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Some estimates for commutators of the fractional maximal function on stratified Lie groups

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Abstract

In this paper, the main aim is to consider the boundedness of the nonlinear commutator $[b, M_\alpha]$ and the maximal commutator $M_{\alpha,b}$ on the Lebesgue spaces over some stratified Lie group $\mathbb{G}$ when the symbol b belongs to the Lipschitz space. As a result, some new characterizations of the Lipschitz spaces on Lie group via $[b, M_\alpha]$ and $M_{\alpha,b}$ are given.

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1 Introduction and main results

During the last several decades, stratified groups have appeared in quantum physics and many branches of mathematics dealing with harmonic analysis, several complex variables, geometry and topology, etc. [13, 29]. Moreover, the stratified groups have such geometric structure that they inherit many analysis properties from the Euclidean spaces [14, 28]. Apart from this, the study of function spaces on stratified groups is more complicated because of the distinction between the geometric structures of Euclidean spaces and stratified groups. It is noteworthy that the fractional maximal operator plays an important role in harmonic analysis and application areas, for instance, partial differential equations (PDEs) and potential theory, since it is closely related to the Riesz potential operator, which is a powerful tool to study the smooth function spaces (see [3, 5, 13]). Meanwhile, the commutators are not only intimately related to the regularity properties of the solutions of certain partial differential equations [4, 7, 9, 26], but also can produce some characterizations of function spaces [17, 25]. On the other hand, most results of the theory of distribution functions and Fourier transforms in Euclidean spaces cannot be extended to groups, so there are still many harmonic analysis problems on stratified Lie groups, which are worth further research.

Suppose T is a classical singular integral operator. The Coifman–Rochberg–Weiss type commutator $[b, T]$ generated by T and a suitable function b is defined by

$$[b, T]f = bT(f) - T(bf). \tag{1.1}$$

A known result indicates that $b \in \mathrm{BMO}(\mathbb{R}^n)$ (the space of bounded mean oscillation functions) if and only if $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. The necessity was given by Coifman et al. [8] and the sufficiency was obtained by Janson [17]. Furthermore, Janson [17] also established some characterizations of the Lipschitz space $\Lambda_\beta(\mathbb{R}^n)$ via commutator (1.1) and proved that $b \in \Lambda_\beta(\mathbb{R}^n)$ if and only if $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < p < n/\beta$ and $1/p - 1/q = \beta/n$ with $0 < \beta < 1$ (see also Paluszyński [25]).

Denote by $\mathbb{G}$ and $\mathbb{R}$ a stratified Lie group and the set of all real numbers, respectively. Let Q be the homogeneous dimension of $\mathbb{G}$, $0 \le \alpha < Q$ and $f : \mathbb{G} \to \mathbb{R}$ be a locally integrable function. Defining the fractional maximal function by

$$M_\alpha(f)(x) = \sup_{\substack{B \ni x \\ B \subset \mathbb{G}}} \frac{1}{|B|^{1-\alpha/Q}} \int_B |f(y)| \, dy,$$

where the supremum is chosen over all $\mathbb{G}$-balls $B \subset \mathbb{G}$ embracing x with radius $r > 0$, and $|B|$ represents the Haar measure of the $\mathbb{G}$-ball B (for the notations and notions, see Sect. 2 below). If $\alpha = 0$, then we simply write M instead of M_0, which is the Hardy-Littlewood maximal function defined as

$$M(f)(x) = \sup_{\substack{B \ni x \\ B \subset \mathbb{G}}} \frac{1}{|B|} \int_B |f(y)| \, dy.$$

Analogous to (1.1), two different forms of commutators generated by the fractional maximal function are given below.

Definition 1.1 Let $0 \le \alpha < Q$ and b be a locally integrable function on $\mathbb{G}$.

(i) The maximal commutator of M_α with b defined as

$$M_{\alpha,b}(f)(x) = \sup_{\substack{B \ni x \\ B \subset \mathbb{G}}} \frac{1}{|B|^{1-\alpha/Q}} \int_B |b(x) - b(y)| |f(y)| \, dy,$$

where the supremum is chosen over all $\mathbb{G}$-balls $B \subset \mathbb{G}$ embracing x.

(ii) The nonlinear commutators generated by M_α and b is given by

$$[b, M_\alpha](f)(x) = b(x)M_\alpha(f)(x) - M_\alpha(bf)(x).$$

We write $[b, M] = [b, M_0]$ and $M_b = M_{0,b}$, when $\alpha = 0$.

