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Estimation of q for ℓ_q-minimization in signal recovery with tight frame

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Abstract

This study aims to reconstruct signals that are sparse with a tight frame from undersampled data by using the ℓ_q-minimization method. This problem can be cast as a ℓ_q-minimization problem with a tight frame subjected to an undersampled measurement with a known noise bound. We proved that if the measurement matrix satisfies the restricted isometry property with $\delta_{2s} \leq 1/2$, there exists a value q_0 such that for any $q \in (0, q_0]$, any signal that is s-sparse with a tight frame can be robustly recovered to the true signal. We estimated q_0 as $q_0 = 2/3$ in the case of $\delta_{2s} \leq 1/2$ and discussed that the value of q_0 can be much higher. We also showed that when $\delta_{2s} \leq 0.3317$, for any $q \in (0, 1]$, robust recovery for signals via ℓ_q-minimization holds, which is consistent with the case of ℓ_q-minimization without a tight frame.

Keywords: Compressed sensing; Tight frame; ℓ_q-minimization

1 Introduction

Sparse representation and sparse signal recovery are derived from signal and image processing [14, 15, 26] and have been extended to other areas, such as sampling theory [21, 27], model identification [23, 36], and sensor networks [20, 30, 32]. Most of these applications search for sparse signals. Here, a signal or vector x is considered s-sparse if $\|x\|_0 \leq s$ and $\|\cdot\|_0$ are the ℓ_0-norm, which counts the nonzero entries of x. Compressed sensing is a sparse signal recovery theory that searches for the sparsest signal in an underdetermined linear system $Ax = y$, where $A \in \mathbb{R}^{n\times N}$ $(n \ll N)$ is the so-called measurement matrix, which is usually full rank, whereas $y \in \mathbb{R}^n$ is the given measurement vector. This procedure can be cast as a ℓ_0-minimization problem. However, the ℓ_0-minimization problem is NP-hard [24], some of which can be extended to ℓ_1-minimization, replacing $\|x\|_0$ with $\|x\|_1$ in ℓ_0-minimization. The ℓ_1-minimization seeks a slightly sparse solution for $y = Ax$. Donoho, Candès, Romberg, and Tao specified the conditions in [4, 5] that solutions of ℓ_1-minimization are the solutions of ℓ_0-minimization. Furthermore, ℓ_1-minimization is a linear programming problem that can be solved using certain algorithms [6, 9, 25, 31, 33].

In some other situations, signal x is not sparse itself, but it is sparse under some bases [29] (such as a Fourier base or wavelet base), frames [11, 12], or redundant dictionaries [10, 28]. In this study, signal x that was sparse in a tight frame was considered. A tight frame is defined as follows.

Definition 1 (Tight frame) [7] Vectors $D_1, D_2, \ldots, D_d \in \mathbb{R}^N$ are said to be a tight frame if they satisfy

$$x = \sum_k \langle x, D_k \rangle D_k, \quad \forall x \in \mathbb{R}^N.$$

Sometimes, we also say that the matrix $D = (D_1, D_2, \ldots, D_d) \in \mathbb{R}^{N\times d}$ is a tight frame. For some signal x, D^*x is either sparse or approximately sparse. In a noisy setting, the sparsity-seeking question can be expressed as

$$\bar{x} = \arg\min_{x\in\mathbb{R}^N} \left\{ \left\| D^*x \right\|_0 : \|Ax - y\| \le \epsilon \right\}, \tag{1}$$

where D^* is the conjugate transpose of D and ϵ is the energy of the known errors. Its ℓ_1-minimization problem is available accordingly [1, 8, 16, 18],

$$\bar{x} = \arg\min_{x\in\mathbb{R}^N} \left\{ \left\| D^*x \right\|_1 : \|Ax - y\|_2 \le \epsilon \right\}. \tag{2}$$

The compliance of the solutions with ℓ_0 and ℓ_1-minimizations has a sufficient condition with a coherent tight frame, which is said to be a restricted isometric property adapted to tight frame D (D-RIP).

Definition 2 (D-RIP) [3] The measurement matrix A satisfies the restricted isometric property adapted to tight frame D with order s if there exists a positive number $\delta_s \in (0, 1)$ such that

$$(1 - \delta_s)\|Dx\|_2^2 \le \|ADx\|_2^2 \le (1 + \delta_s)\|Dx\|_2^2$$

holds for $\forall x \in \sum_s$, where $\sum_s = \{x : \|x\|_0 \le s\}$. Here, δ_s is the restricted isometric constant (RIC) of order s.

Let $\nu_{\max(s)}$ be an operator that returns the s largest coefficients of $\nu \in \mathbb{R}^N$ in magnitude,

$$\nu_{\max(s)} = \arg\min_{\|\tilde{\nu}\|_0 \le k} \|\nu - \tilde{\nu}\|_2.$$

If in D-RIP $D = Id$, where Id is the identity matrix, then D-RIP is the traditional RIP. For the traditional RIP, Cai and Zhang provided a sharp bound for δ_{2s} in [2] as $\delta_{2s} < \sqrt{2}/2$. For D-RIP, Candès, Eldar et al. showed that Gaussian, sub-Gaussian, and boundary matrices satisfy the D-RIP with a high probability in [3]. They also proved that when $\delta_{2s} < 0.08$, the solution of the ℓ_1-minimization satisfies

$$\|\hat{x} - x\|_2 \le C_0 \frac{\|D^*x - (D^*x)_{\max(s)}\|_1}{\sqrt{k}} + C_1\epsilon,$$

where C_0 and C_1 are constants, $\hat{x}$ is the recovered signal and x is the true signal. As shown, the upper boundary of $\|\hat{x} - x\|_2$ is controlled by $\|D^*x - (D^*x)_{\max(s)}\|_1$ and ϵ. If D^*x is s-sparse or approximately s-sparse and ϵ is sufficiently small, the error between the recovered signal and the true signal can be regulated within an acceptable range.

The dynamic relation between ℓ_0 and ℓ_1-minimization is not clear. Thus, we studied ℓ_q-minimization with $0 < q < 1$ [18, 19, 22]. The ℓ_q-minimization problem is

$$\hat{x} = \arg\min_{x\in\mathbb{R}^N} \left\{ \left\|D^*x\right\|_q : \|Ax - y\|_2 \le \epsilon \right\}. \tag{3}$$

When $q \to 0$, ℓ_q-minimization approximates ℓ_0-minimization, while if $q \to 1$, ℓ_q-minimization approximates ℓ_1-minimization.

