Iqbal and Rehman *Journal of Inequalities and Applications* (2023) 2023:132
https://doi.org/10.1186/s13660-023-03038-5

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Giaccardi inequality for generalized convex functions and related results

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Abstract

In this article, the famous Giaccardi inequality is generalized for modified (h,m)-convex functions under the condition that h is a multiplicative positive function. The results are also discussed for different values of h and m. Also, Lagrange and Cauchy type mean value theorems for the functional due to Giaccardi's inequality are obtained.

Mathematics Subject Classification: 26D10; 26D15; 26A51

Keywords: Modified h-convex functions; Modified (h,m)-convex functions; Giaccardi inequality

1 Introduction and preliminaries

Convex functions and their generalizations are essential tools in mathematical analysis, with a vast range of applications and useful properties. In recent years, a number of new classes of convex functions have been introduced and studied. A significant generalization of convex functions is the introduction of h-convex functions by Varošanec [22]. She studied the basic properties and showed that h-convex functions include s-convex functions [6], P-functions [2], and Godunova–Levin functions [5] as its special cases. For further generalization related to convex functions, see [3, 7, 18, 19]. In recent years, a large number of academics and mathematicians have looked into modified h-convex and modified (h,m)-convex functions given by Noor et al. in [9].

Modified h-convex and (h,m)-convex functions are crucial developments in the field of convex analysis. These functions satisfy a combination of convexity and monotonicity conditions. They have numerous applications in areas such as optimization, control theory, and mathematical economics. In 2014, Noor et al. published a paper entitled "Hermite-Hadamard inequalities for modified h-convex functions" [9]. In this paper, the authors study the properties of modified h-convex and (h,m)-convex functions and derive inequalities that relate to their convexity and monotonicity. The purpose of this research is to further explore and expand upon the concepts and results presented in Noor et al.'s paper in [9] and to investigate the potential applications of these functions in various fields.

In the literature, Giaccardi introduced a famous inequality known as the Giaccardi inequality [4]. Let $\boldsymbol{\vartheta} := (\vartheta_1, \vartheta_2, \dots, \vartheta_n)$ be a non-negative n-tuple and let $\boldsymbol{\chi} := (\chi_1, \chi_2, \dots, \chi_n)$

be a positive n-tuple such that $\tilde{\vartheta}_n := \sum_{\mu=1}^{n} \chi_\mu \vartheta_\mu \geq \vartheta_\lambda \geq \vartheta_0$, $\tilde{\vartheta}_n \in [a,b]$, and $(\vartheta_\lambda - \vartheta_0)(\tilde{\vartheta}_n - \vartheta_\lambda) \geq 0$ for $\lambda = 1,\dots,n$. If $\Psi : [a,b] \to \mathbb{R}$ is a convex function, then

$$\sum_{\lambda=1}^{n} \chi_\lambda \Psi(\vartheta_\lambda) \leq \frac{\sum_{\lambda=1}^{n} \chi_\lambda (\vartheta_\lambda - \vartheta_0)}{\tilde{\vartheta}_n - \vartheta_0} \Psi(\tilde{\vartheta}_n) + \frac{\sum_{\lambda=1}^{n} \chi_\lambda (\tilde{\vartheta}_n - \vartheta_\lambda)}{\tilde{\vartheta}_n - \vartheta_0} \Psi(\vartheta_0). \tag{1}$$

Many scholars worked on improvement and generalization of the Giaccardi inequality. Its variant for convex-concave antisymmetric functions was proved in [13]. Pečarić and Perić gave an improvement of this inequality and derived related mean value theorems (MVTs) by considering the non-negative difference of the newly introduced inequality in [11]. This inequality was derived on coordinates by Rehman et al. in [16]. Recently, Liu et al. [8] generalized this inequality for modified h-convex functions. For further information on the Giaccardi inequality, see [4, 10, 14].

Definition 1.1 A function $\Psi : [a,b] \to \mathbb{R}$ is convex if

$$\Psi\big(\lambda\vartheta + (1-\lambda)\varrho\big) \leq \lambda\Psi(\vartheta) + (1-\lambda)\Psi(\varrho), \quad \forall \vartheta, \varrho \in [a,b], \lambda \in [0,1].$$

Definition 1.2 [21] Let J be an interval containing $[0,1]$ and let $h : J \to \mathbb{R}$ be a non-negative function. A function $\Psi : [a,b] \to \mathbb{R}$ is modified h-convex if

$$\Psi\big(\lambda\vartheta + (1-\lambda)\varrho\big) \leq h(\lambda)\Psi(\vartheta) + \big(1 - h(\lambda)\big)\Psi(\varrho), \quad \forall \vartheta, \varrho \in [a,b], \lambda \in [0,1]. \tag{2}$$

Definition 1.3 [9] Let J be an interval containing $[0,1]$ and let $h : J \to \mathbb{R}$ be a non-negative function. A function $\Psi : [0,b] \to \mathbb{R}$ is modified (h,m)-convex, where $m \in [0,1]$, if

$$\Psi\big(\lambda\vartheta + m(1-\lambda)\varrho\big) \leq h(\lambda)\Psi(\vartheta) + m\big(1 - h(\lambda)\big)\Psi(\varrho), \quad \forall \vartheta, \varrho \in [0,b], \lambda \in [0,1]. \tag{3}$$

Remark 1.4 For different values of h and m in (3), one has the following results:

1 Taking $h(\lambda) = \lambda^\alpha$, one gets the (α, m)-convex functions given in [21].
2 Taking $h(\lambda) = \lambda^\alpha$ and $m = 1$ gives the $(\alpha, 1)$-convex functions defined in [9].
3 When $h(\lambda) = \lambda$ and $m = 1$, this gives convex functions.
4 Taking $h(\lambda) = \lambda$ gives the m-convex functions defined in [20].
5 By taking $m = 1$, one gets the modified h-convex functions given in [21].

