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# Second-order optimality conditions for interval-valued functions

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## Abstract

This work is included in the search of optimality conditions for solutions to the scalar interval optimization problem, both constrained and unconstrained, by means of second-order optimality conditions. As it is known, these conditions allow us to reject some candidates to minima that arise from the first-order conditions. We will define new concepts such as second-order gH-derivative for interval-valued functions, 2-critical points, and 2-KKT-critical points. We obtain and present new types of interval-valued functions, such as 2-pseudoinvex, characterized by the property that all their second-order stationary points are global minima. We extend the optimality criteria to the semi-infinite programming problem and obtain duality theorems. These results represent an improvement in the treatment of optimization problems with interval-valued functions.

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## 1 Introduction

Convexity has always been a desirable property of a function because it ensures that local optima are global. Pseudoconvexity is also important as this property ensures that stationary points are optimal. This property is shared by invex functions (see Hanson [12]) and makes it possible to design numerical algorithms that find solutions to optimization problems. These functions have been applied to fields such as fractional programming or programming with interval-valued or fuzzy objective functions (see [1, 22, 23]). In the case of constrained problems, invexity is replaced by KT-invexity [19] to ensure that the KKT stationary points are optimal. Examples of invex functions are  $\theta(x) = \ln(x)$  for  $x > 0$  and  $\theta(x) = x^2 + 3 \sin^2(x)$ .

It is also well known that to obtain the optima, first-order conditions are not enough, and we must resort to second-order conditions to rule out some of the candidates. An example of this is illustrated in Ginchev and Ivanov [8]:

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*Example 1* We consider:

$$\begin{aligned} \text{Minimize } \theta(x) &= \begin{cases} x^2, & x \geq 0, \\ -x^2, & x < 0, \end{cases} \\ \text{subject to } g(x) &= -x \leq 0. \end{aligned}$$

The only stationary point  $x = 0$  is a global minimum, but this problem cannot be solved with the classical sufficient conditions since our goal is second-order pseudoconvex but not pseudoconvex.

Authors such as Antczak [4], Luc [18], and Mishra [20] related invexity with second-order derivatives.

More closely related to our study, Ginchev and Ivanov [8] introduced the parabolic local minimum notion and proved that every parabolic local minimum satisfies the necessary and sufficient conditions for a global minimum with real functions.

In [13] the author utilized second-order Fréchet differentiable functions in  $R^n$  and defined the second-order KT-pseudoconvexity to prove that each KT point of second-order is a global minimum. Later, Ivanov in [14] generalized the first-order sufficient optimality conditions from invex functions to second-order invex ones.

Uncertainty in the coefficients of a programming problem can be trapped through one of these methods:

- Assume that they are random variables and have some known distribution function, but sometimes it is difficult to know their shape.
- Using robust optimization, which always considers the worst-case scenario of the model but does not consider that a good scenario could occur and therefore may not be appropriate to use.
- Using interval analysis through the calculation of the lower and upper extremes of the estimated parameters of historical data.

Since the creation of interval analysis in the 1960s by Ramon Moore [21] to control the imprecision of data, there have been significant advances in the way of working with intervals and in concepts as important as the differentiability of interval-valued functions, see [5, 35]. Differential calculus is essential for finding solutions by the gradient method or for the Karush–Kuhn–Tucker optimality conditions. In [29] Roy, Panda, and Qiu propose a solution search scheme for an interval optimization problem based on the gradient.

Singh et al., in two papers [33, 34], derived KKT optimality conditions for multiobjective optimization problems in which both objective and constraints are assumed to be interval-valued functions. So did Jayswal et al. [16, 17] and Ahmad et al. [2, 3] generalizing first-degree optimality conditions to interval-valued functions in both the differentiable and nondifferentiable cases. But these conditions do not involve second-order conditions as we do in this paper.

Also, Daidai [7] presented a generalization of the first- and second-order approximations to optimality conditions for strongly convex functions. The author solved the Euler equation with second-order approximation data by Newton's method.

Within the field of interval-valued functions, in 2015, Osuna et al. [22] using the concept of gH-differentiability characterized pseudoinvex functions as those in which the

stationary and optimal points of unconstrained multi-objective programming problems coincide. In 2017, Osuna et al. [24] generalized the above results to the multiobjective interval-valued programming problems with constraints. They proved that if this problem is KT-pseudoinvex, then every vector interval Karush–Kuhn–Tucker solution is a strictly weakly efficient solution. In 2022, Osuna et al. [26] proved that if  $x^*$  is an optimum of a gH-differentiable interval-valued function  $F$  on  $\mathbb{R}$ , then  $0 \in F'(x^*)$ , i.e., 0 belongs to this interval. This simple property opens the way to design algorithms for the search of solutions.

In 2018, Ruiz-Garzón et al. [31] obtained first-order optimality conditions for the scalar and vector optimization problem on Riemann manifolds but not second-order conditions, which will be discussed in this article. In 2020, Ruiz-Garzón et al. [30] used Lipschitz functions, and in [32] second-order optimality conditions were given but not for the interval-valued functions.

Semi-infinite programming originated in the 1920s in a work by Haar [11] and was named after Charnes et al. [6]. In this field, the systematization work carried out by Goberna and López is noteworthy [9, 10]. In this line of work it is worth mentioning the recent paper by Tung and Tam [37], Upadhyay, Gosh, Mishra, and Treanță [38, 39] where they study the optimality and duality conditions for multiobjective semi-infinite programming on Hadamard manifolds, obtaining first-order results but not second-order results like the ones we deal with in this paper nor with interval-valued functions.

Motivated by the previous work, our objective is focused on extending the second-order optimality conditions obtained with real functions to interval-valued functions. Therefore, we present necessary and sufficient optimality conditions for both unconstrained and constrained scalar interval optimization problems, looking for the function types for which the second-order critical points and the global minimum points coincide.

To the best of our knowledge, there is no paper to study the second-order optimality conditions for interval optimization problem. This paper is to make in this direction. Our contributions are the following:

- Proposing a new concept of 2-invexity, 2-pseudoinvexity interval-valued functions, and 2-critical points.
- Characterizing the 2-PIX functions as those in which 2-critical points and minima coincide.
- Defining the concept of 2-Karush–Kuhn–Tucker stationary point to constrained scalar interval optimization problem.
- Proposing when the constrained scalar interval optimization problem can be considered to be 2-KKT-pseudoinvex.
- Analyzing the environmental conditions so that the 2-KKT points and the minima coincide.
- Establishing weak and strong duality theorems of a dual Mond–Weir type problem.
- Extend the optimality criteria from the finite case to the semi-infinite case.

Interval problems capture the uncertainty that classical optimization is not able to capture. The advantages of the interval approach over classical multi-objective optimization is that conventional optimization techniques cannot be applied directly to the interval problem because when working with intervals the lower end point must always be smaller than the upper end point. Furthermore, solving an interval problem based separately on

the lower and upper end points of the interval, and not jointly, means that the topological structure of the interval space is not exploited.