Although $[b, T]$ is a linear operator, $[b, M_\alpha]$ is called a nonlinear commutator since it is not even a sub-linear operator. It is worth noting that the maximal commutator $M_{\alpha,b}$ and the nonlinear commutator $[b, M_\alpha]$ are essentially different from each other. For instance, $M_{\alpha,b}$ is not only positive but also sub-linear, while $[b, M_\alpha]$ is neither positive nor sub-linear.

In 1990, Milman and Schonbek [24] used the real interpolation technique to establish a commutator result that applies to both the Hardy-Littlewood maximal function and a large class of nonlinear operators. In 2000, Bastero et al. [1] proved the necessary and sufficient conditions for the nonlinear commutator $[b,M]$ to be bounded on L^p spaces when the symbol belongs to $\mathrm{BMO}(\mathbb{R}^n)$. In 2009, Zhang and Wu [34] further extended the above results to the commutators of fractional maximal function. Subsequently, Zhang and coauthors [31, 36] obtained some characterizations of the Lipschitz spaces via the boundedness of M_b and $[b,M]$ on Lebesgue spaces and Morrey spaces, and $[b,M_\alpha]$ on Orlicz spaces, respectively. Recently, Guliyev [15, 16] extended the mentioned results to Orlicz spaces $L^\Phi(\mathbb{G})$ over some stratified Lie group when the symbols belong to $\mathrm{BMO}(\mathbb{G})$ spaces and Lipschitz spaces $\Lambda_\beta(\mathbb{G})$ respectively, and obtained separately some characterizations for certain subclasses of $\mathrm{BMO}(\mathbb{G})$ and $\Lambda_\beta(\mathbb{G})$. And Liu et al. [21] established the characterization of BMO spaces by the boundedness of some commutators in variable Lebesgue spaces. Meanwhile, Wu and Zhao [30] extended some results of [31] to stratified Lie group when the symbols belong to the Lipschitz spaces.

Inspired by the above literature, the purpose of this paper is to study the mapping properties of the nonlinear commutator $[b,M_\alpha]$ and the maximal commutator $M_{\alpha,b}$ on the Lebesgue spaces in the context of some stratified Lie group $\mathbb{G}$ when $b \in \Lambda_\beta(\mathbb{G})$. As a consequence, we give some new characterizations of the Lipschitz spaces in terms of $M_{\alpha,b}$ and $[b,M_\alpha]$.

To elaborate on the results, we first give the following notations.

Let $\alpha \geq 0$ and $f \in L^1_{\mathrm{loc}}(\mathbb{G})$, for a given $\mathbb{G}$-ball B^*, defining the fractional maximal function with regard to B^* by

$$M_{\alpha,B^*}(f)(x) = \sup_{\substack{B \ni x \\ B \subset B^*}} \frac{1}{|B|^{1-\alpha/Q}} \int_B \left|f(y)\right| \,\mathrm{d}y,$$

where the supremum is chosen over all $\mathbb{G}$-balls B with $x \in B \subset B^*$. We simply replace M_{0,B^*} by M_{B^*} when $\alpha = 0$.

Our main results may be formulated as follows.

Theorem 1.1 *Suppose b is a locally integrable function on $\mathbb{G}$. Let $0 < \beta < 1$, $0 < \alpha < Q$ and $0 < \alpha + \beta < Q$. Then the following assertions are equivalent*:

(A.1) *$b \in \Lambda_\beta(\mathbb{G})$ and $b \geq 0$.*

(A.2) *$[b,M_\alpha]$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ for all p and q satisfy $1 < p < \frac{Q}{\alpha+\beta}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha+\beta}{Q}$.*

(A.3) *$[b,M_\alpha]$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ for some p and q such that $1 < p < \frac{Q}{\alpha+\beta}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha+\beta}{Q}$.*

(A.4) *For some $s \in [1,\infty)$, we have*

$$\sup_B \frac{1}{|B|^{\beta/Q}} \left(\frac{1}{|B|} \int_B \left|b(x) - |B|^{-\alpha/Q} M_{\alpha,B}(b)(x)\right|^s \mathrm{d}x \right)^{1/s} < \infty. \tag{1.2}$$

(A.5) (1.2) *holds for all $s \in [1,\infty)$.*

Remark 1

(i) When $\alpha = 0$, the equivalence of (A.1), (A.2), and (A.4) was proved in [30, Theorem 1.3].