In [8], the authors derived Lagrange and Cauchy type MVTs for the following functional due to Giaccardi's inequality:

$$\begin{aligned} \Im(\Psi; h, \vartheta_0) = {} & \frac{\sum_{\lambda=1}^{n} \chi_\lambda h(\vartheta_\lambda - \vartheta_0)}{h(\tilde{\vartheta}_n - \vartheta_0)} \Psi(\tilde{\vartheta}_n) - \sum_{\lambda=1}^{n} \chi_\lambda \Psi(\vartheta_\lambda) \\ & + \sum_{\lambda=1}^{n} \chi_\lambda \left(1 - \frac{h(\vartheta_\lambda - \vartheta_0)}{h(\tilde{\vartheta}_n - \vartheta_0)}\right) \Psi(\vartheta_0), \end{aligned} \tag{4}$$

where $\boldsymbol{\vartheta} := (\vartheta_1, \vartheta_2, \dots, \vartheta_n)$ is a non-negative n-tuple and $\boldsymbol{\chi} := (\chi_1, \chi_2, \dots, \chi_n)$ is a positive n-tuple such that $\tilde{\vartheta}_n := \sum_{\mu=1}^{n} \chi_\mu \vartheta_\mu \geq \vartheta_\lambda \geq \vartheta_0$, $\tilde{\vartheta}_n \in [a,b]$, for $\lambda = 1,\dots,n$, and h is a multiplicative positive function.

Theorem 1.5 *Consider $\Psi \in C^1([a,b])$ and let h and h' be bounded. Then for the functional $\Im$ defined in* (4), *there exists ϱ in the interior of $[a,b]$ such that*

$$\Im(\Psi;h,\vartheta_0) = \frac{h(\varrho-\vartheta_0)\Psi'(\varrho) - (\Psi(\varrho)-\Psi(\vartheta_0))h'(\varrho-\vartheta_0)}{2\varrho h(\varrho-\vartheta_0) - (\varrho^2-\vartheta_0^2)h'(\varrho-\vartheta_0)}\Im(\mathcal{F};h,\vartheta_0), \tag{5}$$

provided that $\Im(\mathcal{F};h,\vartheta_0)$ is non-zero, where $\mathcal{F}(\vartheta)=\vartheta^2$.

Theorem 1.6 *Consider $\Psi_1,\Psi_2 \in C^1([a,b])$ and let h and h' be bounded. Then for the functional $\Im$ defined in* (4), *there exists ϱ in the interior of $[a,b]$ such that*

$$\frac{\Im(\Psi_1;h,\vartheta_0)}{\Im(\Psi_2;h,\vartheta_0)} = \frac{h(\varrho-\vartheta_0)\Psi_1'(\varrho) - \Psi_1(\varrho)h'(\varrho-\vartheta_0) + \Psi_1(\vartheta_0)h'(\varrho-\vartheta_0)}{h(\varrho-\vartheta_0)\Psi_2'(\varrho) - \Psi_2(\varrho)h'(\varrho-\vartheta_0) + \Psi_2(\vartheta_0)h'(\varrho-\vartheta_0)}, \tag{6}$$

provided that the denominators are non-zero.

The special case of the above functional for a certain class of convex functions has been considered in [17].

This paper is set up as follows.

In the second section, a Giaccardi inequality is derived for modified (h,m)-convex functions. Some previous known results are obtained for different values of h and m.

In the third section, a non-negative linear functional due to the Giaccardi inequality for modified (h,m)-convex functions is defined. With the help of that functional, Lagrange and Cauchy type MVTs are obtained for the Giaccardi inequality for modified (h,m)-convex functions. Some special cases are discussed.

2 Main results

We assume that $\boldsymbol{\vartheta} := (\vartheta_1,\vartheta_2,\dots,\vartheta_n)$ is a non-negative n-tuple, $\boldsymbol{\chi} := (\chi_1,\chi_2,\dots,\chi_n)$ is a positive n-tuple, $\vartheta_0 \in [0,\infty)$ such that $\tilde{\vartheta}_n := \sum_{\mu=1}^n \chi_\mu\vartheta_\mu \geq \vartheta_\lambda \geq m\vartheta_0$, $m\in[0,1]$, and $h:(0,\infty)\to\mathbb{R}$ is a positive multiplicative function in the rest of the paper.

First we give an important lemma, which has an essential role in proving our main result.