One of the fundamental tools for the study of problems with fuzzy-valued functions is their connection with interval problems based on the level sets of a fuzzy interval, which are real intervals. Differentiability concepts developed for fuzzy optimization are often based on differentiability in interval optimization. For example, Osuna-Gómez et al. (2022) in [25] defined a level-wise gH-differentiable fuzzy function when  $n \geq 1$ , using the results obtained for interval-valued functions. The derivative in interval space was quasi-linear, while in real functions it is linear. An interval function can be gH-differentiable even though the extremes are not. Therefore, the results established in the article can be used to solve fuzzy optimization problems.

*Summary.*

Next, we will show the different sections of this article. In Sect. 2, we recall interval arithmetic operations and define the new concept of second-order gH-derivative. In Sect. 3, we define the concepts of 2-invex and 2-pseudoinvex functions, and 2-critical points. We study the relationships between them. Subsequently, we extend some of these concepts to the constrained case. We define the 2-KKT stationary points and the 2-KKT-pseudoinvex problem to identify the conditions under which stationary points and minima coincide. Our challenge is to extend and generalize the classical results in the literature obtained with Fréchet differentiable functions given by Ginchev and Ivanov [8], Ivanov [13, 14], and Ruiz-Garzón [32] to interval-valued functions. In Sect. 4, we study the weak and strong duality theorems, and in Sect. 5 we extending the optimality criteria to the semi-infinite programming problem.

**2 Preliminaries**

**2.1 Notations for intervals**

We denote by  $\mathcal{K}_C$  the family of all bounded closed intervals in  $\mathbb{R}$ . Let  $A = [a^L, a^U]$  and  $B = [b^L, b^U]$  be two closed intervals. By definition, we have the sum of two intervals and the product of a scalar by an interval as follows:

$$(a) \ A + B = [a^L + b^L, a^U + b^U] \text{ and } \lambda A = \begin{cases} [\lambda a^L, \lambda a^U], & \lambda \geq 0 \\ [\lambda a^U, \lambda a^L], & \lambda < 0 \end{cases}, \text{ where } \lambda \in \mathbb{R}.$$

(b) Subtraction of two intervals can also be set broadly. We have that gH-difference between the two intervals [5] is as follows:

$$A \ominus_{gH} B = [\min \{a^L - b^L, a^U - b^U\}, \max \{a^L - b^L, a^U - b^U\}].$$

We need to establish an order between the intervals.

**Definition 1** Let  $A = [a^L, a^U]$  and  $B = [b^L, b^U]$  be two closed intervals in  $\mathbb{R}$ . We write:

- $A \leq B \Leftrightarrow a^L \leq b^L \text{ and } a^U \leq b^U$ ;
- $A \leq B \Leftrightarrow A \leq B \text{ and } A \neq B$ , i.e.,  $a^L \leq b^L \text{ and } a^U \leq b^U$ , with a strict inequality;
- $A < B \Leftrightarrow a^L < b^L \text{ and } a^U < b^U$ .

The function  $f : (a, b) \rightarrow \mathcal{K}_C$  is called an *interval-valued function*, i.e.,  $f(x)$  is a closed interval in  $\mathbb{R}$  for each  $x \in \mathbb{R}$ . We will denote  $f(x) = [f^L(x), f^U(x)]$ , where  $f^L$  and  $f^U$  are real-valued functions and satisfy  $f^L(x) \leq f^U(x)$  for every  $x \in \mathbb{R}$ .

**Definition 2** ([35], Definition 10) Let  $u \in (a, b)$  and  $h$  be such that  $\bar{x} + h \in (a, b)$ . Then the gH-derivative of a function  $f : (a, b) \rightarrow \mathcal{K}_C$  is defined as

$$f'_{gH}(\bar{x}) = \lim_{h \rightarrow 0} \frac{1}{h} (f(\bar{x} + h) \ominus_{gH} f(\bar{x})) \tag{1}$$

if the limit exists. The interval  $f'_{gH}(\bar{x}) \in \mathcal{K}_C$  satisfying (1) is called the generalized Hukuhara derivative of  $f$  (gH-derivative for short) at  $\bar{x}$ .

The gH-derivative can be easily calculated as follows.

**Lemma 1** [5, 28] Let  $D \subseteq \mathbb{R}$  be a nonempty open set and let  $f : D \rightarrow \mathcal{K}_C$  be a continuous function. Then  $f$  is gH-derivative at  $\bar{x} \in D$  if and only if  $f^L$  and  $f^U$  have derivative at  $\bar{x}$ . Furthermore, we have

$$f'_{gH}(\bar{x}) = [\min\{f'^L(\bar{x}), f'^U(\bar{x})\}, \max\{f'^L(\bar{x}), f'^U(\bar{x})\}],$$

where  $f'^L(\bar{x})$  and  $f'^U(\bar{x})$  are the derivatives of  $f^L$  and  $f^U$  at  $\bar{x}$ , respectively.

*Example 2* Suppose that  $S = \{x \in \mathbb{R}, x \geq 1\}$  and  $f : D \subseteq S \rightarrow \mathcal{K}_C$  is defined by

$$f(x) = [x, x^2], \quad \forall x \in S.$$

We have that

$$\begin{aligned} f'_{gH}(\bar{x}) &= \lim_{h \rightarrow 0} \frac{1}{h} (f(\bar{x} + h) \ominus_{gH} f(\bar{x})) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} [\min\{(x + tv - x), (x + tv)^2 - x^2\}, \max\{(x + tv - x), (x + tv)^2 - x^2\})] \\ &= [1, 2x]. \end{aligned}$$

Thus, Stefanini et al. [36] proposed the following definition of second-order derivative, which plays a central role in our study.

**Definition 3** Let  $f : (a, b) \rightarrow \mathcal{K}_C$  be gH-differentiable on  $(a, b)$  and  $\bar{x} \in (a, b)$  and  $h$  be such that  $\bar{x} + h \in (a, b)$ . The second-order gH-derivative of a function  $f(x)$  at  $\bar{x}$  is defined as

$$f''_{gH}(\bar{x}) = \lim_{h \rightarrow 0} \frac{1}{h} (f'_{gH}(\bar{x} + h) \ominus_{gH} f'_{gH}(\bar{x})) \tag{2}$$

if the limit exists. The interval  $f''_{gH}(\bar{x}) \in \mathcal{K}_C$  satisfying (2) is called the second-order generalized Hukuhara derivative of  $f$  (second-order gH-derivative for short) at  $\bar{x}$ .

With all the tools defined above, we are finally in a position where we can start to look for the necessary and sufficient second-order conditions in different contexts.

### 3 Applications to scalar interval optimization problems

In this section, we characterize the functions whose critical points are global optimums in the context of scalar interval optimization problems.

### 3.1 Unconstrained case

We start by considering the unconstrained scalar interval optimization problem:

$$\begin{aligned} \text{(SOP)} \quad \min f(x) &= [f^L(x), f^U(x)] \\ \text{subject to } x &\in (a, b) \subseteq \mathbb{R}, \end{aligned}$$

where  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$  is a differentiable function.