In general, the recovery condition by the ℓ_q-minimization ($0 < q < 1$) is less restrictive than the ℓ_1-minimization. In [34], Zhang and Li proved that if the sensing matrix A satisfies the D-RIP condition $\delta_{2s} < \sqrt{2}/2$, then all signals x with s-sparse with a tight frame can be recovered exactly via the constrained ℓ_1-minimization. For ℓ_q-minimization with tight frame, in [17], Li and Lin showed that for a tight frame D, if $\delta_{2s} < 1/2$, then there exists $q_0 = q_0(\delta_{2k}) \in (0, 1]$, such that for any $q \in (0, q_0)$, the recovered signal $\hat{x}$ via ℓ_q-minimization and the true signal x satisfy

$$\|\hat{x} - x\|_2 \le C_0 \frac{\|D^*x - (D^*x)_{\max(s)}\|_1}{s^{\frac{1}{q}-\frac{1}{2}}} + C_1\epsilon,$$

where C_0 and C_1 are constants that depend on δ_{2s} and q. However, this result does not provide the exact value for q_0. Subsequently, the D-RIP conditions for ℓ_q-minimization with a tight frame are improved. In [35], Zhang and Li showed that if the sensing matrix A satisfies the D-RIP with

$$\delta_{2s} < \frac{\eta}{2 - q - \eta} := \delta(q), \tag{4}$$

where $\eta \in (1 - q, 1 - \frac{q}{2})$ is the only positive solution of the equation

$$\frac{q}{2}\eta^{\frac{2}{q}} + \eta - 1 + \frac{q}{2} = 0,$$

then any s-sparse signal x with a tight frame can be exactly and stably recovered via ℓ_q-minimization in noiseless and noisy cases, respectively. D-RIP condition (4) for ℓ_q minimization is less restrictive than $\delta_{2s} < \sqrt{2}/2$ for ℓ_1 minimization. If let $p = 1/2$, we have $\delta_{2s} < 0.859$ by (4), which is less restrictive than $\delta_{2s} < \sqrt{2}/2$ for ℓ_1-minimization.

We provide an example to illustrate that if $\delta_{2s} > \sqrt{2}/2$, ℓ_1-minimization may fail, but ℓ_q-minimization works. We construct a measurement matrix $A \in \mathbb{R}^{2\times 3}$, and a tight frame $D \in \mathbb{R}^{3\times 5}$, as follows

$$A = \frac{1}{\sqrt{4}}\begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & -\sqrt{3} \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & -\frac{2+\sqrt{2}}{4} & \frac{\sqrt{6-4\sqrt{2}}}{4} & 0 \\ 0 & \frac{2+\sqrt{2}}{4} & 1/2 & 0 & \frac{\sqrt{6-4\sqrt{2}}}{4} \end{pmatrix}. \tag{5}$$

We can calculate that $\delta_2 = 0.75 > \sqrt{2}/2$. Vectors $x^{(1)} = (2, 0, 0)^T$ and $x^{(2)} = (0, 1, 1)^T$ have the same observed vector, namely $Ax^{(1)} = Ax^{(2)}$. We have

$$D^*x^{(1)} = (2, 0, 0, 0, 0)^T,$$

$$D^*x^{(2)} = \left(0, \frac{1}{2} + \frac{2+\sqrt{2}}{4}, \frac{1}{2} - \frac{2+\sqrt{2}}{4}, \frac{\sqrt{6-4\sqrt{2}}}{4}, \frac{\sqrt{6-4\sqrt{2}}}{4}\right)^T.$$

$D^*x^{(1)}$ and $D^*x^{(2)}$ have the same ℓ_1-norm, which means that signal recovery for $x^{(1)}$ through ℓ_1-minimization fails. ℓ_q-minimization is necessary in this case. The general solution of the equations $Ax = Ax^{(1)}$ is $x = (2-2c, c, c)^T$, where c is arbitrary real number. We can derive

$$\left\|D^*x^{(1)}\right\|_q^q = 2^q = (2-2c+c+c)^q \le |2-2c|^q + |c|^q + |c|^q = \left\|D^*x\right\|_q^q,$$

where the first inequality uses the conclusion: if $a > 0$, $b > 0$ and $0 < q < 1$, then $(a+b)^q \le a^q + b^q$. Hence, we have $\|D^*x^{(1)}\|_q < \|D^*x\|_q$ for $0 < q < 1$ and any solution x of the equations $Ax = Ax^{(1)}$. Therefore, ℓ_q-minimization can recover signal $x^{(1)}$.

This study examines signal recovery with a tight frame via ℓ_q-minimization for the case of a restricted isometry constant $\delta_{2s} < 1/2$. The main contribution shows not only the existence of q_0, such that for any $q \in (0, q_0]$, any s-sparse signal with a tight frame can be recovered via ℓ_q-minimization, but also the exact value $q_0 = 2/3$. A computer also demonstrated that the value of q_0 can be increased to $q_0 = 0.97$.

The remainder of this paper is organized as follows. In Sect. 2, some useful lemmas and their proofs are outlined, and Sect. 3 presents the main theorems. We provide the proofs of these main theorems in Sect. 4. Conclusions are presented in Sect. 5.

Notations: Given a signal $x = (x_1, x_2, \dots, x_N)^T$, the ℓ_0-norm is the number of its nonzero entries, that is, $\|x\|_0 = Card(supp(x))$. Here, $Card(\cdot)$ is the cardinality of a vector and $supp(x)$ is the support set of x. The ℓ_1-norm of vector x is the sum of the absolute values of its entries, that is, $\|x\|_1 = \sum_{i\ge1} |x_i|$. We can define its ℓ_q-norm with $0 < q < 1$ as $\|x\|_q = (\sum_{i\ge1} |x_i|^q)^{1/q}$. We can also define ℓ_∞-norm of x as $\|x\|_\infty = \max_{1\le i\le N}\{|x_i|\}$ and $\ell_{-\infty}$-pseudonorm of x as $\|x\|_{-\infty} = \min_{1\le i\le N}\{|x_i|\}$, respectively. Given $x = (x_1, x_2, \dots, x_N)^T \in \mathbb{R}^N$, $x_{\max(s)}$ denotes the vector that maintains the largest s entries in absolute value, and sets the others to zero. For a matrix $D \in \mathbb{R}^{N\times d}$ and index subset $T \subset \{1, 2, \dots, d\}$, D_T is used as the matrix D restricted to the columns indexed by T, D_T^* is the conjugate transpose of D_T and T^C is the complement of T in $\{1, 2, \dots, d\}$. Given a vector $h \in \mathbb{R}^N$, then $D^*h = ((D^*h)_1, (D^*h)_2, \dots, (D^*h)_d)^T \in \mathbb{R}^d$. Suppose $\{j_1, j_2, \dots, j_d\}$ is the rearrangement of $\{1, 2, \dots, d\}$ such that vector D^*h is monotonically decreasing in absolute value, that is, $|(D^*h)_{j_1}| \ge |(D^*h)_{j_2}| \ge \dots \ge |(D^*h)_{j_d}|$, then divide the set $\{j_1, j_2, \dots, j_d\}$ into some subsets with cardinality s starting from its head, if the cardinality of the last subset is less than s then just keep it, that is $T_0 = \{j_1, j_2, \dots, j_s\}$, $T_1 = \{j_{s+1}, j_{s+2}, \dots, j_{2s}\}$, $T_2 = \{j_{2s+1}, j_{2s+2}, \dots, j_{3s}\}$, $\dots$. Here, let $T = T_0$.