Lemma 2.1 *Let $\Psi:[0,\infty)\to\mathbb{R}$ be a function such that $\frac{\Psi(\vartheta)}{h(\vartheta-m\vartheta_0)}$ is increasing for $\vartheta > m\vartheta_0$. Then we have*

$$\sum_{\lambda=1}^n \chi_\lambda\Psi(\vartheta_\lambda) \leq \frac{\sum_{\lambda=1}^n \chi_\lambda h(\vartheta_\lambda - m\vartheta_0)}{h(\tilde{\vartheta}_n - m\vartheta_0)}\Psi(\tilde{\vartheta}_n). \tag{7}$$

Proof Since $\tilde{\vartheta}_n \geq \vartheta_\lambda$ and it is given that $\frac{\Psi(\vartheta)}{h(\vartheta-m\vartheta_0)}$ is increasing for $\vartheta > m\vartheta_0$, we have

$$\frac{\Psi(\vartheta_\lambda)}{h(\vartheta_\lambda - m\vartheta_0)} \leq \frac{\Psi(\tilde{\vartheta}_n)}{h(\tilde{\vartheta}_n - m\vartheta_0)}.$$

As h is assumed to be a positive function, we have

$$h(\tilde{\vartheta}_n - m\vartheta_0)\Psi(\vartheta_\lambda) \leq h(\vartheta_\lambda - m\vartheta_0)\Psi(\tilde{\vartheta}_n).$$

Multiplying the above inequality by χ_λ and taking the sum from $\lambda = 1, \dots, n$, we have

$$h(\tilde{\vartheta}_n - m\vartheta_0) \sum_{\lambda=1}^{n} \chi_\lambda \Psi(\vartheta_\lambda) \leq \Psi(\tilde{\vartheta}_n) \sum_{\lambda=1}^{n} \chi_\lambda h(\vartheta_\lambda - m\vartheta_0).$$

This is equivalent to (7). □

In the following theorem a Giaccardi inequality for modified (h,m)-convex functions is given.

Theorem 2.2 *Let $\Psi : [0,\infty) \to \mathbb{R}$ be a modified (h,m)-convex function. Then*

$$\sum_{\lambda=1}^{n} \chi_\lambda \Psi(\vartheta_\lambda) \leq A\Psi(\tilde{\vartheta}_n) + m\left(\sum_{\lambda=1}^{n} \chi_\lambda - A\right)\Psi(\vartheta_0), \tag{8}$$

where

$$A = \frac{\sum_{\lambda=1}^{n} \chi_\lambda h(\vartheta_\lambda - m\vartheta_0)}{h(\tilde{\vartheta}_n - m\vartheta_0)}. \tag{9}$$

Proof First assume that

$$\Upsilon(\Psi, m, h; \vartheta) = \frac{\Psi(\vartheta) - m\Psi(\vartheta_0)}{h(\vartheta - m\vartheta_0)} \quad \text{for all } \vartheta, \vartheta_0 \in [0,\infty). \tag{10}$$

For $\varrho > \vartheta > m\vartheta_0$ such that $\vartheta = \lambda\varrho + m(1-\lambda)\vartheta_0$, where $\lambda \in (0,1)$, we have

$$\Upsilon(\Psi, m, h; \vartheta) = \frac{\Psi(\lambda\varrho + m(1-\lambda)\vartheta_0) - m\Psi(\vartheta_0)}{h(\lambda\varrho + m(1-\lambda)\vartheta_0 - m\vartheta_0)}.$$

Using the fact that Ψ is a modified (h,m)-convex function, we have

$$\Upsilon(\Psi, m, h; \vartheta) \leq \frac{h(\lambda)\Psi(\varrho) + m(1-h(\lambda))\Psi(\vartheta_0) - m\Psi(\vartheta_0)}{h(\lambda(\varrho - m\vartheta_0))}.$$

Since h is multiplicative, we have

$$\Upsilon(\Psi, m, h; \vartheta) \leq \frac{h(\lambda)\Psi(\varrho) - mh(\lambda)\Psi(\vartheta_0)}{h(\lambda)h(\varrho - m\vartheta_0)}.$$

This implies

$$\Upsilon(\Psi, m, h; \vartheta) \leq \frac{\Psi(\varrho) - m\Psi(\vartheta_0)}{h(\varrho - m\vartheta_0)} = \Upsilon(\Psi, m, h; \varrho),$$

that is, $\Upsilon(\Psi, m, h; \vartheta)$ is increasing on $[0,\infty)$.

Replacing $\Psi(\vartheta)$ by $\Psi(\vartheta) - m\Psi(\vartheta_0)$ in Lemma 2.1, we have

$$\sum_{\lambda=1}^{n} \chi_\lambda \big(\Psi(\vartheta_\lambda) - m\Psi(\vartheta_0)\big) \leq \frac{\sum_{\lambda=1}^{n} \chi_\lambda h(\vartheta_\lambda - m\vartheta_0)}{h(\tilde{\vartheta}_n - m\vartheta_0)} \big(\Psi(\tilde{\vartheta}_n) - m\Psi(\vartheta_0)\big).$$

