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A remark about asymptotic stability in Duffing equations: lateral stability in Comb-drive finger MEMS

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Abstract

In this short paper we tackle two subjects. First, we provide a lower bound for the first eigenvalue of the antiperiodic problem for a Hill's equation based on L^p-conditions, and as a consequence, we introduce an adjusted statement of the main result about the asymptotic stability of periodic solutions for the general Duffing equation in (Torres in Mediterr. J. Math. 1(4):479–486, 2004) (Theorem 4). This appropriate version of the result arises because of one subtlety in the proof provided in (Torres in Mediterr. J. Math. 1(4):479–486, 2004). More precisely, the lower bound of the first antiperiodic eigenvalue associated with Hill's equations of potential $a(t)$ employed there may be negative, thus the conclusion is not completely attained. Hence, the adjustments considered here provide a mathematically correct result. On the other hand, we apply this result to obtain a lateral asymptotic stable periodic oscillation in the Comb-drive finger MEMS model with a cubic nonlinear stiffness term and linear damping. This fact is not typical in Comb-drive finger devices, thus our results could provide a new possibility; a new design principle for stabilization in Comb-drive finger MEMS.

Keywords: MEMS; Comb-drive; Periodic solution; Asymptotic stability; Lower and upper solutions method; Antiperiodic eigenvalues

1 Introduction

This paper aims to provide a lower bound for the first eigenvalue of the antiperiodic problem for a Hill's equation. By combining this estimation with other inequalities we prove a result about asymptotic stability of periodic solutions for certain Duffing equations in the line of [1]. First, we shall introduce some basic notions.

For a given $f \in L^1(0,T)$, we write $f \succ 0$ if $f(t) \geq 0$ for almost every t and it is positive in a subset of positive measure. Let us define the set $W = \{u \in W^{2,1}(0,T) : u(0) = u(T), \dot{u}(0) = \dot{u}(T)\}$. For a given $a \in L^1(0,T)$ and $c \geq 0$, the differential operator $L : W \to L^1(0,T)$ defined as

$$Lu := \ddot{u} + c\dot{u} + a(t)u,$$

is called *inversely positive* if it is invertible and $Lu \succ 0$ implies $u > 0$. The notion of an *inversely positive* operator, also known as a maximum principle, has relevant applications in the study of nonlinear oscillators. In reference [1], Corollary 2.5 provides a sufficient condition that is incorrect because it is based on the use of inequality (2.4) therein. We shall explain the details in Sect. 2 and develop the correct version of the results towards the end of this introduction. In particular, we obtain sufficient conditions over L for the *inversely positive* property (see Theorem 2). In Sect. 3 we shall apply these new theoretical results to the study of the dynamics of some MEMS. More precisely, we obtain a result about the existence of a positive and asymptotically locally stable periodic solution for the Comb-drive finger MEMS model with cubic stiffness and linear damping, a novel result that is not yet reported in the literature to our knowledge.

Finding interesting and deep connections between topological degree and asymptotic stability, the seminal papers [2–4] provide sufficient conditions to obtain at least one asymptotically locally stable periodic solution for differential equations of Liénard type. The author in [1] generalized the results in [4] for Duffing equations $\ddot{x} + c\dot{x} + g(t,x) = 0$ ($c > 0$) with weaker conditions of L^p type over g. The proofs of this author employ a lower bound of the corresponding first eigenvalue λ_0^A for the antiperiodic problem of a Hill's equation due to Zhang ([5]) that only works when $\lambda_0^A \geq 0$. Here, we provide a lower bound of λ_0^A for the negative case (Theorem 5). This permits us to present complete forms of the results in [1] and some consequences. Next, we shall complete the main results in [1] (Corollary 2.4, Corollary 2.5, Lemma 3.2, and Theorem 4.2) by means of a redefinition of the set $\Omega_{p,c}$ there. It seems to us that $\Omega_{p,c}$ has been employed in works like [6–9], so we believe that the results obtained here could be of interest for those authors.