In [15] Jayswal et al. introduced invexity, pseudoinvexity, and quasiinvexity function concepts for interval-valued functions.

**Definition 4** A differentiable  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$  function is said to be an invex (IX) at  $\bar{x} \in (a, b)$  with respect to  $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  if there exists  $\eta(x, \bar{x}) \in \mathbb{R}$  nonidentically zero, such that  $\forall x \in (a, b)$ ,

$$f(x) \ominus_{gH} f(\bar{x}) \succeq f'_{gH}(\bar{x})(\eta(x, \bar{x})).$$

**Definition 5** Let  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$  be a differentiable function. Then  $f$  is said to be pseudoinvex at  $\bar{x} \in (a, b)$  with respect to  $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  if there exists  $\eta(x, \bar{x}) \in \mathbb{R}$  such that  $\forall x \in (a, b)$ ,

$$f(x) < f(\bar{x}) \Rightarrow f'_{gH}(\bar{x})(\eta(x, \bar{x})) < [0, 0].$$

**Definition 6** Let  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$  be a differentiable function. Then  $f$  is said to be quasiinvex at  $\bar{x} \in (a, b)$  with respect to  $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  if there exists  $\eta(x, \bar{x}) \in \mathbb{R}$  such that  $\forall x \in (a, b)$ ,

$$f(x) \preceq f(\bar{x}) \Rightarrow f'_{gH}(\bar{x})(\eta(x, \bar{x})) \preceq [0, 0].$$

*Example 3* [15] Let us consider that  $f : \mathbb{R}_+ \rightarrow \mathcal{K}_C$ , defined by  $f(x) = [3, 4]x + [1, 5]x^3$ , is pseudoinvex and quasiinvex with respect to  $\eta(x, y) = x^2 - y^2$  but not an invex function.

We are interested in characterizing interval-valued functions where the stationary and optimal points coincide.

Thus, we will now propose a generalization of the concept of 2-invexity given by Ivanov [14] for Fréchet differentiable functions in dimensional finite Euclidean space to interval-valued functions.

**Definition 7** A second-order differentiable  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$  function is said to be a 2-invex (2-IX) at  $\bar{x} \in (a, b)$  with respect to  $\eta, \xi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  if there exist  $\eta(x, \bar{x}) \in \mathbb{R}$  non-identically zero,  $\xi(x, \bar{x}) \in \mathbb{R}$  such that  $\forall x \in (a, b)$ ,

$$f(x) \ominus_{gH} f(\bar{x}) \succeq f'_{gH}(\bar{x})(\eta(x, \bar{x})) + f''_{gH}(\bar{x})(\xi(x, \bar{x})).$$

*Remark 1* Note that each invex function is also a 2-invex function.

*Example 4* Let us consider that  $f : \mathbb{R}_+ \rightarrow \mathcal{K}_C$ , defined by  $f(x) = [3, 4]x + [1, 5]x^3$ , is not an invex function with respect to  $\eta(x, y) = x^2 - y^2$  but is a 2-invex function.

We can define a new concept of second-order stationary point for the interval-valued functions.

**Definition 8** Suppose that the function  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$  is second-order differentiable at any  $\bar{x} \in (a, b)$ . A feasible point  $\bar{x}$  for SIOP is said to be a 2-critical point (2-CP) if

$$0 \in f'_{gH}(\bar{x}), \tag{3}$$

$$f''^L_{gH}(\bar{x}) \geq 0. \tag{4}$$

Thus, we can propose and prove the following theorem that characterizes the concept of 2-invex functions.

**Theorem 2** Let  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$  be a second-order differentiable function at any  $\bar{x} \in (a, b)$ . The function  $f$  is 2-invex at  $\bar{x} \in (a, b)$  with respect to  $\eta, \xi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  if and only if each 2-CP is a global minimum of  $f$  on  $(a, b)$ .

*Proof* We will argue by contradiction. Suppose that  $f$  is 2-invex at  $\bar{x} \in (a, b)$  with respect to  $\eta, \xi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\bar{x}$  is a 2-CP but it is not a global minimum. Thus it follows that  $0 \in f'_{gH}(\bar{x})$ . Furthermore,  $f''^L_{gH}(\bar{x}) \geq 0$ , and there is  $x \in (a, b)$  with  $f(x) < f(\bar{x})$ .

By the 2-invexity of  $f$  with respect to  $\eta$  and  $\xi$ , there exist  $\eta(x, \bar{x}) \in \mathbb{R}$  and  $\xi(x, \bar{x}) \in \mathbb{R}$  such that

$$[0, 0] \succ f(x) \ominus_{gH} f(\bar{x}) \succeq f'_{gH}(\bar{x})(\eta(x, \bar{x})) + f''_{gH}(\bar{x})(\xi(x, \bar{x})), \tag{5}$$

and then  $f''^L_{gH}(\bar{x}) < 0$ , and therefore  $f''^L_{gH}(\bar{x}) < 0$ , which is a contradiction.

Now, we will prove the sufficient condition for 2-invexity.

Suppose that each 2-CP  $\bar{x}$  is a global minimum, but  $f$  is not 2-invex at  $\bar{x} \in (a, b)$  with respect to  $\eta, \xi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ . Then there exist  $\eta(x, \bar{x}) \in \mathbb{R}$  nonidentically zero and  $\xi(x, \bar{x}) \in \mathbb{R}$  such that

$$f(x) \ominus_{gH} f(\bar{x}) < f'_{gH}(\bar{x})(\eta(x, \bar{x})) + f''_{gH}(\bar{x})(\xi(x, \bar{x})).$$

Then we choose  $\xi(x, \bar{x}) = 0$  and  $\eta(x, \bar{x}) = -tf'_{gH}(\bar{x})$ , where  $t$  is arbitrary positive real. Thus, it follows that

$$f(x) \ominus_{gH} f(\bar{x}) < -t \|f'_{gH}(\bar{x})\|^2, \quad \forall t > 0,$$

which contradicts with the base assumption made of  $\bar{x}$  being a global minimum. □

This result extends Theorem 2.6 given by Ivanov [14] and Theorem 2 given by Ruiz-Garzón et al. [32] of scalar functions to interval-valued functions.

*Example 5* Let us consider  $f : \mathbb{R} \rightarrow \mathcal{K}_C$  defined by  $f(x) = [x^4, e^{x^2}]$ . We can calculate:

$$f'_{gH}(x) = [4x^3, 2xe^{x^2}], \quad f''_{gH}(x) = [12x^2, 2e^{x^2} + 4x^2e^{x^2}].$$

The point  $\bar{x} = 0$  for SIOP is a 2-critical point (2-CP) because

$$0 \in f'_{gH}(\bar{x}) = [0, 0], \tag{6}$$

$$f''^L_{gH}(\bar{x}) \geq 0, \tag{7}$$

and a global minimum of  $f$ , then, by Theorem 2, the function  $f$  is 2-invex at  $\bar{x}$ .