(ii) Additionally, it was shown in Theorem 1.3 of [30] that $b \in \Lambda_\beta(\mathbb{G})$ and $b \geq 0$ if and only if

$$\sup_B |B|^{-\beta/Q} \left(|B|^{-1} \int_B \left| b(x) - M_B(b)(x) \right|^q \, dx \right)^{1/q} < \infty \tag{1.3}$$

holds (see also Lemma 2.4 below). In contrast to (1.3), (1.2) yields a new characterization for nonnegative Lipschitz functions.

Now, we consider the mapping properties of $M_{\alpha,b}$ on Lebesgue spaces over some stratified Lie group $\mathbb{G}$ when b belongs to a Lipschitz space.

Theorem 1.2 *Let $b \in L^1_{\mathrm{loc}}(\mathbb{G})$, $0 < \beta < 1$ and $0 < \alpha < \alpha + \beta < Q$. Then the following assertions are equivalent*:

(B.1) $b \in \Lambda_\beta(\mathbb{G})$.

(B.2) *$M_{\alpha,b}$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ for all p, q with $1 < p < \frac{Q}{\alpha+\beta}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha+\beta}{Q}$.*

(B.3) *$M_{\alpha,b}$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ for some p, q with $1 < p < \frac{Q}{\alpha+\beta}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha+\beta}{Q}$.*

(B.4) *For some $s \in [1, \infty)$, we have*

$$\sup_B \frac{1}{|B|^{\beta/Q}} \left(\frac{1}{|B|} \int_B \left| b(x) - b_B \right|^s \, dx \right)^{1/s} < \infty. \tag{1.4}$$

(B.5) (1.4) *holds for all $s \in [1, \infty)$.*

Remark 2

(i) The equivalence of (B.1), (B.2), and (B.3) was proved in [30, Theorem 1.1] (for $\alpha = 0$). The equivalence of (B.1), (B.4), and (B.5) is implied in Lemma 2.3 below.

(ii) When $\mathbb{G} = \mathbb{R}^n$, the above equivalence was proved in [33] (see Corollary 1.3).

(iii) When $\alpha = 0$ and $\mathbb{G} = \mathbb{R}^n$, the analogous results in Lebesgue spaces were obtained in [31] for the case of constant exponent, while in [32, 33] for the case of variable exponent.

This paper is organized as follows. In Sect. 2, we will recall some basic definitions and known results. In Sect. 3, we will prove main results.

In this paper, the letter C always represents a constant without relation to the main parameters involved and whose value may vary from line to line. Moreover, here and hereafter L^p $(1 \leq p \leq \infty)$ will always denote the standard L^p-space with respect to the Haar measure dx, with the L^p-norm $\|\cdot\|_p$. Denote by χ_E the characteristic function of a measurable set E of $\mathbb{G}$. And for any $f \in L^1_{\mathrm{loc}}(\mathbb{G})$, set $f_E = \frac{1}{|E|} \int_E f(x) \, dx$.

2 Preliminaries and lemmas

To prove the principal results, we first review some necessary concepts and properties. Below we give some preliminaries concerning stratified Lie groups (or so-called Carnot groups). We refer the reader to [3, 13, 28].

2.1 Lie group $\mathbb{G}$

Definition 2.1 Let $m \in \mathbb{Z}^+$, $\mathcal{G}$ be a finite-dimensional Lie algebra, and $[X, Y] = XY - YX \in \mathcal{G}$ be Lie bracket with $X, Y \in \mathcal{G}$.

(i) If $Z \in \mathcal{G}$ is an m^{th} order Lie bracket and $W \in \mathcal{G}$, then $[Z, W]$ is an $(m+1)^{\text{st}}$ order Lie bracket.

(ii) We call $\mathcal{G}$ an m-step nilpotent Lie algebra if m is the smallest integer such that all Lie brackets of order $m+1$ are zero.

(iii) A Lie algebra $\mathcal{G}$ is said to be stratified if there exists a direct sum vector space decomposition

$$\mathcal{G} = \bigoplus_{j=1}^{m} V_j = V_1 \oplus \cdots \oplus V_m \tag{2.1}$$

such that $\mathcal{G}$ is m-step nilpotent, that is,

$$[V_1, V_j] = \begin{cases} V_{j+1} & 1 \le j \le m-1, \\ 0 & j \ge m \end{cases}$$

holds.