The solution is simple. The set $\Omega_{p,c}$ in [1] should be redefined by the following: For any $p \in [1,\infty]$ and $c > 0$ consider

$$\hat{\Omega}_{p,c} := \left\{ a \in L^\infty(0,T) : a \succ 0,\ \|a\|_p < \left(1 + \frac{c^2}{4\|a\|_\infty}\right) K\left(2p^*, T\right) \right\}, \tag{1}$$

where $K(p,T)$ is the best Sobolev constant in the inequality

$$C\|u\|_p^2 \leq \|\dot{u}\|_2^2,$$

for all $u \in H_0^1(0,T)$ and p^* be the conjugated exponent of p.

Therefore, the Corollary 2.4, Corollary 2.5, and Lemma 3.2 in [1] could be changed with the following alternative inequality:

$$\|A\|_p < \left(1 + \frac{c^2}{4\|a\|_\infty}\right) K\left(2p^*, T\right),$$

with $A = a^+$, a, and a^+, respectively.

The cornerstone result in the referred article consists in assuming conditions in order to guarantee that $\lambda_0^A[a] + \frac{c^2}{4} > 0$. Thus, we obtain our first main result that relies on our key statement about a lower bound of the eigenvalue for the antiperiodic problem in Hill's equations (Theorem 5 in Sect. 2).

Theorem 1 *Let* $p \in [1,\infty]$ *and assume that* $a \in \hat{\Omega}_{p,c}$. *Then,* $\lambda_0^A[a] + \frac{c^2}{4} > 0$.

As a consequence, we formulate a result that should be considered as a corrected version of Lemma 2.3 in [1].

Theorem 2 *Let $p \in [1,\infty]$ and assume that $a \in \hat{\Omega}_{p,c}$. Then, the operator L is inversely positive.*

In this way, the main result in [1] (Theorem 4.2) can be reformulated as follows. First, we shall introduce a definition from the Lower and Upper Solutions approach. Consider the general Duffing equation

$$\ddot{x} + c\dot{x} + g(t,x) = 0, \tag{2}$$

where $c > 0$ and $g : [0,T] \times]x_1, x_2[\to \mathbb{R}$ is a L^1-Carathéodory function ($x_1 < x_2$) such that the partial derivative g_x exists and it is also L^1-Carathéodory. We say that a function $x(t) \equiv \ell(t)$ (resp., $x(t) \equiv u(t)$) is a lower (resp., upper) solution of the periodic problem associated with (2) if the left-hand side of (2) evaluated in $x(t)$ is ≥ 0 (resp., ≤ 0). Additionally, when the strict inequality is verified then these solutions are called strict lower (resp., upper) solutions.

Theorem 3 *Let $\ell > u$ be a couple of strict lower and upper solutions of equation (2). Assume that for a given $p \in [1,\infty]$ there exists $a \in \hat{\Omega}_{p,c}$ verifying that*

$$g_x(t,x) \leq a(t), \quad \text{for a.e. } t \in [0,T], \forall x \in [u(t), \ell(t)].$$

Then, (2) has at least an asymptotically locally stable T-periodic solution $v(t)$ such that $u < v < \ell$, provided that the number of T-periodic solutions between u and ℓ is finite.

Remark 1 The conditions in the precedent theorem can we rewritten as

$$\frac{\|a\|_\infty \|a\|_p}{\|a\|_\infty + \frac{c^2}{4}} \leq K\big(2p^*, T\big), \tag{3}$$

thus we have a control by means of two measures over a and the friction. The condition trivially holds when c is very large and also when a is "very small". On the other hand, note that

$$\frac{\|a\|_\infty \|a\|_p}{\|a\|_\infty + \frac{c^2}{4}} \leq G_{p,c}\big(\|a\|_\infty\big),$$

where $G_{p,c}(u) = \frac{u^2 T^{1/p}}{u^2 + \frac{c^2}{4}}$ is an increasing function on $[0,\infty]$ with $G_{p,c}(0) = 0$. Thus, there exists a unique $\zeta_p \in]0,\infty[$ such that $G_{p,c}(\zeta_p) = K(2p^*, T)$ and $G_{p,c}(u) < K(2p^*, T)$ for all $u \in]0, \zeta_p[$. In particular, we have the following sufficient condition:

$$a \succ 0, \quad \|a\|_\infty < \zeta_p.$$

For the case $p = \infty$, the set $\hat{\Omega}_{\infty,c}$ is defined by the above condition with

$$\zeta_\infty := \frac{K + \sqrt{K(K + c^2)}}{2}, \quad K = K(2,T) = \left(\frac{\pi}{T}\right)^2.$$