We will now define the new concept of the 2-pseudoinvex function for the interval-valued functions.

**Definition 9** Let  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$  be a second-order differentiable function at any  $\bar{x} \in (a, b)$ . A differentiable  $f$  function is said to be a 2-pseudoinvex (2-PIX) at  $\bar{x} \in (a, b)$  with respect to  $\eta, \xi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  if there exists  $\eta(x, \bar{x}) \in \mathbb{R}$  nonidentically zero such that

$$f(x) \ominus_{gH} f(\bar{x}) < [0, 0] \Rightarrow \begin{cases} f'_{gH}(\bar{x})(\eta(x, \bar{x})) \leq [0, 0], \\ 0 \in f'_{gH}(\bar{x}) \Rightarrow f''^{UL}_{gH}(\bar{x}) < 0. \end{cases}$$

We will now discuss when the 2-PIX and 2-IX functions coincide.

**Theorem 3** Suppose that:

- (a)  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$  is second-order  $gH$ -differentiable at every  $\bar{x} \in (a, b)$ ;
  - (b)  $x \in (a, b)$  such that if  $f(x) < f(\bar{x})$  then  $0 \in f'_{gH}(\bar{x})$ .
- If  $f$  is a 2-pseudoinvex function at  $\bar{x} \in S_1$ , then  $f$  is also 2-invex at  $\bar{x} \in S_1$ .

*Proof* Let  $\bar{x}, x \in (a, b)$  be two points such that  $f(x) < f(\bar{x})$ .

If  $f'_{gH}(\bar{x})(\eta(x, \bar{x})) < [0, 0]$ , then the inequality

$$f'_{gH}(\bar{x})(\eta(x, \bar{x})) + f''_{gH}(\bar{x})(\xi(x, \bar{x})) < [0, 0] \tag{8}$$

is ensured with  $\xi(x, \bar{x}) = 0$ .

If  $0 \in f'_{gH}(\bar{x})$ , then inequality (8) holds since  $f$  is 2-pseudoinvex with respect to  $\eta, \xi$ .

Inequality (8) implies the 2-invexity of  $f$  since if  $f$  is not a 2-invex function then there exist  $\eta(x, \bar{x}), \xi(x, \bar{x}) \in \mathbb{R}$  such that

$$f(x) \ominus_{gH} f(\bar{x}) < f'_{gH}(\bar{x})(\eta(x, \bar{x})) + f''_{gH}(\bar{x})(\xi(x, \bar{x})) < [0, 0],$$

a contradiction with all  $\bar{x}$  being a minimum. □

We can move forward and get the following result.

**Corollary 4** Suppose that:

- (a)  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C$  is second-order  $gH$ -differentiable at every  $\bar{x} \in (a, b)$ ;
- (b)  $x \in (a, b)$  such that if  $f(x) < f(\bar{x})$  then  $0 \in f'_{gH}(\bar{x})$ .

The function  $f$  is 2-pseudoinvex at  $\bar{x} \in S_1$  if and only if each 2-CP is a global minimum of  $f$  on  $S_1$ .



*Proof* On the one hand, from Theorem 3, the 2-pseudoinvexity implies the 2-invexity. On the other hand, the 2-invexity implies the 2-pseudoinvexity. Together with Theorem 2, we obtain our assertion.  $\square$

The previous corollary extends the results obtained by Ivanov, Theorems 2.12 and 2.14 in [14], by Ruiz-Garzón et al. [32], Corollary 1, from an environment of convexity and scalar functions to a more general environment of invexity and interval-valued functions. The 2-PIX functions are characterized by the fact that the minimum points and the 2-CP points coincide.

### 3.2 Constrained case

In this section, we consider the constrained scalar interval optimization problem of the form:

$$\begin{aligned}
 \text{(CIOP)} \quad \min f(x) &= [f^L(x), f^U(x)] \\
 \text{subject to} \quad g_j(x) &\leq 0, j = 1, 2, \dots, m \\
 x &\in \mathbb{R},
 \end{aligned}$$

where  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C, g_j : (a, b) \subseteq \mathbb{R} \rightarrow \mathbb{R}, j = 1, 2, \dots, m$ , is a set of differentiable functions. Hence, let us consider

$$S_1 = \{x \in \mathbb{R}, g_j(x) \leq 0, j = 1, 2, \dots, m\},$$

and let  $I(x)$  be the set of active constraints.

Similar to the unconstrained case, our aim is to find the kind of functions for which the Karush–Kuhn–Tucker points and the optimums coincide.

**Definition 10** Suppose that the functions  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C, g_j : (a, b) \subseteq \mathbb{R} \rightarrow \mathbb{R}, j = 1, 2, \dots, m$ , are second-order differentiable at any  $\bar{x} \in S_1$ . A feasible point  $\bar{x}$  for CIOP is said to be a 2-Karush–Kuhn–Tucker stationary point (in short, 2-KKT point) if there exist nonnegative multipliers  $\lambda = (\lambda^L, \lambda^U), \mu_1, \dots, \mu_m$  with  $(\lambda, \mu) \neq (0, 0)$  such that

$$L'_x(\eta(x, \bar{x})) = \lambda f'_{gH}(\bar{x})(\eta(x, \bar{x})) + \sum_{j=1}^n \mu_j g'_j(\bar{x})(\eta(x, \bar{x})) = [0, 0], \tag{9}$$

$$\mu_j g_j(\bar{x}) = 0, \quad j = 1, 2, \dots, m, \tag{10}$$

$$\lambda f'_{gH}(\bar{x})(\eta(x, \bar{x})) = [0, 0], \tag{11}$$

$$\mu_j g'_j(\bar{x})(\eta(x, \bar{x})) = 0, \quad j \in I(\bar{x}), \tag{12}$$

$$L''_x(\eta(x, \bar{x})) = \lambda f''_{gH}(\bar{x})(\eta(x, \bar{x})) + \sum_{j \in I(\bar{x})} \mu_j g''_j(\bar{x})(\eta(x, \bar{x})) \geq 0, \tag{13}$$

where  $L = \lambda f + \sum_{j=1}^n \mu_j g_j$  is the Lagrange function.

These conditions are an extension of the classical KKT conditions by two further conditions.

In this section, as in the previous section, our aim to analyze the conditions under which the KKT points and minima coincide.

In the following, our focus is to extend the kind of KT-invex functions created by Martin [19] and other later ones introduced by Osuna et al. [27] to generalized invexity interval-valued functions. To do so, let us set the following definitions.