This gives

$$\sum_{\lambda=1}^{n}\chi_\lambda\Psi(\vartheta_\lambda)-m\Psi(\vartheta_0)\sum_{\lambda=1}^{n}\chi_\lambda\le\frac{\sum_{\lambda=1}^{n}\chi_\lambda h(\vartheta_\lambda-m\vartheta_0)}{h(\tilde{\vartheta}_n-\vartheta_0)}\Psi(\tilde{\vartheta}_n)-m\Psi(\vartheta_0)\frac{\sum_{\lambda=1}^{n}\chi_\lambda h(\vartheta_\lambda-m\vartheta_0)}{h(\tilde{\vartheta}_n-\vartheta_0)}.$$

This completes the proof. □

For a modified (h,m)-convex function Ψ, the mapping $\Upsilon(\Psi,m,h;\vartheta)$ defined in (10) is increasing if and only if

$$\frac{d}{d\vartheta}\Upsilon(\Psi,m,h;\vartheta)\ge 0,$$

provided that the derivatives exist. This gives

$$\Upsilon'(\vartheta)\ge\frac{\Psi(\vartheta)-m\Psi(\vartheta_0)}{h(\vartheta-m\vartheta_0)}\cdot h'(\vartheta-m\vartheta_0). \tag{11}$$

The result given by Pečarić et al. in [12, p. 09] for convex functions can be obtained by taking h as identity function and $m=1$. Another result for m-convex functions has been obtained by Bakula et al. in [1], which can easily be obtained by taking $h(\vartheta)=\vartheta$ in (11). Also, taking $m=1$ in (11) gives the result for modified h-convex functions given by Liu et al. in [8, Remark 1].

Remark 2.3 Now we discuss the cases when h and m have different particular values in Theorem 2.2.

1 One gets the Giaccardi inequality for (α,m)-convex functions when $h(\vartheta)=\vartheta^\alpha$.
2 Taking $h(\vartheta)=\vartheta^\alpha$ and $m=1$ gives [8, Corollary 1].
3 The Giaccardi inequality can be derived by taking $h(\vartheta)=\vartheta$ and $m=1$ (see [4] or [12, p. 152]).
4 A result obtained by taking $h(\vartheta)=\vartheta$ can be considered as a Giaccardi inequality for m-convex functions. Bakula et al. have given a similar result in [1, Theorem 3.3].
5 A case when $m=1$ has been proved by Liu et al. in [8, Theorem 4].

A Petrović inequality for modified (h,m)-convex functions is given in the following theorem.

Theorem 2.4 *Let $\Psi:[0,\infty)\to\mathbb{R}$ be a modified (h,m)-convex function. Then*

$$\sum_{\lambda=1}^{n}\chi_\lambda\Psi(\vartheta_\lambda)\le\frac{\sum_{\lambda=1}^{n}\chi_\lambda h(\vartheta_\lambda)}{h(\tilde{\vartheta}_n)}\Psi(\tilde{\vartheta}_n)+m\left(\sum_{\lambda=1}^{n}\chi_\lambda-\frac{\sum_{\lambda=1}^{n}\chi_\lambda h(\vartheta_\lambda)}{h(\tilde{\vartheta}_n)}\right)\Psi(0). \tag{12}$$

Proof It is an easy consequence of Theorem 2.2. Taking $\vartheta_0=0$ in inequality (8) gives the required result. □

Remark 2.5 Different values of h and m in Theorem 2.4 give results having significant importance in the literature.

1 One gets the Petrović inequality for (α, m)-convex functions when $h(\vartheta) = \vartheta^\alpha$.
2 One gets the result proved by Liu et al. in [8, Corollary 2] by taking $h(\vartheta) = \vartheta^\alpha$ and $m = 1$. It can be considered as the Petrović inequality for $(\alpha, 1)$-convex functions.
3 The result given by Petrović in [15] can be obtained by taking $h(\vartheta) = \vartheta$ and $m = 1$.
4 The result obtained by taking $h(\vartheta) = \vartheta$ can be considered as the Petrović inequality for m-convex functions. In [1, Corollary 3.4], Bakula et al. proved a similar result.
5 A case when $m = 1$ has been proved by Liu et al. in [8, Corollary 14].

3 Mean value theorems

For the non-negative difference of inequality (8), we define the linear functional as follows in order to provide the MVTs.

For $\Psi : [0, \infty) \to \mathbb{R}$, we define a functional

$$\Im(\Psi; m, h, \vartheta_0) = A\Psi(\tilde{\vartheta}_n) - \sum_{\lambda=1}^{n} \chi_\lambda \Psi(\vartheta_\lambda) + m\left(\sum_{\lambda=1}^{n} \chi_\lambda - A\right)\Psi(\vartheta_0), \tag{13}$$

where A is defined in (9) and h is a multiplicative positive function.

The linear functional due to Giaccardi's inequality for (α, m)-convex functions can be obtained by taking $h(\vartheta) = \vartheta^\alpha$ in (13), which is as follows:

$$\mathcal{A}(\Psi; m, \vartheta_0) = \Im\big(\Psi; m, \vartheta^\alpha, \vartheta_0\big). \tag{14}$$

Taking $\vartheta_0 = 0$ in (13) yields the linear functional for the Petrović inequality:

$$\begin{aligned} \mathfrak{P}(\Psi; m, h) = {} & \frac{\sum_{\lambda=1}^{n} \chi_\lambda h(\vartheta_\lambda)}{h(\tilde{\vartheta}_n)} \Psi(\tilde{\vartheta}_n) - \sum_{\lambda=1}^{n} \chi_\lambda \Psi(\vartheta_\lambda) \\ & + m\left(\sum_{\lambda=1}^{n} \chi_\lambda - \frac{\sum_{\lambda=1}^{n} \chi_\lambda h(\vartheta_\lambda)}{h(\tilde{\vartheta}_n)}\right)\Psi(0). \end{aligned} \tag{15}$$

In this section, we give MVTs for functionals defined in (13) and (15). First we introduce two modified (h, m)-convex functions using the fact that a function is modified (h, m)-convex if and only if the corresponding function defined in (10) is increasing.