Since each element of V_j $(2 \le j \le m)$ is a linear combination of $(j-1)^{\text{th}}$ order Lie bracket of the elements of V_1, it is not difficult to find that the above V_1 generates the totality of the Lie algebra $\mathcal{G}$ by taking Lie brackets.

With the help of the related notions of Lie algebra (see Definition 2.1), the following definition can be obtained.

Definition 2.2 Let $\mathbb{G}$ be a finite-dimensional, connected and simply-connected Lie group associated with Lie algebra $\mathcal{G}$. Then

(i) $\mathbb{G}$ is said to be nilpotent if its Lie algebra $\mathcal{G}$ is nilpotent.

(ii) $\mathbb{G}$ is called stratified if its Lie algebra $\mathcal{G}$ is stratified.

(iii) We call $\mathbb{G}$ homogeneous if it is a nilpotent Lie group whose Lie algebra $\mathcal{G}$ has a family of dilations $\{\delta_r\}$, namely, for $r > 0$, $X_k \in V_k$ $(k = 1, \ldots, m)$,

$$\delta_r\left(\sum_{k=1}^{m} X_k\right) = \sum_{k=1}^{m} r^k X_k,$$

which are automorphisms of the Lie algebra.

Remark 3 Let $\mathcal{G} = \mathcal{G}_1 \supset \mathcal{G}_2 \supset \cdots \supset \mathcal{G}_{m+1} = \{0\}$ represent the lower central series of $\mathcal{G}$, and $X = \{X_1, \ldots, X_n\}$ be a basis of V_1 for $\mathcal{G}$.

(a) (see [37]) One can construct the direct sum decomposition (2.1) by identifying each $\mathcal{G}_j$ as a vector subspace of $\mathcal{G}$ and setting $V_m = \mathcal{G}_m$ and $V_j = \mathcal{G}_j \setminus \mathcal{G}_{j+1}$ for $j = 1, \ldots, m-1$.

(b) (see [12]) One call the number $Q = \operatorname{trace} A = \sum_{j=1}^{m} j \dim(V_j)$ the homogeneous dimension of $\mathcal{G}$, where A is a diagonalizable linear transformation of $\mathcal{G}$ with positive eigenvalues.

(c) (see [37] or [12]) The number Q is also known as the homogeneous dimension of $\mathbb{G}$ since $\mathrm{d}(\delta_r x) = r^Q \,\mathrm{d}x$ for all $r > 0$, and

$$Q = \sum_{j=1}^{m} j \dim(V_j) = \sum_{j=1}^{m} \dim(\mathcal{G}_j).$$

The following properties can be found in [27](see Proposition 1.1.1 or Proposition 1.2 in [13]).

Proposition 2.1 *Set $\mathcal{G}$ be a nilpotent Lie algebra, and $\mathbb{G}$ represent its corresponding connected and simply-connected nilpotent Lie group. Then we have*

(i) *The exponential map* $\exp : \mathcal{G} \to \mathbb{G}$ *is a diffeomorphism. Furthermore, if $\mathbb{G}$ is identified with $\mathcal{G}$ via* exp, *then the group law $(x, y) \mapsto xy$ is a polynomial map.*
(ii) $\exp \lambda$ *is called a bi-invariant Haar measure on $\mathbb{G}$ if λ is a Lebesgue measure on $\mathcal{G}$ (or a bi-invariant Haar measure* $\mathrm{d}x$ *on $\mathbb{G}$ is just the lift of Lebesgue measure on $\mathcal{G}$ by* exp).

Thereafter, set Q be the homogeneous dimension of $\mathbb{G}$, y^{-1} represent the inverse of $y \in \mathbb{G}$, and $y^{-1}x$ stand for the group multiplication of y^{-1} by x. Moreover, the identity element of the group $\mathbb{G}$ is called the origin, represented by e.

A homogenous norm $\rho : x \to \rho(x)$ defined on $\mathbb{G}$ is a continuous function from $\mathbb{G}$ to $[0, \infty)$, which is C^∞ on $\mathbb{G} \setminus \{e\}$ and satisfies

$$\begin{cases} \rho(x^{-1}) = \rho(x), \\ \rho(\delta_t x) = t\rho(x) \quad \text{for all } x \in \mathbb{G} \text{ and } t > 0, \\ \rho(e) = 0. \end{cases}$$

Moreover, there exists a constant $c_0 \geq 1$ such that $\rho(xy) \leq c_0(\rho(x) + \rho(y))$ for all $x, y \in \mathbb{G}$.