**Definition 11** The CIOP problem is said to be 2-KKT-pseudoinvex (2-KKT-PIX) with respect to  $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and for all feasible points  $x, \bar{x}$  for CIOP, if  $f(x) \ominus_{gH} f(\bar{x}) \prec [0, 0]$  then

$$f'_{gH}(\bar{x})(\eta(x, \bar{x})) \subseteq [0, 0], \tag{14}$$

$$0 \in f'_{gH}(\bar{x}) \Rightarrow f''_{gH}(\bar{x})(\eta(x, \bar{x})) \prec [0, 0], \tag{15}$$

$$g'_j(\bar{x})(\eta(x, \bar{x})) \leq 0, \quad \forall j \in I(\bar{x}), \tag{16}$$

$$g'_j(\bar{x})(\eta(x, \bar{x})) = 0, \quad \forall j \in I(\bar{x}) \Rightarrow g''_j(\bar{x})(\eta(x, \bar{x})) \leq 0, \tag{17}$$

where  $I(\bar{x}) = \{j = 1, \dots, m : g_j(\bar{x}) = 0\}$ .

*Remark 2* In [24] Osuna-Gómez et al. defined a multiobjective interval optimization problem (MIVOP) as KT-pseudoinvex-I problem if expressions (14) and (16) hold.

We now can obtain the sufficient condition for global optimality.

**Theorem 5** Suppose that the functions  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C, g_j : (a, b) \subseteq \mathbb{R} \rightarrow \mathbb{R}, j = 1, 2, \dots, m$ , are second-order differentiable at any  $\bar{x} \in S_1$ . Furthermore, assume that the constrained scalar interval optimization problem CIOP is 2-KKT pseudoinvex problem with respect to  $\eta$ . Then each 2-KKT point is a global minimum.

*Proof* Suppose that the constrained scalar interval optimization problem CIOP is a 2-KKT pseudoinvex problem with respect to  $\eta$  and that  $\bar{x} \in S_1$  is a 2-KKT point, and we need to prove that  $\bar{x}$  is a global minimum. By reductio ad absurdum, let us assume the opposite, and thus, that there is  $x \in S_1$  with  $f(x) \prec f(\bar{x})$ . By 2-KKT-PIX we have that  $f'_{gH}(\bar{x})(\eta(x, \bar{x})) \subseteq [0, 0]$  and  $g'_j(\bar{x})(\eta(x, \bar{x})) \leq 0, j \in I(\bar{x})$ .

Since  $x$  is a 2-KKT stationary point, there exist  $\lambda > 0$  and  $\mu_j \geq 0, j \in I(\bar{x})$  such that expressions (9)–(13) hold. Then we conclude from  $L'_x(\eta(x, \bar{x})) = 0$  that

$$f'_{gH}(\bar{x})(\eta(x, \bar{x})) = g'_j(\bar{x})(\eta(x, \bar{x})) = [0, 0], \quad \forall j \in I(\bar{x})$$

such that  $\mu_j > 0$ .

From the 2-KKT-PIX with respect to  $\eta$  of CIOP, we have that  $f''_{gH}(\bar{x})(\xi(x, \bar{x})) \prec [0, 0]$  and  $g''_j(\bar{x})(\eta(x, \bar{x})) \leq 0$  for all  $j \in I(\bar{x})$  with  $\mu_j \geq 0$ . Thus, on the one hand, we get  $L''_x(\eta(x, \bar{x})) \prec [0, 0]$ , and on the other hand, we get expression (13), a contradiction.  $\square$

*Remark 3* Osuna-Gómez et al. [24] proved that if MIVOP is a KT-pseudoinvex-I problem, then every interval KKT solution is a strictly weakly efficient solution.

We now can obtain the sufficient condition for global optimality.

**Theorem 6** *Suppose that:*

- (a) *The functions  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C, g_j : (a, b) \subseteq \mathbb{R} \rightarrow \mathbb{R}, j = 1, 2, \dots, m$ , are second-order differentiable at any  $\bar{x} \in S_1$ ;*
- (b) *The functions  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C, g_j : (a, b) \subseteq \mathbb{R} \rightarrow \mathbb{R}, j = 1, 2, \dots, m$ , are quasiinvex differentiable at  $\bar{x} \in S_1$  with respect to  $\eta$  nonidentically zero.*

*If each 2-KKT point is a global minimum, then the problem CIOP is 2-KKT pseudoinvex.*

*Proof* Suppose that each 2-KKT stationary point is a global minimum, we will prove that CIOP is 2-KKT pseudoinvex. Given two  $x, \bar{x} \in S_1$  points with

$$f(x) \prec f(\bar{x}). \tag{18}$$

According to the quasiinvexity of  $f$  at  $\bar{x} \in S_1$  with respect to  $\eta$ , the expression  $f'_{gH}(\bar{x})(\eta(x, \bar{x})) \subseteq [0, 0]$  holds.

If  $0 \in f'_{gH}(\bar{x})$ , we can prove that  $f''_{gH}(\bar{x})(\xi(x, \bar{x})) \prec [0, 0]$ .

By reductio ad absurdum, suppose that  $f''_{gH}(\bar{x})(\eta(x, \bar{x})) \geq [0, 0]$ , then  $\bar{x}$  is a 2-KKT point, which implies, by the hypothesis, that  $\bar{x}$  is a global minimum, which is in contradiction with expression (18).

We need to prove that  $g'_j(\bar{x})(\eta(x, \bar{x})) \leq 0$  and the expression

$$g'_j(\bar{x})(\eta(x, \bar{x})) = 0, \quad \forall j \in I(\bar{x}) \Rightarrow g''_j(\bar{x})(\eta(x, \bar{x})) \leq 0.$$

But it follows directly from the assumption  $x \in S_1, j \in I(\bar{x})$  and the quasiinvexity of  $g_j$  at  $\bar{x} \in S_1$  with respect to  $\eta$ .

In conclusion, all this shows that equations (14)–(17) hold, and then the problem CIOP is 2-KKT pseudoinvex with respect to  $\eta$ . □

And we obtain the following corollary.

**Corollary 7** *Suppose that:*

- (a) *The functions  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C, g_j : (a, b) \subseteq \mathbb{R} \rightarrow \mathbb{R}, j = 1, 2, \dots, m$ , are second-order differentiable at any  $\bar{x} \in S_1$ .*
- (b) *The functions  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C, g_j : (a, b) \subseteq \mathbb{R} \rightarrow \mathbb{R}, j = 1, 2, \dots, m$ , are quasiinvex differentiable at  $\bar{x} \in S_1$  with respect to  $\eta$  nonidentically zero.*

*Then, each 2-KKT point is a global minimum if and only if the problem CIOP is 2-KKT pseudoinvex with respect to  $\eta$ .*

Then, in quasiinvexity environments, if the problem is 2-KKT-PIX, the minima coincide with 2-KKT-points. This result generalizes Ivanov’s [13] Theorems 3.1 and 3.2, and Ruiz-Garzón et al., Corollary 2, [32] of scalar functions to interval-valued functions.

We now illustrate Corollary 7 with an example.

*Example 6* The functions  $f : \mathbb{R} \rightarrow \mathcal{K}_C, g : \mathbb{R} \rightarrow \mathbb{R}$  are differentiable and second-order gH-directionally differentiable at  $\bar{x} = 0$  with respect to  $\eta(x, \bar{x}) = (x^2 - \bar{x}^2) \in \mathbb{R}$ .