Lemma 3.1 *Consider two functions $\Omega_1, \Omega_2 : [0, \infty) \to \mathbb{R}$ defined as*

$$\Omega_1(\vartheta) = \mathfrak{M}\vartheta^2 - \Psi(\vartheta) \quad \textit{and} \quad \Omega_2(\vartheta) = \Psi(\vartheta) - \mathfrak{m}\vartheta^2$$

such that $\Psi : [0, \infty) \to \mathbb{R}$ is a differentiable function satisfying

$$\mathfrak{m} \le \frac{h(\vartheta - m\vartheta_0)\Psi'(\vartheta) - (\Psi(\vartheta) - m\Psi(\vartheta_0))h'(\vartheta - m\vartheta_0)}{2\vartheta h(\vartheta - m\vartheta_0) - (\vartheta^2 - m\vartheta_0^2)h'(\vartheta - m\vartheta_0)} \le \mathfrak{M}, \quad \forall \vartheta \in [0, \infty), \tag{16}$$

where h is a positive differentiable function on $(0, \infty)$. Then Ω_1, Ω_2 are modified (h, m)-convex functions for $\vartheta > m\vartheta_0$.

Proof First assume that

$$\mathcal{G}(\Omega_1, m, h; \vartheta) = \frac{\Omega_1(\vartheta) - m\Omega_1(\vartheta_0)}{h(\vartheta - m\vartheta_0)}.$$

Putting the value of Ω_1 in the above expression, we have

$$\begin{aligned}\mathcal{G}(\Omega_1,m,h;\vartheta) &= \frac{\mathfrak{M}\vartheta^2 - \Psi(\vartheta) - m\mathfrak{M}\vartheta_0^2 + m\Psi(\vartheta_0)}{h(\vartheta - m\vartheta_0)} \\ &= \frac{\mathfrak{M}(\vartheta^2 - m\vartheta_0^2)}{h(\vartheta - m\vartheta_0)} - \frac{\Psi(\vartheta) - m\Psi(\vartheta_0)}{h(\vartheta - m\vartheta_0)}.\end{aligned}$$

Differentiating with respect to ϑ, we have

$$\begin{aligned}\frac{d}{d\vartheta}\mathcal{G}(\Omega_1,m,h;\vartheta) = \mathfrak{M}&\frac{h(\vartheta - m\vartheta_0)2\vartheta - (\vartheta^2 - m\vartheta_0^2)h'(\vartheta - m\vartheta_0)}{h^2(\vartheta - m\vartheta_0)} \\ &- \frac{h(\vartheta - m\vartheta_0)\Psi'(\vartheta) - (\Psi(\vartheta) - m\Psi(\vartheta_0))h'(\vartheta - m\vartheta_0)}{h^2(\vartheta - m\vartheta_0)}.\end{aligned}$$

From the right inequality in (16), we have

$$\begin{aligned}0 &\le \mathfrak{M}\bigl(2\vartheta h(\vartheta - m\vartheta_0) - \bigl(\vartheta^2 - m\vartheta_0^2\bigr)h'(\vartheta - m\vartheta_0)\bigr) \\ &\quad - h(\vartheta - m\vartheta_0)\Psi'(\vartheta) - \bigl(\Psi(\vartheta) - m\Psi(\vartheta_0)\bigr)h'(\vartheta - m\vartheta_0).\end{aligned}$$

As h is a positive function, one can divide the above inequality by $h^2(\vartheta - m\vartheta_0)$ on both sides. This gives

$$\begin{aligned}0 &\le \mathfrak{M}\frac{2\vartheta h(\vartheta - m\vartheta_0) - (\vartheta^2 - m\vartheta_0^2)h'(\vartheta - m\vartheta_0)}{h^2(\vartheta - m\vartheta_0)} \\ &\quad - \frac{h(\vartheta - m\vartheta_0)\Psi'(\vartheta) - (\Psi(\vartheta) - m\Psi(\vartheta_0))h'(\vartheta - m\vartheta_0)}{h^2(\vartheta - m\vartheta_0)}.\end{aligned}$$

Hence, $\frac{d}{d\vartheta}\mathcal{G}(\Omega_1,m,h;\vartheta) \ge 0$.

Similarly, one can show that $\frac{d}{d\vartheta}\mathcal{G}(\Omega_2,m,h;\vartheta) \ge 0$.

This shows that $\mathcal{G}(\Omega_1,m,h;\vartheta)$ and $\mathcal{G}(\Omega_2,m,h;\vartheta)$ are increasing for $\vartheta > m\vartheta_0$. Hence, Ω_1 and Ω_2 are modified (h,m)-convex functions. □

The Lagrange type MVT for the functional defined in (13) is given in the following theorem.