2 Some useful lemmas

First, we provide the relationship between ℓ_1 and the ℓ_q-norm, which is used to estimate the error bound.

Lemma 3 ([17]) *Let* $0 < q \le 1$, $x \in \mathbb{R}^N$,*then*

$$0 \le \|x\|_1 - \frac{\|x\|_q}{N^{1/q-1}} \le Q_q N\big(\|x\|_\infty - \|x\|_{-\infty}\big), \tag{6}$$

where $Q_q = q^{\frac{q}{1-q}} - q^{\frac{1}{1-q}}$. *Additionally,* Q_q *is a monotonous and convex function. The two limitations of this function with* $q \to 0^+$ *and* $q \to 1^-$, *are respectively*:

$$Q_0 := \lim_{q\to 0^+} Q_q = 1, \qquad Q_1 := \lim_{q\to 1^-} Q_q = 0.$$

The relationship between ℓ_2 and the ℓ_q-norm is also required during the estimation of the error bound.

Lemma 4 *For a fixed* $x \in \mathbb{R}^N$ *and* $0 < q \le 1$, *the following inequalities hold*

$$0 \le \|x\|_2 - N^{\frac{1}{2}-\frac{1}{q}}\|x\|_q \le \sqrt{N}\left(Q_q + \frac{1}{4}\right)\left(\|x\|_\infty - \|x\|_{-\infty}\right). \tag{7}$$

Proof According to the Cauchy–Schwarz inequality,

$$\|x\|_2 \ge \frac{\|x\|_1}{\sqrt{N}}. \tag{8}$$

In [13], the relationship between the ℓ_1 and ℓ_2 norms is

$$\|x\|_2 \le \frac{\|x\|_1}{\sqrt{N}} + \frac{\sqrt{N}}{4}\left(\|x\|_\infty - \|x\|_{-\infty}\right). \tag{9}$$

Using Lemma 3, inequalities (8) and (9), we can derive the result. □

For index set $T \subset \{1,2,\dots,N\}$, denote $D_T^* x := (D_T)^* x$. Suppose that $\hat{x}$ is the solution to problem (3) and $x \in \mathbb{R}^N$ satisfies $\|y - Ax\|_2 \le \epsilon$. Let

$$h = \hat{x} - x, \tag{10}$$

then $D^*h = ((D^*h)_1, (D^*h)_2, \dots, (D^*h)_d)^T$. Without generality, let $\{j_1, j_2, \dots, j_d\}$ be a rearrangement of $\{1,2,\dots,d\}$ such that

$$\left|\left(D^*h\right)_{j_1}\right| \ge \left|\left(D^*h\right)_{j_2}\right| \ge \cdots \ge \left|\left(D^*h\right)_{j_d}\right|.$$

Then denote

$$\begin{aligned} &T = T_0 = \{j_1, j_2, \dots, j_s\}, \qquad T_1 = \{j_{s+1}, j_{s+2}, \dots, j_{2s}\}, \\ &T_2 = \{j_{2s+1}, j_{2s+2}, \dots, j_{3s}\}, \qquad \dots. \end{aligned} \tag{11}$$

Clearly, $D^*h = \sum_{j\ge 0} D_{T_i}^* h$. Define ω and Ψ as follows

$$\omega := \frac{\|D_{T_1}^* h\|_q^q}{\sum_{i\ge 1}\|D_{T_i}^* h\|_q^q}, \tag{12}$$

$$\Psi := \sqrt{\sum_{i\ge 2}\left\|D_{T_i}^* h\right\|_2^2 + \delta_{2s}\left(\sum_{i\ge 2}\left\|D_{T_i}^* h\right\|_2\right)^2}. \tag{13}$$

Thus, $0 \le \omega \le 1$ and $\sum_{i\ge 2}\|D_{T_i}^* h\|_q^q = (1-\omega)(\sum_{i\ge 1}\|D_{T_i}^*\|_q^q)$.

Li showed the following lemma in [17], which gives the bound of the ℓ_2-norm square of $D^*_{T_i}h$ with $i \geq 2$. These results can be obtained from Lemma 4.1 and (3.5) of [17].

Lemma 5 (*Lemma* 4.1 *and inequality* (3.5) *in* [17]) *Let* $0 < q \leq 1$, h, $\{T_i, i \geq 0\}$, *and* Ψ *be defined as* (10),(11), *and* (13), *respectively, then the following inequalities hold*:

$$\sum_{i\geq 2}\left\|D^*_{T_i}h\right\|_2^2 \leq \frac{(1-\omega)\omega^{(2-q)/q}}{s^{(2-q)/q}}\left(\sum_{i\geq 1}\left\|D^*_{T_i}h\right\|_q^q\right)^{2/q}, \tag{14}$$

$$\left\|D^*_{T_0\cup T_1}h\right\|_2^2 \leq \frac{(2\epsilon+\Psi)^2}{1-\delta_{2s}}, \tag{15}$$

where s denotes sparsity.

The bound of the ℓ_2-norm of $D^*_{T_i}h$ with $i \geq 2$ is also required and is given by the following lemma.

Lemma 6 *Let* $0 < q \leq 1$, h, $\{T_i, i \geq 0\}$, *and* ω *be defined by* (10), (11), *and* (12). *Then*,

$$\sum_{i\geq 2}\left\|D^*_{T_i}h\right\|_2 \leq \frac{(1-\omega)^{1/q}+(Q_q+1/4)\omega^{1/q}}{s^{1/q-1/2}}\left(\sum_{i\geq 1}\left\|D^*_{T_i}h\right\|_q^q\right)^{1/q}. \tag{16}$$

Proof According to the relation between the ℓ_2-norm and ℓ_q-norm in Lemma 4, we have

$$\left\|D^*_{T_i}h\right\|_2 \leq s^{1/2-1/q}\left\|D^*_{T_i}h\right\|_q + \sqrt{s}(Q_q+1/4)\left(\left\|D^*_{T_i}h\right\|_\infty - \left\|D^*_{T_i}h\right\|_{-\infty}\right).$$