Let the constrained scalar interval optimization problem CIOP1 be defined as follows:

$$(CIOP1) \quad \text{Min}f(x) = [x^3, e^{x^2}]$$

subject to:

$$g(x) = x \geq 0$$

$$x \in \mathbb{R}.$$

We can calculate

$$f'_{gH}(\bar{x}) = [3x^2, 2xe^{x^2}], \quad f''_{gH}(\bar{x}) = [6x, 2e^{x^2} + 4x^2e^{x^2}],$$

$$g'(\bar{x}) = 1, \quad g''(\bar{x}) = 0.$$

We will see that  $\bar{x} = 0$  is a 2-KKT-point and a global minimum to the constrained scalar interval optimization problem CIOP1.

$$L'_x(\eta(x, \bar{x})) = \lambda f'_{gH}(\bar{x})(\eta(x, \bar{x})) + \sum_{j=1}^n g'_j(\bar{x})(\eta(x, \bar{x})) = [0, 0], \tag{19}$$

$$\mu_j g_j(\bar{x}) = 0, \quad j = 1, 2, \dots, m, \tag{20}$$

$$\lambda f'_{gH}(\bar{x})(\eta(x, \bar{x})) = [0, 0], \tag{21}$$

$$\mu_j g'_j(\bar{x})(\eta(x, \bar{x})) = 0, \quad j \in I(\bar{x}), \tag{22}$$

$$L''_x(\eta(x, \bar{x})) = \lambda f''_{gH}(\bar{x})(\eta(x, \bar{x})) + \sum_{j \in I(\bar{x})} \mu_j g''_j(\bar{x})(\eta(x, \bar{x})) \geq 0, \tag{23}$$

where  $L = \lambda f + \sum_{j=1}^n \mu_j g_j$  is the Lagrange function.

Now, according to conditions (19) and (23) of the theorem, we have to solve the following simultaneous equations:

$$\lambda^L (3\bar{x}^2)(x^2 - \bar{x}^2) + \mu_1 (x^2 - \bar{x}^2) = 0,$$

$$\lambda^U (2\bar{x}e^{\bar{x}^2})(x^2 - \bar{x}^2) + \mu_1 (x^2 - \bar{x}^2) = 0,$$

$$\mu_1(\bar{x}) = 0,$$

$$\lambda^L (3\bar{x}^2)(x^2 - \bar{x}^2) = 0,$$

$$\lambda^U (2\bar{x}e^{\bar{x}^2})(x^2 - \bar{x}^2) = 0,$$

$$\mu_1(x^2 - \bar{x}^2) = 0,$$

$$[\lambda^L, \lambda^U][6\bar{x}, 2e^{\bar{x}^2} + 4\bar{x}^2e^{\bar{x}^2}](x^2 - \bar{x}^2) + 0 \geq [0, 0].$$

We obtain  $\bar{x} = 0, \lambda^L = \lambda^U = 1, \mu_1 = 0$ .

Since  $f$  and  $g$  are quasiinvex at  $\bar{x} = 0$ , according to the previous Corollary 7, the constrained scalar interval optimization problem CIOP is then 2-KKT-PIX.

### 4 Duality

We consider the second-order interval Mond–Weir dual problem of the form:

$$(SOIDP) \quad \max \widehat{f}(u) = [f^L(u), f^U(u)]$$

$$\text{subject to } L''_u(\eta(x, u)) \geq 0$$

$$\sum_{j \in J} \mu_j g_j(u) \leq 0, j = 1, 2, \dots, m$$

$$x, u \in \mathbb{R}$$

$$\lambda = (\lambda^L, \lambda^U) \in \mathbb{R}_+^2 \setminus \{0\}, \mu_j \in \mathbb{R}_+, \text{ with } (\lambda, \mu) \neq (0, 0),$$

where  $L''_u(\eta(x, u)) = \lambda f''_{gH}(u)(\eta(x, u)) + \sum_{j \in I(u)} \mu_j g''_j(u)(\eta(x, u))$ . Hence, let us consider  $S_2$  the feasible set of SOIDP.

Our objective is to establish weak and strong duality theorems.

**Theorem 8** (Weak duality) *Let  $x \in S_1$  and  $(u, \lambda, \mu) \in S_2$ . Suppose that:*

- (a) *The functions  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C, g_j : (a, b) \subseteq \mathbb{R} \rightarrow \mathbb{R}, j = 1, 2, \dots, m$ , are second-order differentiable at any  $u \in S_2$ .*
- (b) *Let SOIDP be 2-KKT pseudoinvex problem with respect to  $\eta$  nonidentically zero.*

*If  $u$  is a 2-KKT point of SOIDP, then*

$$f(x) \leq \widehat{f}(u).$$

*Proof* By contradiction,  $f(x) \leq \widehat{f}(u) = f(u)$  since  $u$  is a 2-KKT-stationary point

$$L'_u(\eta(x, u)) = \lambda f'_{gH}(u)(\eta(x, u)) + \sum_{j=1}^n g'_j(u)(\eta(x, u)) = [0, 0], \tag{24}$$

$$\mu_j g_j(u) = 0, \quad j = 1, 2, \dots, m, \tag{25}$$

$$\lambda f'_{gH}(u)(\eta(x, u)) = [0, 0], \tag{26}$$

$$\mu_j g'_j(u)(\eta(x, u)) = 0, \quad j \in I(u), \tag{27}$$

$$L''_u(\eta(x, u)) = \lambda f''_{gH}(u)(\eta(x, u)) + \sum_{j \in I(u)} \mu_j g''_j(u)(\eta(x, u)) \geq 0. \tag{28}$$

From (26) and (27) it follows that

$$f'_{gH}(u)(\eta(x, u)) = [0, 0] \quad \text{such that } \lambda \in \mathbb{R}_+^2 \setminus \{0\}, \eta \neq 0, \tag{29}$$

$$g'_j(u)(\eta(x, u)) = 0, \quad j \in I(u) \text{ such that } \mu_j > 0. \tag{30}$$

From the 2-KKT-pseudoinvexity of (CIOP) it follows that

$$f''_{gH}(u)(\eta(x, u)) < [0, 0] \quad \text{such that } \lambda > 0, \tag{31}$$

$$g''_j(u)(\eta(x, u)) \leq 0, \quad j \in I(u) \text{ such that } \mu_j > 0. \tag{32}$$

From (31) and (32) we obtain that

$$L''_u(\eta(x, u)) = \lambda f''_{gH}(u)(\eta(x, u)) + \sum_{j \in I(u)} \mu_j g''_j(u)(\eta(x, u)) < [0, 0]. \tag{33}$$

Contradiction with (28). □

**Theorem 9** (Strong duality) *Let  $\bar{x} \in S_1$  be a 2-KKT point of CIOP. Let us assume that the functions  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C, g_j : (a, b) \subseteq \mathbb{R} \rightarrow \mathbb{R}, j = 1, 2, \dots, m$ , are second-order differentiable at any  $\bar{x} \in S_1$ . Then there exist  $\lambda = (\lambda^L, \lambda^U) \in \mathbb{R}_+^2 \setminus \{0\}, \mu_j \in \mathbb{R}_+$ , with  $(\lambda, \mu) \neq (0, 0)$  such that  $(\bar{x}, \lambda, \mu) \in S_2$  and*

$$f(\bar{x}) = \widehat{f}(\bar{x}).$$

Further, let the assumptions of Theorem 8 hold, then  $(\bar{x}, \lambda, \mu)$  is a solution of SOIDP.