Theorem 3.2 *Let h and h' be bounded functions and let $\Im$ be a functional defined in* (13). *If Ψ and Ψ' are bounded and continuous on $[0,\infty)$, then there exists ϱ in the interior of $[0,\infty)$ such that*

$$\Im(\Psi;m,h,\vartheta_0) = \frac{h(\varrho - m\vartheta_0)\Psi'(\varrho) - (\Psi(\varrho) - m\Psi(\vartheta_0))h'(\varrho - m\vartheta_0)}{2\varrho h(\varrho - m\vartheta_0) - (\varrho^2 - m\vartheta_0^2)h'(\varrho - m\vartheta_0)}\Im(\mathcal{F};m,h,\vartheta_0), \quad (17)$$

provided that $\Im(\mathcal{F};m,h,\vartheta_0)$ is non-zero, where $\mathcal{F}(\vartheta) = \vartheta^2$.

Proof By the given condition h and h' are bounded and Ψ and Ψ' are bounded and continuous on $[0,\infty)$, so there exist real numbers $\mathfrak{m}$ and $\mathfrak{M}$ such that

$$\mathfrak{m} \le \frac{h(\vartheta - m\vartheta_0)\Psi'(\vartheta) - (\Psi(\vartheta) - m\Psi(\vartheta_0))h'(\vartheta - m\vartheta_0)}{2\vartheta h(\vartheta - m\vartheta_0) - (\vartheta^2 - m\vartheta_0^2)h'(\vartheta - m\vartheta_0)} \le \mathfrak{M}, \quad \forall\vartheta \in [0,\infty).$$

Let Ω_2 be a modified (h,m)-convex function defined in Lemma 3.1. Then

$$\Im(\Omega_2; m, h, \vartheta_0) \geq 0,$$

that is,

$$\Im\big(\Psi(\vartheta) - \mathfrak{m}\vartheta^2; m, h, \vartheta_0\big) \geq 0.$$

This implies

$$\Im(\Psi; m, h, \vartheta_0) \geq \mathfrak{m}\Im(\mathcal{F}; m, h, \vartheta_0). \tag{18}$$

By considering the function Ω_1 defined in Lemma 3.1, one can show that

$$\mathfrak{M}\Im(\mathcal{F}; m, h, \vartheta_0) \geq \Im(\Psi; m, h, \vartheta_0). \tag{19}$$

From inequalities (18) and (19), we have

$$\mathfrak{m} \leq \frac{\Im(\Psi; m, h, \vartheta_0)}{\Im(\mathcal{F}; m, h, \vartheta_0)} \leq \mathfrak{M}.$$

So there exists ϱ in the interior of $[0,\infty)$ such that

$$\frac{\Im(\Psi; m, h, \vartheta_0)}{\Im(\mathcal{F}, m, h; \vartheta_0)} = \frac{h(\varrho - m\vartheta_0)\Psi'(\varrho) - (\Psi(\varrho) - m\Psi(\vartheta_0))h'(\varrho - m\vartheta_0)}{2\varrho h(\varrho - m\vartheta_0) - (\varrho^2 - m\vartheta_0^2)h'(\varrho - m\vartheta_0)}.$$

This completes the proof. □

Remark 3.3 Now we discuss the cases when h and m have different particular values in Theorem 3.2.

1 Taking $h(\vartheta) = \vartheta^\alpha$, one gets the Lagrange type MVT for the functional defined in (14).
2 The result obtained by taking $h(\vartheta) = \vartheta^\alpha$ and $m = 1$ has been proved by Liu et al. in [8, Corollary 18].
3 By taking $h(\vartheta) = \vartheta$ and $m = 1$, one gets the result given by Rehman et al. in [17, Corollary 2.2].
4 A case when $h(\vartheta) = \vartheta$ has been proved by Rehman et al. in [17, Theorem 2.2].
5 The result given in Theorem 1.5 can be obtained by taking $m = 1$.

The following theorem consists of a Lagrange type MVT relating to the functional caused by the Petrović inequality.

Theorem 3.4 *Assume that h and h' are bounded functions. If Ψ and Ψ' are bounded and continuous on $[0,\infty)$, then there exist ϱ in the interior of $[0,\infty)$ such that*

$$\mathfrak{P}(\Psi; m, h) = \frac{h(\varrho)\Psi'(\varrho) - (\Psi(\varrho) - m\Psi(0))h'(\varrho)}{2\varrho h(\varrho) - \varrho^2 h'(\varrho)}\mathfrak{P}(\mathcal{F}; m, h), \tag{20}$$

where $\mathcal{F}(\vartheta) = \vartheta^2$ and $\mathfrak{P}$ is a functional defined in (15), provided that $\Im(\mathcal{F}, m, h; \vartheta_0)$ is non-zero.

Proof It is a simple consequence of Theorem 3.2, for $\vartheta_0 = 0$. □

Remark 3.5 We analyze the instances for different values of h and m in Theorem 3.4.

