}^{\alpha,\beta,1,1}(f; x, y)$ and $\tilde{K}_{90,90}^{\alpha,\beta}(f; x, y)$ for different values of α and β

$\alpha_1 = \alpha_2$	$\beta_1 = \beta_2$	$\ \tilde{K}_{90,90}^{\alpha,\beta,1,1}(f) - f\ $	$\ \tilde{K}_{90,90}^{\alpha,\beta}(f) - f\ $
1	5	0.0520	0.1970
5	10	0.0507	0.2709
10	10	0.0507	0.7197
10	100	0.0394	1.6340

6 Conclusion

In this work, we construct the exponential bivariate Bernstein–Kantorovich–Stancu operators. Then, we calculate the rate of convergence with the modulus of continuity of the functions defined on $C(S_{\mu,\nu})$. Also, we give the Voronovskaya-type theorem. Finally, the error tables of the exponential bivariate Bernstein–Kantorovich operators are given for different values of m, n, μ, ν, α , and β .

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Data availability

All data generated or analyzed during this study are included in this published article.

Declarations

Competing interests

The authors declare no competing interests.

Author contributions

Lian-Ta Su: Construction of main problem and funding. Kadir Kanat: Construction of main problem, computer data analysis. Melek Sofyalioglu: computer data analysis, illustrations of figures. Merve Kisakol: made calculations, wrote the main manuscript text. All authors reviewed the manuscript.

Author details

¹Fujian Provincial Key Laboratory of Data-Intensive Computing, Key Laboratory of Intelligent Computing and Information Processing, School of Mathematics and Computer Science, Quanzhou Normal University, Quanzhou, China. ²Department of Mathematics, Polatlı Faculty of Science and Letters, Ankara Hacı Bayram Veli University, 06900, Ankara, Turkey.

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