2 Proofs

Next, we present the proofs of the Theorems in the Introduction. Let us consider the following second-order differential equation

$$\ddot{u} + \big(a(t) + \lambda\big)u = 0, \tag{4}$$

where $a(t)$ is a real-valued T-periodic function for $T > 0$ such that $a \succ 0$ ($a \in L^\infty(0,T)$) and $\lambda \in \mathbb{R}$. Henceforth, we shall consider the following boundary conditions associated with (4)

$$u(0) = u(T) = 0 \tag{5}$$

and

$$u(0) = -u(T), \qquad \dot{u}(0) = -\dot{u}(T). \tag{6}$$

Let $\lambda_0^D[a]$ and $\lambda_0^A[a]$ be the first eigenvalues associated to the corresponding homogeneous Dirichlet problem (problem (4)+(5)) and the antiperiodic problem (problem (4)+(6)), respectively.

From the variational characterization of the spectrum of (4) we have that

$$\lambda_0^A[a] = \min\big\{\lambda_0^D[a_s] : s \in \mathbb{R}\big\}, \tag{7}$$

where $a_s(t) \equiv a(t+s)$ and

$$\lambda_0^D[a] = \min_{u \in H_0^1(0,T),\, \|u\|_2 = 1} q(u), \tag{8}$$

where q is the quadratic form

$$q(t) = \int_0^T \dot{u}^2(t)\,dt - \int_0^T a(t)u^2(t)\,dt. \tag{9}$$

We remark that $\lambda_0^D[a]$ is simple, and if ϕ is an eigenfunction associated with the homogeneous Dirichlet problem then the equality in (8) is satisfied. In that sense, there exists exactly one function $\phi_0 \in H_0^1(0,T)$ such that $\|\phi_0\|_2 = 1$, $\phi_0 > 0$ on $]0,T[$ and ϕ_0 is a solution of (4)+(5).

Next, we recall a lower bound of $\lambda_0^D[a]$ that is based on the ideas in [5] and [10]. First, let us consider for any $p \in [1,\infty]$ the best Sobolev constant in the inequality

$$C\|u\|_p^2 \le \|\dot{u}\|_2^2,$$

for all $u \in H_0^1(0,T)$ and let p^* be the conjugated exponent of p. Thus, we have that for $u \in H_0^1(0,T)$ such that $\|u\|_2 = 1$, the Hölder inequality and the definition of the best Sobolev constant $K(2p^*, T)$ imply that

$$\begin{aligned} q(u) &= \|\dot{u}\|_2^2 - \int_0^T a(t)u^2(t)\,dt \\ &\geq \|\dot{u}\|_2^2 - \|a\|_p \|u\|_{2p^*}^2 \\ &\geq \|\dot{u}\|_2^2 \left(1 - \frac{\|a\|_p}{K(2p^*, T)}\right). \end{aligned} \tag{10}$$

As a consequence of the previous inequality we have the following result ([5] and [10]).

Theorem 4 *Let $p \in [1, \infty]$, and consider the equation (4) where the potential $a(t)$ is a real-valued T-periodic function in $L^\infty(0, T)$ for $T > 0$, and $a \succ 0$. If $\|a\|_p \leq K(2p^*, T)$, then for some $s_0 \in \mathbb{R}$, $\lambda_0^A[a] = \lambda_0^D[a_{s_0}] \geq K(2, T)(1 - \frac{\|a\|_p}{K(2p^*,T)}) \geq 0$.*

Now, we are able to present our second main result.

Theorem 5 *Let p and s_0 be as in Theorem 4. Assume that $\lambda_0^A[a] = \lambda_0^D[a_{s_0}] < 0$. Then, $\|a\|_p > K(2p^*, T)$ and $\lambda_0^A[a] \geq \|a\|_\infty(1 - \frac{\|a\|_p}{K(2p^*,T)})$.*