1 By assuming $h(\vartheta) = \vartheta^\alpha$, one gets the result for modified (α, m)-convex functions.
2 To get the result for $(\alpha, 1)$-convex functions, take $h(\vartheta) = \vartheta^\alpha$ and $m = 1$.
3 Setting $h(\vartheta) = \vartheta$ and $m = 1$ gives the result for convex functions.
4 The result for m-convex functions can be obatined by taking $h(\vartheta) = \vartheta$.
5 In a case when $m = 1$, one gets the result for modified h-convex functions. It has been proved by Liu et al. in [8, Corollary 19].

The following theorem consists of a Cauchy type MVT related to the functional defined in (13).

Theorem 3.6 *Let h and h' be bounded functions. If Ψ_1 and Ψ_2 are bounded and continuous functions on $[0, \infty)$, then there exists ϱ in the interior of $[0, \infty)$ such that*

$$\frac{\Im(\Psi_1; m, h, \vartheta_0)}{\Im(\Psi_2; m, h, \vartheta_0)} = \frac{h(\varrho - m\vartheta_0)\Psi_1'(\varrho) - \Psi_1(\varrho)h'(\varrho - m\vartheta_0) + m\Psi_1(\vartheta_0)h'(\varrho - m\vartheta_0)}{h(\varrho - m\vartheta_0)\Psi_2'(\varrho) - \Psi_2(\varrho)h'(\varrho - m\vartheta_0) + m\Psi_2(\vartheta_0)h'(\varrho - m\vartheta_0)}, \tag{21}$$

provided that the denominators are non-zero, where $\Im$ is a functional defined in (13).

Proof Suppose that

$$\mathcal{F} = t_1\Psi_1 - t_2\Psi_2, \quad \text{where } t_1 = \Im(\Psi_2; m, h, \vartheta_0) \text{ and } t_2 = \Im(\Psi_1; m, h, \vartheta_0),$$

with the assumption that Ψ_1 and Ψ_2 are bounded and continuous functions on $[0, \infty)$. Replacing Ψ with $\mathcal{F}$ in Theorem 3.2, we have

$$\begin{aligned} 0 = &h(\varrho - m\vartheta_0)\big((t_1\Psi_1 - t_2\Psi_2)(\varrho)\big)' - (t_1\Psi_1 - t_2\Psi_2)(\varrho)h'(\varrho - m\vartheta_0) \\ &+ m(t_1\Psi_1 - t_2\Psi_2)(\vartheta_0)h'(\varrho - m\vartheta_0). \end{aligned}$$

With a little computation, one has

$$\frac{t_2}{t_1} = \frac{h(\varrho - m\vartheta_0)\Psi_1'(\varrho) - \Psi_1(\varrho)h'(\varrho - m\vartheta_0) + m\Psi_1(\vartheta_0)h'(\varrho - m\vartheta_0)}{h(\varrho - m\vartheta_0)\Psi_2'(\varrho) - \Psi_2(\varrho)h'(\varrho - m\vartheta_0) + m\Psi_2(\vartheta_0)h'(\varrho - m\vartheta_0)}.$$

Putting the values of t_1 and t_2 completes the proof. □

Remark 3.7 Now we discuss the cases when h and m have different particular values in Theorem 3.6.

1 In a case when $h(\vartheta) = \vartheta^\alpha$, the result for (α, m)-convex functions can be obtained.
2 For $h(\vartheta) = \vartheta^\alpha$ and $m = 1$, one gets the result proved by Liu et al. in [8, Corollary 23].
3 A Cauchy type MVT for convex functions can be obtained by setting $h(\vartheta) = \vartheta$ and $m = 1$.
4 To get the result for m-convex functions proved by Rehman et al. in [17, Theorem 2.3], take $h(\vartheta) = \vartheta$.
5 The result given in Theorem 1.6 can be obtained by setting $m = 1$.

A Cauchy type MVT is presented for the functional defined in (15) in the following theorem.

Theorem 3.8 *Let h and h' be bounded functions. If Ψ_1 and Ψ_2 are bounded and continuous functions on $[0,\infty)$, then there exists ϱ in the interior of $[0,\infty)$ such that*

$$\frac{\mathfrak{P}(\Psi_1;m,h)}{\mathfrak{P}(\Psi_2;m,h)}=\frac{h(\varrho)\Psi_1'(\varrho)-\Psi_1(\varrho)h'(\varrho)+m\Psi_1(0)h'(\varrho)}{h(\varrho)\Psi_2'(\varrho)-\Psi_2(\varrho)h'(\varrho)+m\Psi_2(0)h'(\varrho)}, \tag{22}$$

provided that the denominators are non-zero, where $\mathfrak{P}$ is a functional defined in (15).

Proof It is a particular case of Theorem 3.6 for $\vartheta_0=0$. □

Remark 3.9 In Theorem 3.8, we discuss the cases for different values of h and m.

1. To get the result for (α,m)-convex functions, take $h(\vartheta)=\vartheta^\alpha$.
2. Taking $h(\vartheta)=\vartheta^\alpha$ and $m=1$, one gets the result given by Liu et al. in [8, Corollary 24].
3. A case where $h(\vartheta)=\vartheta$ and $m=1$ can be considered as a Cauchy type MVT for the Petrović inequality for convex functions.
4. By assuming $h(\vartheta)=\vartheta$, one gets the result for m-convex functions.
5. The result obtained by taking $m=1$ has been proved by Liu et al. in [8, Corollary 23].