Summing up for i, we have

$$\begin{aligned}\sum_{i\geq 2}\left\|D^*_{T_i}h\right\|_2 &\leq s^{1/2-1/q}\sum_{i\geq 2}\left\|D^*_{T_i}h\right\|_q + \sqrt{s}(Q_q+1/4)\sum_{i\geq 2}\left(\left\|D^*_{T_i}h\right\|_\infty - \left\|D^*_{T_i}h\right\|_{-\infty}\right) \\ &\leq s^{1/2-1/q}\sum_{i\geq 2}\left\|D^*_{T_i}h\right\|_q + \sqrt{s}(Q_q+1/4)\left\|D^*_{T_2}h\right\|_\infty.\end{aligned} \tag{17}$$

Note that

$$\left\|D^*_{T_1}h\right\|_q = \left(\left|\left(D^*h\right)_{s+1}\right|^q + \cdots + \left|\left(D^*h\right)_{2s}\right|^q\right)^{1/q} \geq \left(s\left\|D^*_{T_2}h\right\|_\infty^q\right)^{1/q} = s^{1/q}\left\|D^*_{T_2}h\right\|_\infty.$$

We have

$$\left\|D^*_{T_2}h\right\|_\infty \leq s^{-1/q}\left\|D^*_{T_1}h\right\|_q. \tag{18}$$

By substituting (18) into (17) and combining (12), we can derive

$$\begin{aligned}&\sum_{i\geq 2}\left\|D^*_{T_i}h\right\|_2 \\ &\quad\leq s^{1/2-1/q}\sum_{i\geq 2}\left\|D^*_{T_i}h\right\|_q + s^{1/2-1/q}\left(Q_q+\frac{1}{4}\right)\left\|D^*_{T_1}h\right\|_q\end{aligned}$$

$$
\begin{aligned}
&\le s^{1/2-1/q}\left(\left(\sum_{i\ge 2}\|D^*_{T_i}h\|_q^q\right)^{1/q}+\left(Q_q+\frac{1}{4}\right)\left(\|D^*_{T_1}h\|_q^q\right)^{1/q}\right)\\
&\le s^{1/2-1/q}\left((1-\omega)^{1/q}\left(\sum_{i\ge 1}\|D^*_{T_i}h\|_q^q\right)^{1/q}+\left(Q_q+\frac{1}{4}\right)\omega^{1/q}\left(\sum_{i\ge 1}\|D^*_{T_1}h\|_q^q\right)^{1/q}\right)\\
&\le \frac{(1-\omega)^{1/q}+(Q_q+\frac{1}{4})\omega^{1/q}}{s^{1/q-1/2}}\left(\sum_{i\ge 1}\|D^*_{T_i}h\|_q^q\right)^{1/q}. \qquad (19)
\end{aligned}
$$

Note that the second inequality in (19) uses the following conclusion: if $a > 0$, $b > 0$ and $0 < q < 1$, then $(a+b)^q \le a^q + b^q$. The third inequality in (19) uses the definition of ω in (12). Moreover, in the first term of the second line in (19),

$$
\begin{aligned}
\sum_{i\ge 2}\|D^*_{T_i}h\|_q^q &= \sum_{i\ge 1}\|D^*_{T_i}h\|_q^q-\|D^*_{T_1}h\|_q^q=\sum_{i\ge 1}\|D^*_{T_i}h\|_q^q-\omega\sum_{i\ge 1}\|D^*_{T_i}h\|_q^q\\
&=(1-\omega)\sum_{i\ge 1}\|D^*_{T_i}h\|_q^q.
\end{aligned}
$$

Then we obtain the third line in the inequalities (19). The proof is completed. □

Two functions are defined as follows: $0 \le \omega \le 1$ and

$$
\alpha(\omega) := (1-\omega)\omega^{\frac{2-q}{q}}+\delta_{2s}\left[(1-\omega)^{\frac{1}{q}}+\left(Q_q+\frac{1}{4}\right)\omega^{\frac{1}{q}}\right]^2, \qquad (20)
$$

$$
\beta(\omega) := \alpha(\omega)-(1-\delta_{2s})\omega^{2/q}. \qquad (21)
$$

According to the definition of Ψ and lemmas 5 and 6, we derive

$$
\begin{aligned}
\Psi^2 &= \sum_{i\ge 2}\|D^*_{T_i}h\|_2^2+\delta_{2s}\left(\sum_{i\ge 2}\|D^*_{T_i}h\|_2\right)^2 \qquad (22)\\
&\le s^{1-2/q}(1-\omega)\omega^{2/q-1}\left(\sum_{i\ge,1}\|D^*_{T_i}h\|_q^q\right)^{2/q}\\
&\quad+\delta_{2s}s^{1-2/q}\left((1-\omega)^{1/q}+\left(Q_q+\frac{1}{4}\right)\omega^{1/q}\right)^2\left(\sum_{i\ge 1}\|D^*_{T_i}h\|_q^q\right)^{2/q}\\
&= s^{1-2/q}\alpha(\omega)\left(\sum_{i\ge 1}\|D^*_{T_i}h\|_q^q\right)^{2/q},
\end{aligned}
$$

and we have

$$
s^{2/q-1}\Psi^2-(1-\delta_{2s})\|D^*_{T_1}h\|_q^2 \le \beta(\omega)\left(\sum_{i\ge 1}\|D^*_{T_i}h\|_q^q\right)^{2/q}. \qquad (23)
$$

From the fact below

$$
\|D^*_{T_0\cup T_1}h\|_2^2=\|D^*_{T_0}h\|_2^2+\|D^*_{T_1}h\|_2^2 \ge s^{1-2/q}\left(\|D^*_{T_0}h\|_q^2+\|D^*_{T_1}h\|_q^2\right),
$$

and by combining Lemma 5, we can derive

$$\|D^*_{T_0}h\|_q^2 \le s^{2/q-1}\|D^*_{T_0\cup T_1}h\|_2^2 - \|D^*_{T_1}h\|_q^2 \le \frac{s^{2/q-1}(2\epsilon+\Psi)^2}{1-\delta_{2s}} - \|D^*_{T_1}h\|_q^2,$$

which means that

$$(1-\delta_{2s})\|D^*_{T_0}h\|_q^2 \le 4s^{2/q-1}\epsilon^2 + 4s^{2/q-1}\epsilon\Psi + \bigl(s^{2/q-1}\Psi^2 - (1-\delta_{2s})\|D^*_{T_1}h\|_q^2\bigr).$$

Substituting inequalities (22) and (23) into the inequality above, we obtain

$$\begin{aligned}&(1-\delta_{2s})\|D^*_{T_0}h\|_q^2\\&\quad\le 4s^{2/q-1}\epsilon^2 + 4\epsilon s^{1/q-1/2}\sqrt{\alpha(\omega)}\Bigl(\sum_{i\ge1}\|D^*_{T_i}h\|_q^q\Bigr)^{1/q} + \beta(\omega)\Bigl(\sum_{i\ge1}\|D^*_{T_i}h\|_q^q\Bigr)^{2/q}.\end{aligned} \tag{24}$$