Proof The first assertion is a direct consequence of Theorem 4. In order to prove the second part, let us consider ϕ_0 defined above for the coefficient a_{s_0}. Hence,

$$\int_0^T \ddot{\phi}_0(t)\phi_0(t)\,dt = -\int_0^T \left(a_{s_0}(t) + \lambda_0^D[a_{s_0}]\right)\phi_0^2(t)\,dt \tag{11}$$

and thus

$$\int_0^T \dot{\phi_0}^2(t)\,dt = \int_0^T \left(a(t+s_0) + \lambda_0^D[a_{s_0}]\right)\phi_0^2(t)\,dt \tag{12}$$

and since the hypothesis implies that $(a_{s_0}(t) + \lambda_0^D[a_{s_0}])\phi_0^2(t) \leq a_{s_0}(t)\phi_0^2(t)$ for all $t \in [0, T]$ and $\|\phi_0\|_2 = 1$ we conclude the following

$$\begin{aligned} \|\dot{\phi}_0\|_2^2 &\leq \int_0^T a(t+s_0)\phi_0^2(t)\,dt \\ &\leq \|a\|_\infty. \end{aligned} \tag{13}$$

Finally, the inequality in (10) with ϕ_0 shows that

$$\begin{aligned} \lambda_0^A[a] &= \lambda_0^D[a_{s_0}] \\ &= q(\phi_0) \\ &\geq \|\dot{\phi}_0\|_2^2 \left(1 - \frac{\|a\|_p}{K(2p^*, T)}\right) \\ &\geq \|a\|_\infty \left(1 - \frac{\|a\|_p}{K(2p^*, T)}\right), \end{aligned} \tag{14}$$

as $\frac{\|a\|_p}{K(2p^*,T)} > 1$. □

Proof of Theorem 1 It is a straightforward consequence of Theorem 5 and the definition in (1). More precisely, if $\lambda_0^A[a] \geq 0$ then we have nothing to prove. If $\lambda_0^A[a] < 0$, then

$$\lambda_0^A[a] + \frac{c^2}{4} \geq \|a\|_\infty \left(1 - \frac{\|a\|_p}{K(2p^*, T)}\right) + \frac{c^2}{4} > 0$$

and the last inequality is equivalent to the hypothesis about the p-norm of a. □

3 Applications: lateral asymptotic locally stable periodic oscillations in MEMS

In this section, we shall introduce our main results regarding the applications to microelectromechanical systems (MEMS). These are micro-scale devices that integrate mechanical and electronic components in a common substrate, and that are employed in different fields as the automotive industry, the medical and biomedical industry, the aerospace sector, etc. (see for example [12]). In this case, we shall consider an interdigitated Comb-drive MEMS modeled with a cubic stiffness term. This device consists of tow comb-like structures (with interdigitated fingers or electrodes); the first structure is movable and it is attached to flexures, whereas the second comb is stationary. Because the device is actuated by a nonconstant periodic input voltage, we have that, in appropriate units of distance and time, the spring–mass model that describes the behavior of the movable component is given by (see [11, 12])

$$\ddot{x} + c\dot{x} + x\left(1 + \alpha x^2 - \frac{\theta V^2(t)}{(1 - x^2)^2}\right) = 0, \quad |x| < 1, \tag{15}$$

where $c > 0$, $\alpha > 2$, and $\theta > 0$ are physical constants of the system, and $V(t) = V_{dc} + \delta \cos(\omega t)$, $\delta \in]0, V_{dc}[$, is the nonconstant positive T-periodic input AC–DC voltage function. Moreover, we shall consider

$$V_m := \min_{t \in [0,T]} V(t), \qquad V_M := \max_{t \in [0,T]} V(t).$$

The mathematical study of the dynamics for Comb-drive finger devices and other MEMS has been of remarkable interest in recent years, see for instance [8, 11, 13–18]. In fact, regarding the stability of positive, periodic solutions (or lateral solutions) in MEMS actuators, the authors in [11] studied a cubic Comb-drive finger MEMS without damping and proved the existence and linear stability of at least one periodic solution under some conditions on the parameters of the system. Here, we are going to tackle the damped problem and the possibility of asymptotic stability in this frame.