*Proof* As  $\bar{x}$  is a 2-KKT point, there exist  $\lambda = (\lambda^L, \lambda^U) \in \mathbb{R}_+^2 \setminus \{0\}, \mu_j \in \mathbb{R}_+$  with  $(\lambda, \mu) \neq (0, 0)$  such that

$$\mu_j g_j(\bar{x}) = 0, \quad j = 1, 2, \dots, m, \tag{34}$$

$$L''_{\bar{x}}(\eta(x, \bar{x})) = \lambda f''_{gH}(\bar{x})(\eta(x, \bar{x})) + \sum_{j \in I(\bar{x})} \mu_j g''_j(\bar{x})(\eta(x, \bar{x})) \geq 0, \tag{35}$$

where  $L = \lambda f + \sum_{j=1}^m \mu_j g_j$  is the Lagrange function.

And therefore,  $(\bar{x}, \lambda, \mu) \in S_2$ , hence  $f(\bar{x}) = \widehat{f}(\bar{x})$ .

On the other hand, if  $(\bar{x}, \lambda, \mu)$  is not a solution of SOIDP, then there exists  $(u, \lambda, \mu) \in S_2$  such that

$$f(\bar{x}) = \widehat{f}(\bar{x}) \leq \widehat{f}(u).$$

Contradiction with Theorem 8. □

*Example 7* Let us consider the constrained scalar interval optimization problem CIOP1 as defined in Example 6. We denote the feasible set of CIOP1 by  $S_1$ .

The Mond–Weir dual problem related to CIOP1, denoted by SOIDP1, may be formulated as follows:

$$\begin{aligned} \text{(SOIDP1)} \quad & \max \widehat{f}(u) = [f^L(u), f^U(u)] = [u^3, e^{u^2}] \\ & \text{subject to } L''_u(\eta(x, u)) \geq 0 \\ & \sum_{j \in J} \mu_j g_j(u) \leq 0, j = 1, 2, \dots, m \\ & x, u \in \mathbb{R} \\ & \lambda = (\lambda^L, \lambda^U) \in \mathbb{R}_+^2 \setminus \{0\}, \mu_j \in \mathbb{R}_+, \text{ with } (\lambda, \mu) \neq (0, 0), \end{aligned}$$

where

$$\begin{aligned} L''_u(\eta(x, u)) &= \lambda f''_{gH}(u)(\eta(x, u)) + \sum_{j \in I(u)} \mu_j g''_j(u)(\eta(x, u)) \\ &= [\lambda^L, \lambda^U][6u, 2e^{u^2} + 4u^2 e^{u^2}](x^2 - u^2). \end{aligned}$$

Hence, let us consider  $S_2$  the feasible set of SOIDP1. The point  $\bar{x} = 0 \in S_1$  is a 2-KKT point of CIOP1 because conditions (19)–(23) hold, as we have seen in Example 6. Thus,

we see that all the assumptions for strong duality of Mond–Weir dual problem are satisfied. Hence, there exist  $\lambda^L = \lambda^U = 1, \mu_1 = 0$  such that  $(\bar{x}, \lambda, \mu)$  is a feasible point of SOIDP1 and  $f(\bar{x}) = \widehat{f}(\bar{x})$ .

Now, from Example 6, we see SOIDP1 is a 2-KKT-PIX problem, then  $(\bar{x}, \lambda, \mu)$  is a solution of SOIDP1.

### 5 Extension: semi-infinite case

Let us consider the semi-infinite interval optimization problem (SIOP) defined as follows:

$$\begin{aligned} \text{(SIOP)} \quad & \min f(x) = [f^L(x), f^U(x)] \\ & \text{subject to:} \\ & g_t(x) \leq 0, \quad t \in T, \\ & x \in \mathbb{R}, \end{aligned}$$

where  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C, g_t : (a, b) \subseteq \mathbb{R} \rightarrow \mathbb{R}, t \in T$  is a set of differentiable functions. We denote the feasible solution set of (SIOP):

$$S_2 = \{x \in \mathbb{R} \mid g_t(x) \leq 0, t \in T\}.$$

The index set  $T$  is an arbitrary nonempty set, not necessarily finite and  $T(\bar{x}) = \{t \in T \mid g_t(\bar{x}) = 0\}$ . The set of active constraint multipliers at  $\bar{x} \in S$  is

$$\Lambda(\bar{x}) = \{\mu \in \mathbb{R}_+^{|T|} \mid \mu_t g_t(\bar{x}) = 0, \forall t \in T\}.$$

As in the finite case, we can define the following.

**Definition 12** Suppose that the functions  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C, g_t : (a, b) \subseteq \mathbb{R} \rightarrow \mathbb{R}, t \in T$ , are second-order differentiable at any  $\bar{x} \in S_2$ . A feasible point  $\bar{x}$  for SIOP is said to be a 2-Karush–Kuhn–Tucker stationary point (in short, 2-KKT point) if there exist nonnegative multipliers  $\lambda = (\lambda^L, \lambda^U), \mu \in \Lambda(\bar{x})$  with  $(\lambda, \mu) \neq (0, 0)$  such that

$$L'_x(\eta(x, \bar{x})) = \lambda f'_{gH}(\bar{x})(\eta(x, \bar{x})) + \sum_{t \in T} \mu_t g'_t(\bar{x})(\eta(x, \bar{x})) = [0, 0], \tag{36}$$

$$\mu_t g_t(\bar{x}) = 0, \quad t \in T, \tag{37}$$

$$\lambda f'_{gH}(\bar{x})(\eta(x, \bar{x})) = [0, 0], \tag{38}$$

$$\mu_t g'_t(\bar{x})(\eta(x, \bar{x})) = 0, \quad t \in T(\bar{x}), \tag{39}$$

$$L''_x(\eta(x, \bar{x})) = \lambda f''_{gH}(\bar{x})(\eta(x, \bar{x})) + \sum_{t \in T(\bar{x})} \mu_t g''_t(\bar{x})(\eta(x, \bar{x})) \geq 0, \tag{40}$$

where  $L = \lambda f + \sum_{t \in T} \mu_t g_t$  is the Lagrange function.