Now, let

$$\begin{aligned}\omega_0 &:= \arg\max\bigl\{\alpha(\omega): 0\le\omega\le1\bigr\},\\ \omega_1 &:= \arg\max\bigl\{\beta(\omega): 0\le\omega\le1\bigr\},\end{aligned}$$

and

$$\lambda := \frac{\alpha(\omega_0)}{\beta(\omega_1)}. \tag{25}$$

Because of

$$\alpha(\omega_0) \ge \alpha(\omega_1) = \beta(\omega_1) + (1-\delta_{2s})\omega_1^{2q} \ge \beta(\omega_1),$$

we have $\lambda \ge 1$ and

$$\begin{aligned}(1-\delta_{2s})\|D^*_{T_0}h\|_q^2 &\le 4s^{\frac{2-q}{q}}\epsilon^2 + 4\epsilon s^{1/q-1/2}\sqrt{\alpha(\omega_0)}\Bigl(\sum_{i\ge1}\|D^*_{T_i}h\|_q^q\Bigr)^{1/q}\\&\quad+\beta(\omega_1)\Bigl(\sum_{i\ge1}\|D^*_{T_i}h\|_q^q\Bigr)^{2/q}\\&\le \bigl(2s^{1/q-1/2}\epsilon\sqrt{\lambda}\bigr)^2 + 2\bigl(2s^{1/q-1/2}\epsilon\sqrt{\lambda}\bigr)\Bigl(\sqrt{\beta(\omega_1)}\Bigl(\sum_{i\ge1}\|D^*_{T_i}h\|_q^q\Bigr)^{1/q}\Bigr)\\&\quad+\Bigl(\sqrt{\beta(\omega_1)}\Bigl(\sum_{i\ge1}\|D^*_{T_i}h\|_q^q\Bigr)^{1/q}\Bigr)^2.\end{aligned}$$

The above inequalities imply that

$$\|D^*_{T_0}h\|_q \le 2\sqrt{\frac{\lambda}{1-\delta_{2s}}}s^{1/q-1/2}\epsilon + \sqrt{\frac{\beta(\omega_1)}{1-\delta_{2s}}}\Bigl(\sum_{i\ge1}\|D^*_{T_i}h\|_q^q\Bigr)^{1/q}. \tag{26}$$

Therefore, the following conclusion is drawn:

$$\left\|D^*_{T_0}h\right\|_q^q \le 2^q s^{1-q/2}\left(\frac{\lambda}{1-\delta_{2s}}\right)^{q/2}\epsilon^q + \left(\frac{\beta(\omega_1)}{1-\delta_{2s}}\right)^{q/2}\left(\sum_{i\ge 1}\left\|D^*_{T_i}h\right\|_q^q\right). \tag{27}$$

Here, the inequality is used again, that is, if $a > 0$, $b > 0$, and $0 < q < 1$, then $(a+b)^q \le a^q + b^q$.

For any index set Ω with $|\Omega| \le s$, we have

$$\begin{aligned}
&\left\|D^*x\right\|_q^q \ge \left\|D^*\hat{x}\right\|_q^q,\\
&\left\|D^*x\right\|_q^q = \left\|D^*_\Omega x\right\|_q^q + \left\|D^*_{\Omega^c}x\right\|_q^q,\\
&\left\|D^*\hat{x}\right\|_q^q = \left\|D^*_\Omega \hat{x}\right\|_q^q + \left\|D^*_{\Omega^c}\hat{x}\right\|_q^q,
\end{aligned}$$

and

$$\begin{aligned}
&\left\|D^*_\Omega \hat{x}\right\|_q^q = \left\|D^*_\Omega h + D^*_\Omega x\right\|_q^q \ge \left\|D^*_\Omega x\right\|_q^q - \left\|D^*_\Omega h\right\|_q^q,\\
&\left\|D^*_{\Omega^c} \hat{x}\right\|_q^q = \left\|D^*_{\Omega^c} h + D^*_{\Omega^c} x\right\|_q^q \ge \left\|D^*_{\Omega^c} h\right\|_q^q - \left\|D^*_{\Omega^c} x\right\|_q^q,
\end{aligned}$$

which means that

$$\left\|D^*_{\Omega^c}h\right\|_q^q \le 2\left\|D^*_{\Omega^c}x\right\|_q^q + \left\|D^*_\Omega h\right\|_q^q. \tag{28}$$

Specifically, if the cardinality of Ω is s, that is, $|\Omega| = s$, and it satisfies $D^*_\Omega x = D^*x - (D^*x)_{\max(s)}$, then we have

$$\begin{aligned}
\sum_{i\ge 1}\left\|D^*_{T_i}h\right\|_q^q &= \left\|D^*_{T_0^c}h\right\|_q^q \le 2\left\|D^*_{\Omega^c}x\right\|_q^q + \left\|D^*_\Omega h\right\|_q^q\\
&\le 2\left\|D^*x - \left(D^*x\right)_{\max(s)}\right\|_q^q + 2^q s^{1-q/2}\left(\frac{\lambda}{1-\delta_{2s}}\right)\epsilon^q\\
&\quad + \left(\frac{\beta(\omega_1)}{1-\delta_{2s}}\right)^{q/2}\left(\sum_{i\ge 1}\left\|D^*_{T_i}h\right\|_q^q\right).
\end{aligned}$$

We can derive

$$\begin{aligned}
&\left[1-\left(\frac{\beta(\omega_1)}{1-\delta_{2s}}\right)^{q/2}\right]\left[\sum_{i\ge 1}\left\|D^*_{T_i}h\right\|_q^q\right]\\
&\quad \le 2^q s^{1-q/2}\left(\frac{\lambda}{1-\delta_{2s}}\right)^{q/2}\epsilon^q + 2\left\|D^*x - \left(D^*x\right)_{\max(s)}\right\|_q^q.
\end{aligned} \tag{29}$$

Define

$$\rho(q) := \min_{0\le\omega\le 1}\left\{\frac{1-\omega^{2/q-1}+2\omega^{2/q}}{1+\omega^{2/q}+[(1-\omega)^{1/q}+(Q_q+1/4)\omega^{1/q}]^2}\right\}. \tag{30}$$