Let us consider the auxiliary function for this model that is defined by

$$\varphi_\alpha(x) = \left(1 + \alpha x^2\right)\left(1 - x^2\right)^2, \tag{16}$$

and consider the following quantities for each $\alpha > 2$

$$\eta_m := \frac{1}{\sqrt{\theta}}, \qquad \eta_M := \sqrt{\frac{\varphi_\alpha(C_\alpha)}{2\theta}}, \quad C_\alpha := \sqrt{\frac{\alpha - 2}{3\alpha}}.$$

Hereafter, we shall use the following claim about the increasing behavior of the maximum value of the auxiliary function that is given by $q(\alpha) := \varphi_\alpha(C_\alpha)$ for $\alpha > 2$. This assertion is a direct consequence of the fact that $\frac{dq}{d\alpha}(\alpha) = \frac{\partial\varphi_\alpha}{\partial\alpha}(C_\alpha) + \frac{\partial\varphi_\alpha}{\partial x}(C_\alpha)\frac{dC_\alpha}{d\alpha} = C_\alpha^2(1-C_\alpha^2)^2 > 0$, $\lim_{\alpha\to\infty} q(\alpha) = \infty$, and $\lim_{\alpha\to 2^+} q(\alpha) = 1$.

Claim 1 *The function $q(\alpha)$ is a monotonic increasing function. Moreover, there exists $\alpha^* > 2$ such that for $\alpha > \alpha^*$ $\varphi_\alpha(C_\alpha) > 2$. In other words, α^* is the unique root of $q(\alpha) = 2$ for $\alpha > 2$.*

Next, we follow the lower and upper solution approach. We recall from Sect. 1 that a constant function $x(t) \equiv \ell$ (resp., $x(t) \equiv u$) is a lower (resp., upper) solution of the periodic problem associated with (15) if the left-hand side of (15) evaluated in $x(t)$ is ≥ 0 (resp., ≤ 0). Equivalently, when $\ell > 0$ ($u > 0$) the following inequality holds

$$\varphi_\alpha(\ell) - \theta V_M^2 \geq 0 \quad \bigl(\text{resp., } \varphi_\alpha(u) - \theta V_m^2 \leq 0\bigr).$$

Thus, we obtain the following result regarding the existence of nonstrict constant lower and upper solutions for (15).

Lemma 1 *Assume that $\alpha > \alpha^*$ for α^* defined in Claim 1 and that*

$$V_m, \quad V_M \in]\eta_m, \eta_M[. \tag{17}$$

Then, there exist roots u_i (resp., ℓ_i) $(i = 1, 2)$ on $]0, 1[$ for the equation $\varphi_\alpha(x) - h = 0$ with $h = \theta V_m^2$ (resp., $h = \theta V_M^2$), verifying that

$$0 < u_1 < \ell_1 < C_\alpha < \ell_2 < u_2 < 1. \tag{18}$$

Proof Note that from Claim 1, if $\alpha > \alpha^*$ then $\eta_m < \eta_M$. In addition, the condition (17) implies that $\varphi_\alpha(0) = 1 < \theta V_m^2$ and $\theta V_M^2 < \varphi_\alpha(C_\alpha)$. Moreover, the function $\varphi_\alpha(x)$ is even, it has a local minimum at $x = 0$ and exactly one local maximum on $]0, 1[$ that is reached at $x = C_\alpha$, thus $\varphi'_\alpha(x) > 0$ on $]0, C_\alpha[$ and $\varphi'_\alpha(x) < 0$ on $]C_\alpha, 1[$. Then, the Intermediate Value Theorem guarantees the existence of exactly one root of $\varphi_\alpha(x) - h = 0$ on $]0, C_\alpha[$ and $]C_\alpha, 1[$ for $h = \theta V_m^2$ and $h = \theta V_M^2$, respectively. Moreover, the order of the roots in (18) is easy to deduce from the previous remarks about $\varphi_\alpha(x)$. □

Remark 2 The roots ℓ_i (resp., u_i) $(i = 1, 2)$ given by the former lemma are nonstrict constant lower (resp., upper) solutions of (15). Nevertheless, hereafter we shall also consider the corresponding strict lower L_i (resp., upper U_i) solutions defined by $L_i = \ell_i + (-1)^{i+1}\epsilon_i$ (resp., $U_i = u_i + (-1)^i\epsilon_i$) for suitable fixed, positive constants $\epsilon_i = \epsilon_i(\alpha) \ll 1$, and such that $0 < U_1(\alpha) < L_1(\alpha) < C_\alpha < L_2(\alpha) < U_2(\alpha) < 1$.