**Definition 13** The SIOP problem is said to be 2-KKT-pseudoinvex (2-KKT-PIX) with respect to  $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and for all feasible points  $x, \bar{x}$  for SIOP, if  $f(x) \ominus_{gH} f(\bar{x}) < [0, 0]$

then

$$f'_{gH}(\bar{x})(\eta(x, \bar{x})) \leq [0, 0], \tag{41}$$

$$0 \in f'_{gH}(\bar{x}) \Rightarrow f''_{gH}(\bar{x})(\eta(x, \bar{x})) < [0, 0], \tag{42}$$

$$g'_t(\bar{x})(\eta(x, \bar{x})) \leq 0, \quad \forall t \in T(\bar{x}), \tag{43}$$

$$g'_t(\bar{x})(\eta(x, \bar{x})) = 0, \quad \forall t \in T(\bar{x}) \Rightarrow g''_t(\bar{x})(\eta(x, \bar{x})) \leq 0, \tag{44}$$

where  $T(\bar{x}) = \{t \in T | g_t(\bar{x}) = 0\}$ .

And we obtain the following corollary.

**Corollary 10** *Suppose that:*

- (a) *The functions  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C, g_t : (a, b) \subseteq \mathbb{R} \rightarrow \mathbb{R}, t \in T$ , are second-order differentiable at any  $\bar{x} \in S_2$ ;*
- (b) *The functions  $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathcal{K}_C, g_t : (a, b) \subseteq \mathbb{R} \rightarrow \mathbb{R}, t \in T$ , are quasiinvex differentiable at  $\bar{x} \in S_2$  with respect to  $\eta$  nonidentically zero.*

*Then each 2-KKT point is a global minimum if and only if the problem SIOP is 2-KKT pseudoconvex with respect to  $\eta$ .*

*Proof* The proof of this result follows the lines of the theorems obtained for the finite case. □

We now illustrate the previous corollary 10 with an example.

*Example 8* The functions  $f : \mathbb{R} \rightarrow \mathcal{K}_C, g : \mathbb{R} \rightarrow \mathbb{R}$  are differentiable and second-order gH-directionally differentiable at  $\bar{x} = 1$  with respect to  $\eta(x, \bar{x}) = (x^2 - \bar{x}^2) \in \mathbb{R}$ .

Let the semi-infinite scalar interval optimization problem SIOP be defined as follows:

$$(SIOP) \quad \min f(x) = [f^L(x), f^U(x)] = [x^3, e^{x^2}]$$

subject to:

$$g_t(x) = kx - k - 1 \leq 0, \quad t \in T = [-1, 1],$$

$$x \in S_2 \subseteq \mathbb{R},$$

then  $g_t(x) \leq 0, \forall t \in T \Leftrightarrow x \in [0, 2], S_2 = [0, 2]$ .

We can calculate:

$$f'_{gH}(\bar{x}) = [3x^2, 2xe^{x^2}], \quad f''_{gH}(\bar{x}) = [6x, 2e^{x^2} + 4x^2e^{x^2}],$$

$$g'(\bar{x}) = \{k\}, \quad g''(\bar{x}) = 0.$$

We will see that  $\bar{x} = 0$  is a 2-KKT-point and a global minimum to the semi-infinite scalar interval optimization problem SIOP.

$$L'_{\bar{x}}(\eta(x, \bar{x})) = \lambda f'_{gH}(\bar{x})(\eta(x, \bar{x})) + \sum_{t \in T} g'_t(\bar{x})(\eta(x, \bar{x})) = [0, 0], \tag{45}$$



$$\mu_t g_t(\bar{x}) = 0, \quad t \in T, \tag{46}$$

$$\lambda f'_{gH}(\bar{x})(\eta(x, \bar{x})) = [0, 0], \tag{47}$$

$$\mu_t g'_t(\bar{x})(\eta(x, \bar{x})) = 0, \quad t \in T(\bar{x}), \tag{48}$$

$$L''_{\bar{x}}(\eta(x, \bar{x})) = \lambda f''_{gH}(\bar{x})(\eta(x, \bar{x})) + \sum_{t \in T(\bar{x})} \mu_t g''_t(\bar{x})(\eta(x, \bar{x})) \geq 0, \tag{49}$$

where  $L = \lambda f + \sum_{t \in T} \mu_t g_t$  is the Lagrange function.

Now, according to conditions (45) and (49) of the theorem, we have to solve the following simultaneous equations:

$$\begin{aligned} \lambda^L (3\bar{x}^2)(x^2 - \bar{x}^2) + \mu_1 k(x^2 - \bar{x}^2) &= 0, \\ \lambda^U (2\bar{x}e^{\bar{x}^2})(x^2 - \bar{x}^2) + \mu_1 k(x^2 - \bar{x}^2) &= 0, \\ \mu_1 (k\bar{x} - k + 1) &= 0, \\ \lambda^L (3\bar{x}^2)(x^2 - \bar{x}^2) &= 0, \\ \lambda^U (2\bar{x}e^{\bar{x}^2})(x^2 - \bar{x}^2) &= 0, \\ \mu_1 k(x^2 - \bar{x}^2) &= 0, \\ [\lambda^L, \lambda^U][6\bar{x}, 2e^{\bar{x}^2} + 4\bar{x}^2 e^{\bar{x}^2}](x^2 - \bar{x}^2) + 0 &\geq [0, 0]. \end{aligned}$$

We obtain  $\bar{x} = 0, \lambda^L = \lambda^U = 1, \mu_1 = 0$ .

Since  $f$  and  $g$  are quasiinvex at  $\bar{x} = 0$ , according to the previous Corollary 10, the semi-infinite scalar interval optimization problem SIOP is then 2-KKT-PIX. Thus, we have extended the optimality conditions for the finite optimization problem to the semi-infinite case.

### 6 Conclusions

Throughout this paper, we have obtained second-order optimality conditions for the scalar interval optimization problems in both the constrained and unconstrained cases. This has been done by extending the notions from the literature of Ginchev and Ivanov [8], Ivanov [13, 14], and Ruiz-Garzón [32] of scalar functions to interval-valued functions. We have analyzed the precise conditions for the critical or stationary points to coincide with the minimums of the interval optimization problem.

To do so, our work has proposed and made use of:

- An adequate second-order gH-differential definition of an interval-valued function.
- An extension of concepts such as 2-invexity, 2-pseudoinvexity, and 2-critical points to the case of interval-valued functions.
- Identifying the 2-invex with the 2-pseudoinvex interval-valued functions and characterizing them according to the equivalence of 2-critical points and minimums.
- An adequate 2-KKT stationary point and 2-KKT-pseudoinvex problem for the constrained interval optimization problem has allowed us to identify the conditions under which stationary points and minima coincide.
- A dual problem model of the Mond–Weir type, deriving the weak and strong duality theorems related to the primal problem.

- A generalization of the optimality conditions for the semi-infinite scalar interval optimization problem SIOP.

One possible way forward may be to implement the theoretical conditions under which such coincidences occur in numerical software to find the minima of the interval optimization problem.

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#### Availability of data and materials

Not applicable.

#### Declarations

##### Competing interests

The authors declare no competing interests.

##### Author contributions

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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