Denote $(\frac{\beta(\omega_1)}{1-\delta_{2s}})^{q/2}$ by σ, that is, $\sigma = (\frac{\beta(\omega_1)}{1-\delta_{2s}})^{q/2}$. When $\delta_{2s} < \rho(q)$, to prove $1-\sigma > 0$ is equivalent to prove $\beta(\omega_1)/(1-\delta_{2s}) < 1$. By the definitions of $\alpha(\omega)$ and $\beta(\omega)$, we have

$$\begin{aligned}
&\beta(\omega_1)/(1-\delta_{2s}) < 1 \\
&\Leftrightarrow \quad \frac{(1-\omega_1)\omega_1^{\frac{2-q}{q}}}{1-\delta_{2s}} + \frac{\delta_{2s}}{1-\delta_{2s}}\left[(1-\omega_1)^{\frac{1}{q}} + \left(Q_q + \frac{1}{4}\right)\omega_1^{\frac{1}{q}}\right]^2 - \omega_1^{\frac{2}{q}} < 1 \\
&\Leftrightarrow \quad (1-\omega_1)\omega_1^{\frac{2-q}{q}} + \delta_{2s}\left[(1-\omega_1)^{\frac{1}{q}} + \left(Q_q + \frac{1}{4}\right)\omega_1^{\frac{1}{q}}\right]^2 - (1-\delta_{2s})\omega_1^{\frac{2}{q}} < 1-\delta_{2s} \\
&\Leftrightarrow \quad \delta_{2s}\left\{1 + \omega_1^{2/q} + \left[(1-\omega_1)^{1/q} + (Q_q + 1/4)\omega_1^{1/q}\right]^2\right\} < 1 - \omega_1^{2/q-1} + 2\omega_1^{2/q} \\
&\Leftrightarrow \quad \delta_{2s} < \frac{1-\omega_1^{2/q-1} + 2\omega_1^{2/q}}{1+\omega_1^{2/q} + [(1-\omega_1)^{1/q} + (Q_q + 1/4)\omega_1^{1/q}]^2}.
\end{aligned}$$

If $\delta_{2s} < \rho(q)$, we have

$$\delta_{2s} < \rho(q) \le \frac{1-\omega_1^{2/q-1} + 2\omega_1^{2/q}}{1+\omega_1^{2/q} + [(1-\omega_1)^{1/q} + (Q_q + 1/4)\omega_1^{1/q}]^2}.$$

Then we can derive that $\beta(\omega_1)/(1-\delta_{2s}) < 1$. Hence, we know that $1-\sigma > 0$, if $\delta_{2s} < \rho(q)$. Therefore, by the inequality (29), we have

$$\sum_{i\ge 1}\left\|D^*_{T_i}h\right\|_q^q \le \frac{2^q s^{1-q/2}}{1-\sigma}\left(\frac{\lambda}{1-\delta_{2s}}\right)^{q/2}\epsilon^q + \frac{2}{1-\sigma}\left\|D^*x - \left(D^*x\right)_{\max(s)}\right\|_q^q. \tag{31}$$

The following lemma is simple, but useful for estimating the error bound in the signal recovery.

Lemma 7 *Let* $0 < q \le 1$, *then*

$$\left(a^q + b^q\right)^{1/q} \le 2^{1/q-1}(a+b), \tag{32}$$

$$\sqrt{(a+\sqrt{b})^2 + c} \le a + \sqrt{b+c}, \tag{33}$$

hold for all $a \ge 0$, $b \ge 0$, *and* $c \ge 0$.

Proof Inequality (32) can be shown using Lemma 3 with $N = 2$, whereas (33) holds if both sides of the inequality are squared. □

3 Main results

We provide the error bound between the recovered signal $\hat{x}$ and any solution to $Ax = y$. This error bound is measured by the noise term ϵ and sparse term $\|D^*x - (D^*x)_{\max(s)}\|_q$.

Theorem 8 *Let D be the matrix with the columns forming a tight frame and $\hat{x}$ be the solution of ℓ_q-minimization. Then, for any fixed* $0 < q \le 1$ *and D-RIP constant* $\delta_{2s} < \rho(q)$, *we have*

$$\|\hat{x} - x\|_2 \le C_0\epsilon + C_1\frac{\|D^*x - (D^*x)_{\max(s)}\|_q}{s^{1/q-1/2}}, \tag{34}$$

where

$$C_0 = \frac{1}{\sqrt{1-\delta_{2s}}}\left\{2 + \left(\frac{2}{1-\sigma}\right)^{1/q}\sqrt{\lambda\left(1+\frac{\alpha(\omega_0)}{1-\delta_{2s}}\right)}\right\},$$
$$C_1 = 2^{1/q-1}\left(\frac{2}{1-\sigma}\right)^{1/q}\sqrt{1+\frac{\alpha(\omega_0)}{1-\delta_{2s}}}. \tag{35}$$

In this error bound, if the noise term $\epsilon = 0$, it is a noiseless setting. If there exists a solution x that is s-sparse with tight frame D, the true signal x is recovered exactly in a noiseless setting.

Remark 9 In [17], Li and Lin solved the existence problem of q_0 to recover a signal with coherent tight frames via ℓ_q-minimization. However, the q_0 was not provided in their paper. Actually, the value of q_0 can be estimated.

If $\omega = 0$, then $D^*x = 0$, Theorem 8 holds true. For $0 < \omega \leq 1$, the following conclusion can be drawn.

Theorem 10 *If the measurement matrix A satisfies the restricted isometry property with tight frame D and* $\delta_{2s} < 0.3317$, *then for any* $q \in (0,1]$, *we have*

$$\|\hat{x} - x\|_2 \leq C_0\varepsilon + C_1\frac{\|D^*x - (D^*x)_{\max(s)}\|_q}{s^{1/q-1/2}}, \tag{36}$$

where C_0 *and* C_1 *are the constants in Theorem* 8.

Remark 11 In fact, δ_{2s} can take values much larger than 0.3317, i.e., if $\delta_{2s} < 0.493$, q can be arbitrary in the range of $(0,1]$, then ℓ_q-minimization recovers the signal robustly with a coherent tight frame. Thus, the conclusion of Theorem 10 holds. However, this requires different proof.

In [17], Li and Lin showed that if $\delta_{2s} < 1/2$, there exists a value q_0 such that the signals can be recovered via ℓ_q-minimization. The following theorem improves this result and provides an exact value for q_0.

Theorem 12 *If the measurement matrix A satisfies the restricted isometry property with tight frame D and* $\delta_{2s} < 1/2$, *then there exists a value* $q_0 = 2/3$, *such that for any* $q \in (0, 2/3]$, $\delta_{2s} < 1/2 \leq \rho(q)$ *holds. Furthermore,*

$$\|\hat{x} - x\|_2 \leq C_0\varepsilon + C_1\frac{\|D^*x - (D^*x)_{\max(s)}\|_q}{s^{1/q-1/2}}, \tag{37}$$

where C_0 *and* C_1 *are the constants in Theorem* 8.

Remark 13 In [17], Li and Lin proved the existence of q_0. However, there has been no estimation of q_0 in [17]. For this problem, we not only prove a result similar to that in [17], but also estimate $q_0 = 2/3$.