Now let us consider the function $g : \mathbb{R}\times]-1, 1[\to \mathbb{R}$ defined by

$$g(t, x) = x + \alpha x^3 - \frac{\theta V^2(t)x}{(1-x^2)^2}.$$

With a view to applying the main result of the previous section, we are interested in studying multiplicity for periodic solutions between the lower solution U_1 and the upper solution L_1. Hence, we need to compute an α-interval for which g_{xx} is different from zero on

$\mathbb{R}\times\,]U_1, L_1[$. A straightforward computation shows that for $f(x) = \frac{1+x^2}{(1-x^2)^4}$, we obtain

$$g_{xx}(t,x) = 6x\big(\alpha - 2\theta V^2(t)f(x)\big), \tag{19}$$

so the sign of g_{xx} on $\mathbb{R}\times\,]U_1, L_1[$ is the sign of the function $\mathcal{A}(t,x) := \alpha(1-x^2)^4 - 2\theta V^2(t)(1+x^2)$ on the same domain.

Therefore, let us assume the same hypotheses as in Lemma 1 and note that $f'(x) = \frac{2x(5+3x^2)}{(1-x^2)^5} > 0$ for all $x \in\,]0,1[$, thus $\mathcal{A}(t,x) > 0$ on $\mathbb{R} \times [0, C_\alpha]$ if and only if α verifies on the same domain the following

$$\alpha > 2\theta V^2(t)f(x),$$

which is equivalent to having that

$$\alpha > 2\theta V_M^2 f(C_\alpha)$$

and then by condition (17) this last inequality is satisfied whenever

$$\alpha > q(\alpha)f(C_\alpha). \tag{20}$$

It is not difficult to check that $w(x) := \alpha - \varphi_\alpha(x)f(x)$ is a decreasing function on $]0,1[$ that vanishes at $x = r_\alpha$ for $r_\alpha = \sqrt{\frac{\alpha-1}{3\alpha+1}}$ and $w(0) = \alpha - 1 > 0$. On the other hand, an easy computation shows that $C_\alpha < r_\alpha$ so (20) holds. Then, we have proved the next lemma.

Lemma 2 *Assume that α, V_m, and V_M verify the hypotheses in Lemma 1. Then, $g_{xx} > 0$ on $\mathbb{R} \times [0, C_\alpha]$.*

Now, let us define for $\alpha > \alpha^*$ the following function

$$h(x) := \frac{1+3x^2}{(1-x^2)^3}. \tag{21}$$

A straightforward computation shows that $h'(x) = 12xf(x)$ for $f(x)$ defined above. Thus, h is increasing on $]0,1[$.

Theorem 6 *Assume that $\alpha > \alpha^*$ and let V_m, V_M verify the condition (17). If there exists some $p \in [1,\infty]$ such that $a(t) := \alpha - 1 - \theta V^2(t)$ verifies*

$$\|a\|_p < \left(1 + \frac{c^2}{4(\alpha - 1 - \theta V_m^2)}\right)K\big(2p^*, T\big). \tag{22}$$

Then, (15) has a positive asymptotic locally stable periodic solution $v(t)$, such that

$$U_1 < v(t) < L_1.$$

Proof The proof will be made in several steps.

Step 1. Multiplicity:

Here, we aim to prove that the equation (15) admits at most two T-periodic solutions with range in $]U_1, L_1[$. This is a consequence of a known result of disconjugacy for linear second-order differential equations and a multiplicity result for a Duffing equation with convex potential in a finite domain. More precisely, if $b \in \hat{\Omega}_{p,c}$ is a T-periodic function, then Theorem 1 reveals that the first eigenvalue of the antiperiodic problem associated with the linear equation $\ddot{x} + c\dot{x} + b(t)x = 0$ is positive. Therefore, Lemma 2.1 and the subsequent remark in [1] imply that the distance between consecutive zeros of any nontrivial solution of this last linear equation is greater than T (disconjugacy property). We note that, under certain conditions, the disconjugacy property holds for linear differential equations with potentials bounded above by $b(t)$. The following claim summarizes this result.

Claim 2 *Assume that b_1 and b are T-periodic functions in $L^\infty(\mathbb{R})$ such that $b_1 \leq b$ with $b \in \hat{\Omega}_{p,c}$. Then, $\ddot{x} + c\dot{x} + b_1(t)x = 0$ enjoys the disconjugacy property.*

On the other hand, by following the main ideas of Lemma 7.2 part b) in [19], where the condition (106) can be replaced by (22) because the essential argument in that work is the disconjugacy property, we obtain the next result.