4 Conclusion

In this article, we generalized the Giaccardi and Petrović inequalities. Also, Lagrange and Cauchy type MVTs related to the functional caused by these inequalities are obtained. It is shown that some previous results have been obtained for particular values of h and m. For example, for $m=1$, one gets the result given in [8]. Considering h to be an identity function gives the result given in [17]. For h being an identity and $m=1$, one gets the result given in [13]. This is an interesting research topic for the future. The findings of this work may encourage additional research in a variety of fields of pure and applied sciences.

Acknowledgements
Not applicable.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Both authors have contributed equally to the research and preparation of this manuscript. Specifically, both authors were involved in the conception and design of the study, and both equally contributed to the development of the mathematical models, the analysis of the results, and the interpretation of the findings. Both authors were involved in the drafting of the manuscript and its critical revision. Both have given their final approval of the version to be submitted.

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Received: 28 February 2023 Accepted: 18 September 2023 Published online: 16 October 2023

References

1. Bakula, M.K., Pečarić, J.E., Ribičić, M.: Companion inequalities to Jensen's inequality for m-convex and (α, m)-convex functions. J. Inequal. Pure Appl. Math. **7**(5), 1–15 (2006)
2. Dragomir, S.S., Pečarić, J.E., Persson, L.E.: Some inequalities of Hadamard type. Soochow J. Math. **21**, 335–341 (1995)
3. Eken, Z., Kemali, S., Tınaztepe, G., Adilov, G.: The Hermite-Hadamard inequalities for p-convex functions. Hacet. J. Math. Stat. **50**(5), 1268–1279 (2021)
4. Giaccardi, F.: Su alcune disuguaglilianze. Giorn. Mat. Finanz. **1**(4), 139–153 (1953)
5. Godunova, E.K., Levin, V.I.: Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions. (Russian) Numerical mathematics and mathematical physics (Russian). Moskov. Gos. Ped. Inst., Moscow, **166**, 138–142 (1985)
6. Hudzik, A., Maligranda, L.: Some remarks on s-convex functions. Aequ. Math. **48**, 100–111 (1994)
7. Kemali, S., Tinaztepe, G., Adilov, G.: New type inequalities for B-1-convex functions involving Hadamard fractional integral. Facta Univ., Ser. Math. Inform. **33**(5), 697–704 (2018)
8. Liu, Y., Iqbal, W., Rehman, A.U., Farid, G., Nonlaopon, K.: Giaccardi inequality for modified h-convex functions and mean value theorems. J. Funct. Spaces **2022**, 1–7 (2022)
9. Noor, M.A., Noor, K.I., Awan, M.U.: Hermite Hadamard inequalities for modified h-convex functions. Transylv. J. Math. Mech. **6**, 1–10 (2014)
10. Pečarić, J.E., Čuljak, V.: Inequality of Petrović's and Giaccardi for convex function of higher order. Southeast Asian Bull. Math. **26**(1), 57–61 (2003)
11. Pečarić, J.E., Perić, J.: Improvements of the Giaccardi and the Petrović's inequality and related Stolarsky type means. An. Univ. Craiova, Ser. Mat. Inform. **39**(1), 65–75 (2012)
12. Pečarić, J.E., Proschan, F., Tong, Y.L.: Convex Functions, Partial Orderings and Statistical Applications. Academic Press, New York (1991)
13. Pečarić, J.E., Rehman, A.U.: Giaccardi inequality for convex-concave antisymmetric functions and applications. J. Inequal. Appl. **2008**, 863–874 (2008)
14. Pečarić, J.E., Rehman, A.U.: On logarithmic convexity for Giaccardi's difference. Rad Hazu. **17**, 1–10 (2013)
15. Petrović's, M.: Sur une fonctionnelle. Publ. Math. Univ. Belgrade (1932)
16. Rehman, A.U., Farid, G., Akbar, H.: On Giaccardi inequality and associated functional in the plane. Int. J. Anal. Appl. **16**(2), 1–16 (2018)
17. Rehman, A.U., Farid, G., Iqbal, W.: More about Petrović's inequality on coordinates via m-convex functions and related results. Kragujev. J. Math. **44**(3), 335–351 (2020)
18. Tınaztepe, G., Kemali, S., Sezer, S., Eken, Z.: The sharper form of a Brunn-Minkowski type inequality for boxes. Hacet. J. Math. Stat. **50**(2), 377–386 (2020)
19. Tınaztepe, R., Tınaztepe, G., Eken, Z., Sezer, S., Kemali, S., Işık İ, Y., Sezer Evcan, S.: The sharper version for generalized power mean inequalities with negative exponent. J. Math. Inequal. **17**(1), 99–111 (2023)
20. Toader, G.: Some generalizations of the convexity. In: Proc. Colloq. Approx. Optim., Cluj-Napoca (Romania), pp. 329–338, 1984
21. Toader, G.: The hierarchy of convexity and some classic inequalities. J. Math. Inequal. **3**(3), 305–313 (2009)
22. Varošanec, S.: On h-convexity. J. Math. Anal. Appl. **326**(1), 303–311 (2007)

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