Remark 14 $q_0 = 2/3$ is not the best value for q_0, and can be much larger. The curve of $\rho(q)$ drawn using MATLAB demonstrates that there exists $q_0 = 0.97$ such that $\delta_{2s} < 1/2 \le \rho(q)$ holds; thus, Theorem 12 holds. However, this is considerably more difficult to achieve.

4 Proof of main results

We give here the proof procedure for each theorem.

4.1 Proof of theorem 8

Proof Using inequality (15) in Lemma 5, we have

$$\begin{aligned}\|\hat{x}-x\|_2^2 &= \|h\|_2^2 = \left\|D^*_{T_0\cup T_1}h\right\|_2^2 + \sum_{i\ge 2}\left\|D^*_{T_i}h\right\|_2^2 \le \frac{(2\epsilon+\Psi)^2}{1-\delta_{2s}} + \sum_{i\ge 2}\left\|D^*_{T_i}h\right\|_2^2 \\ &\le \left(\frac{2\epsilon}{\sqrt{1-\delta_{2s}}} + \sqrt{\frac{s^{1-2/q}\alpha(\omega)(\sum_{i\ge 1}\|D^*_{T_i}h\|_q^q)^{2/q}}{1-\delta_{2s}}}\right)^2 + \sum_{i\ge 2}\left\|D^*_{T_i}h\right\|_2^2,\end{aligned}$$

where the last inequality uses the result in (22). Therefore, by Lemma 7, we have

$$\begin{aligned}\|h\|_2 &\le \sqrt{\left(\frac{2\epsilon}{1-\delta_{2s}} + \sqrt{\frac{s^{1-2/q}\alpha(\omega)(\sum_{i\ge 1}\|D^*_{T_i}h\|_q^q)^{2/q}}{1-\delta_{2s}}}\right)^2 + \sum_{i\ge 2}\left\|D^*_{T_i}h\right\|_2^2} \\ &\le \frac{2\epsilon}{\sqrt{1-\delta_{2s}}} + \sqrt{\frac{s^{1-2/q}\alpha(\omega)(\sum_{i\ge 1}\|D^*_{T_i}h\|_q^q)^{2/q}}{1-\delta_{2s}} + \sum_{i\ge 2}\left\|D^*_{T_i}h\right\|_2^2} \\ &= \frac{2\epsilon}{\sqrt{1-\delta_{2s}}} + \frac{1}{\sqrt{1-\delta_{2s}}}\sqrt{s^{1-2/q}\alpha(\omega)\left(\sum_{i\ge 1}\left\|D^*_{T_i}h\right\|_q^q\right)^{2/q} + (1-\delta_{2s})\sum_{i\ge 2}\left\|D^*_{T_i}h\right\|_2^2} \\ &\le \frac{2\epsilon}{\sqrt{1-\delta_{2s}}} + s^{1/2-1/q}\sqrt{\frac{\alpha(\omega)}{1-\delta_{2s}} + (1-\omega)\omega^{2/q-1}}\left(\sum_{i\ge 1}\left\|D^*_{T_i}h\right\|_q^q\right)^{1/q} \\ &\le \left[2 + \left(\frac{2}{1-\sigma}\right)^{1/q}\sqrt{\lambda\left(\frac{\alpha(\omega)}{1-\delta_{2s}} + (1-\omega)\omega^{2/q-1}\right)}\right]\frac{\epsilon}{\sqrt{1-\delta_{2s}}} \\ &\quad + 2^{1/q-1}\sqrt{\frac{\alpha(\omega)}{1-\delta_{2s}} + (1-\omega)\omega^{2/q-1}}\left(\frac{2}{1-\sigma}\right)^{1/q}\frac{\|D^*x-(D^*x)_{\max(s)}\|_q}{s^{1/q-1/2}} \\ &\le C_0\epsilon + C_1\frac{\|D^*x-(D^*x)_{\max(s)}\|_q}{s^{1/q-1/2}},\end{aligned}$$

where C_0 and C_1 are given by (35), the second inequality uses the inequality (33) in Lemma 7, the third inequality uses inequality (14) in Lemma 5, and the fourth inequality uses inequality (32) in Lemma 7 and (31). □

4.2 Proof of theorem 10

Proof We discuss the case where $0 < \omega \leq 1$. Theorem 8 shows that this conclusion holds as long as $\delta_{2s} < 0.3317 \leq \rho(q)$. According to the definition of $\rho(q)$,

$$\frac{1-\omega^{2/q-1}+2\omega^{2/q}}{1+\omega^{2/q}+[(1-\omega)^{1/q}+(Q_q+1/4)\omega^{1/q}]^2} \geq \frac{1-\omega^{2/q-1}+2\omega^{2/q}}{1+\omega^{2/q}+[(1-\omega)^{1/q}+\frac{5}{4}\omega^{1/q}]^2}.$$

Therefore, Theorem 10 holds if for any $0 < \omega \leq 1$ and all $q \in (0,1]$, the following inequality holds:

$$\frac{1-\omega^{2/q-1}+2\omega^{2/q}}{1+\omega^{2/q}+[(1-\omega)^{1/q}+\frac{5}{4}\omega^{1/q}]^2} \geq 0.3317. \tag{38}$$

Let $a := 1/q \in [1,+\infty)$, then let

$$\frac{1-\omega^{2a-1}+2\omega^{2a}}{1+\omega^{2a}+[(1-\omega)^{a}+\frac{5}{4}\omega^{a}]^2} \geq \frac{n_1}{n_2}, \tag{39}$$

for all $0 < \omega \leq 1$ and all $a \in [1,+\infty)$, where $n_1, n_2 \in N^+$ and $n_1 \leq n_2$.

The following procedure estimates the lower bound of n_1/n_2. Inequality (39) is equivalent to

$$(n_2-n_1)-n_2\omega^{2a-1}+(2n_2-n_1)\omega^{2a} \geq n_1\left[(1-\omega)^a+\frac{5}{4}\omega^a\right]^2. \tag{40}$$

Inequality (40) holds if the infimum on the left is greater than or equal to the supremum on the right side. Let $f(\omega,a) = (n_2-n_1)-n_2\omega^{2a-1}+(2n_2-n_1)\omega^{2a}$, $g(\omega,a) = (1-\omega)^a+\frac{5}{4}\omega^a$, calculate the partial derivatives of the two functions and let them be zeros. Then, we have

$$\begin{cases} \frac{\partial f}{\partial \omega} = -n_2(2a-1)\omega^{2a-2}+2(2n_2-n_1)a\omega^{2a-1} = 0, \\ \frac{\partial f}{\partial a} = -2n_2\omega^{2a-1}\ln\omega+2(2n_2-n_1)\omega^{2a}\ln\omega = 0, \end{cases} \tag{41}$$

$$\begin{cases} \frac{\partial g}{\partial \omega} = -a(1-\omega)^{a-1}+\frac{5}{4}a\omega^{a-1} = 0, \\ \frac{\partial g}{\partial a} = (1-\omega)^a\ln(1-\omega)+\frac{5}{4}\omega^a\ln\omega = 0. \end{cases} \tag{42}$$

From equations (41) it can be derived that $2an_2 = 2an_2 - n_2$, which does not hold because $n_2 \in N^+$. Therefore, $f(\omega,a)$ has no stationary points. In equations (42), because $\frac{\partial g}{\partial a} < 0$, $g(\omega,a)$ also has no stationary points.