Claim 3 *Assume that b_i and b are T-periodic functions in $L^\infty(\mathbb{R})$ verifying that $b_i \leq b$ for $i = 1, 2$ and $b \in \hat{\Omega}_{p,c}$. If $b_1 \ll b_2$, then the linear equations $\ddot{x} + c\dot{x} + b_i(t)x = 0$ do not admit nontrivial T-periodic solutions simultaneously.*

From Claim 3 and the fact that $g_{xx} > 0$ on $\mathbb{R} \times]U_1, L_1[$ (as a consequence of Lemma 2), it is not difficult to prove the conclusion of this step by following the ideas in [19, 20]. For the sake of completeness, let us clarify the last assertion. We can adapt the ideas in [20] (see Lemma 11 there) and use the disconjugacy property to show that the periodic solutions of (15) are ordered. Thus, the proof of the multiplicity easily follows: if (15) admits at least three different periodic solutions, then we obtain two linear differential equations with ordered potentials as in Claim 3 (by using the Mean Value Theorem, the monotonicity of g_x, and the inequality in (23) from next step), and such that the consecutive differences between the periodic solutions of (15) solve the corresponding linear equations. This contradicts Claim 3.

Step 2. *Existence and stability properties*:

First, we have that

$$g_x(t, x) = 1 + 3\alpha x^2 - \theta V^2(t) h(x),$$

so that the former discussion implies that

$$g_x(t, x) \leq a(t), \quad \forall (t, x) \in \mathbb{R} \times [U_1, L_1], \tag{23}$$

as $h \geq 1$ and $1 + 3\alpha C_\alpha^2 = \alpha - 1$.

Hence, all hypotheses of Theorem 3 hold. In fact, first, from *Step* 1 there are only a finite number of periodic solutions with range in $]U_1, L_1[$. Secondly, $\theta V_M^2 < \alpha - 1$ as $\varphi_\alpha(C_\alpha) < \alpha - 1$ for $\alpha > 2$, and from hypothesis $\theta V_M^2 < \varphi_\alpha(C_\alpha)$. This assertion is straightforward since $C_2 = 0$, $\varphi_2(C_2) = 1$, and $\frac{\partial \varphi_\alpha}{\partial \alpha}(C_\alpha) = C_\alpha^2(1 - C_\alpha^2)^2 < 1$. Thus, $a(t) > 0$ for all $t \in \mathbb{R}$. This implies that $a \in \hat{\Omega}_{p,c}$ because of (22) and $\|a\|_\infty = \alpha - 1 - \theta V_m^2$. A direct application of Theorem 3 implies the existence of a periodic solution $v(t)$ with range in $]U_1, L_1[$ that is locally asymptotically stable. □

Remark 3 Note that under the hypotheses of Theorem 6 the equilibrium $x \equiv 0$ is unstable (hyperbolic) for c small enough because the linearized equation at $x \equiv 0$ is of the form

$$\ddot{u} + c\dot{u} + \left(1 - \theta V^2(t)\right)u = 0,$$

with $1 - \theta V^2(t) < 0$ for all $t \in \mathbb{R}$. Typically, a linear Comb-drive finger MEMS stabilizes at the origin, but in this operation regime the stability is switched to some lateral periodic solution.

4 Concluding remarks

In this paper we have provided sufficient conditions of L^p-type over a Duffing equation periodically forced with linear damping to obtain at least one asymptotically locally stable periodic solution, based on the computation of a lower bound for the first eigenvalue of the antiperiodic problem for Hill's equations. This result fills some gaps present in the literature. As an application, we have provided a new design principle for Comb-drive finger actuators driven by a periodic voltage that enables the appearance of stable, periodic, lateral oscillations, an uncommon feature in Comb-drive finger MEMS.

On the other hand, by applying Lemma 1 and a classical result of the Lower and Upper Solutions Method, it is possible to prove that there exists a positive periodic solution of (15) with range in $]L_2, U_2[$. This solution could be typically unstable, e.g., if we have uniqueness of the periodic solution with that range (see [21]). Are there more periodic solutions for this model? To our knowledge the multiplicity of positive periodic solutions for this problem still remains open.

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