By calculating the value of the bounds, we know that $f(\omega,a)$ achieves its minimum value at $a = 1$, whereas $g(\omega,a)$ has its maximum value at $\omega = 1$. It is not difficult to compute this for all $0 < \omega \leq 1$ and all $a \in [1,+\infty)$,

$$\inf_{\omega,a} f(\omega,a) = \frac{7n_2^2+4n_1^2-12n_1n_2}{4(2n_2-n_1)},$$

$$\sup_{\omega,a} g(\omega,a) = \frac{25}{16}n_1.$$

Inequality $\inf_{\omega,a} f(\omega,a) \geq \sup_{\omega,a} g(\omega,a)$, i.e., $\frac{7n_2^2+4n_1^2-12n_1n_2}{4(2n_2-n_1)} \geq \frac{25}{16}n_1$, is equivalent to

$$41\left(\frac{n_1}{n_2}\right)^2 - 98\left(\frac{n_1}{n_2}\right) + 28 \geq 0. \tag{43}$$

Because $0 < n_1/n_2 \leq 1$, inequality (43) holds when $0 < n_1/n_2 \leq 0.3317$. In other words, 0.3317 is the lower bound of $\rho(q)$, so $\delta_{2s} < 0.3317 \leq \rho(q)$ holds. The proof is complete. □

4.3 Proof of theorem 12

Proof Let $a := 1/q \in [3/2, +\infty)$. According to the definitions of $\rho(q)$ and a, we only need to prove that for all $0 < \omega \leq 1$ and all $a \in [3/2, +\infty)$, the following inequality holds:

$$\frac{1-\omega^{2a-1}+2\omega^{2a}}{1+\omega^{2a}+[(1-\omega)^a+\frac{5}{4}\omega^a]^2} \geq \frac{1}{2}. \tag{44}$$

Inequality (44) is equivalent to

$$1-2\omega^{2a-1}+3\omega^{2a}-\left[(1-\omega)^a+\frac{5}{4}\omega^a\right]^2 \geq 0. \tag{45}$$

Let $f(\omega,a) = 1-2\omega^{2a-1}+3\omega^{2a}-[(1-\omega)^a+\frac{5}{4}\omega^a]^2$, then its partial derivative with respect to a is calculated as

$$\begin{aligned}\frac{\partial f}{\partial a} &= -4\omega^{2a-1}\ln\omega + \frac{23}{8}\omega^{2a}\ln\omega - 2(1-\omega)^{2a}\ln(1-\omega) - \frac{5}{2}\omega^a(1-\omega)^a\ln\bigl[\omega(1-\omega)\bigr] \\ &= \left(\frac{23}{8}\omega-4\right)\omega^{2a-1}\ln\omega - 2(1-\omega)^{2a}\ln(1-\omega) - \frac{5}{2}\omega^a(1-\omega)^a\ln\bigl[\omega(1-\omega)\bigr] \\ &> 0.\end{aligned} \tag{46}$$

Therefore, $f(\omega,a)$ has no stationary point and an extreme point at the bounds, and it is known that $f(\omega,a)$ reaches its minimum value at $a = 3/2$. To prove that $f(\omega,a) \geq 0$ for all ω and a, we must prove that for all ω, the following inequality holds,

$$f\left(\omega,\frac{3}{2}\right) = 3\omega - 5\omega^2 + \frac{39}{16}\omega^3 - \frac{5}{2}\bigl[\omega(1-\omega)\bigr]^{\frac{3}{2}} \geq 0. \tag{47}$$

Inequality (47) is equivalent to

$$\left(3-5\omega+\frac{39}{16}\omega^2\right)^2 \geq \frac{25}{4}\omega(1-\omega)^3,$$

and we derive,

$$9-\frac{145}{4}\omega+58\omega^2-\frac{85}{2}\omega^3+\left(\frac{39^2}{16^2}+\frac{25}{4}\right)\omega^4 \geq 0. \tag{48}$$

The coefficient of ω^4 is separated into two parts,

$$9-\frac{145}{4}\omega+58\omega^2-\frac{85}{2}\omega^3+\left(\frac{38^2}{16^2}+\frac{25}{4}\right)\omega^4+\frac{39^2-38^2}{16^2}\omega^4 \geq 0. \tag{49}$$

Let $g(\omega) = 9 - \frac{145}{4}\omega + 58\omega^2 - \frac{85}{2}\omega^3 + (\frac{38^2}{16^2} + \frac{25}{4})\omega^4$, and its derivative is

$$\frac{dg}{d\omega} = \frac{1}{16}\left(-580 + 1856\omega - 2040\omega^2 + 761\omega^3\right).$$

For $0 < \omega \leq 1$, because $\frac{d^2g}{d\omega^2} > 0$, we have $\frac{dg}{d\omega} < 0$, which means that $g(\omega)$ decreases monotonically. Therefore, we know that $g(\omega) > 0$, for any $0 < \omega \leq 1$. Because $\frac{39^2-38^2}{16^2}\omega^4 > 0$, inequality (49) holds for any $0 < \omega \leq 1$. The proof is complete. □

5 Conclusion

As for the q value problem of sparse signal recovery using ℓ_q-minimization, the existence of q value has been proven, that is, if the measurement matrix satisfies D-RIP with $\delta_{2s} \leq 1/2$, then there exists a value q_0 such that for any $q \in (0, q_0]$, any signal that is s-sparse with a tight frame can be robustly recovered to the true signal. In this work, we mainly estimated q_0 as $q_0 = 2/3$ in the case of $\delta_{2s} \leq 1/2$ and discussed that the value of q_0 can be much higher. We also proved that if $\delta_{2s} \leq 0.3317$, for any $q \in (0, 1]$, robust recovery for signals via ℓ_q-minimization holds, which is consistent with the case of ℓ_q-minimization without a tight frame.

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Declarations

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The authors declare no competing interests.

Author contributions
Kaihao Liang proved the main theorem and wrote the manuscript, Chaolong Zhang provided the proof of some lemmas and the discussion of the result of the main theorem, Wenfeng Zhang provided the funding for this project, as well as the research ideas. All authors reviewed the manuscript.

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