With the norm above, we define the $\mathbb{G}$ ball centered at x with radius r by $B(x, r) = \{y \in \mathbb{G} : \rho(y^{-1}x) < r\}$, and by λB represent the ball $B(x, \lambda r)$ with $\lambda > 0$, let $B_r = B(e, r) = \{y \in \mathbb{G} : \rho(y) < r\}$ be the open ball centered at e and of radius r, which is the mapping image under δ_r of $B(e, 1)$. And by ${}^{\complement}B(x, r) = \mathbb{G} \setminus B(x, r) = \{y \in \mathbb{G} : \rho(y^{-1}x) \geq r\}$ denote the complement of $B(x, r)$. Let $|B(x, r)|$ stand for the Haar measure of the ball $B(x, r) \subset \mathbb{G}$, and there exists $c_1 = c_1(\mathbb{G})$ such that

$$\left|B(x, r)\right| = c_1 r^Q, \quad x \in \mathbb{G}, r > 0.$$

In addition, the doubling condition is also satisfied by the Haar measure of a homogeneous Lie group $\mathbb{G}$ (see pages 140 and 501, [11]), i.e., $\forall x \in \mathbb{G}$, $r > 0$, $\exists C$, such that

$$\left|B(x, 2r)\right| \leq C\left|B(x, r)\right|.$$

In a stratified Lie group, the most fundamental partial differential operator is the sub-Laplacian associated to $X = \{X_1, \ldots, X_n\}$, i.e., the second-order partial differential operator on $\mathbb{G}$ given by

$$\mathfrak{L} = \sum_{i=1}^{n} X_i^2.$$

In the context of a Lie groups $\mathbb{G}$, when Young function $\Phi(t) = t^p$ and its complementary function $\Psi(t) = t^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, the following results can be inferred from [15] by elementary calculations.

Lemma 2.1 (Hölder's inequality on $\mathbb{G}$) *Let $1 \le p, q \le \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, $\Omega \subset \mathbb{G}$ be a measurable set and measurable functions $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then there exists a positive constant C such that*

$$\int_\Omega \left|f(x)g(x)\right| \,\mathrm{d}x \le C\|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}.$$

The following property can also be deduced from [15] by elementary calculations when Young function $\Phi(t) = t^p$.

Lemma 2.2 (Norms of characteristic functions) *Let $0 < p < \infty$ and $\Omega \subset \mathbb{G}$ be a measurable set with finite Haar measure. Then*

$$\|\chi_\Omega\|_{L^p(\mathbb{G})} = \|\chi_\Omega\|_{WL^p(\mathbb{G})} = |\Omega|^{1/p}.$$

2.2 Lipschitz spaces on $\mathbb{G}$

Next we give the definition of the Lipschitz spaces on $\mathbb{G}$, and state some basic properties and useful lemmas.

Definition 2.3 (Lipschitz-type spaces on $\mathbb{G}$)

(i) Let $0 < \beta < 1$ and ρ be the homogenous norm. We say a function b belongs to the Lipschitz space $\Lambda_\beta(\mathbb{G})$ if there exists a constant $C > 0$ such that

$$\left|b(x) - b(y)\right| \le C\left(\rho\left(y^{-1}x\right)\right)^\beta$$

for all $x, y \in \mathbb{G}$, where the smallest constant C is called the Lipschitz norm of b and denoted by $\|b\|_{\Lambda_\beta(\mathbb{G})}$.

(ii) (see [22]) Let $0 < \beta < 1$ and $1 \le p < \infty$. A locally integrable function b is said to belong to the space $\mathrm{Lip}_{\beta,p}(\mathbb{G})$ if there exists a positive constant C, such that

$$\sup_{B \ni x} \frac{1}{|B|^{\beta/Q}} \left(\frac{1}{|B|}\int_B \left|b(x) - b_B\right|^p \,\mathrm{d}x\right)^{1/p} \le C,$$

where the supremum is taken over every ball $B \subset \mathbb{G}$ containing x and $b_B = \frac{1}{|B|}\int_B b(x)\,\mathrm{d}x$. The least constant C satisfying the conditions above shall be denoted by $\|b\|_{\mathrm{Lip}_{\beta,p}(\mathbb{G})}$.

Remark 4

(a) In addition to the form of Definition 2.3 (i), we also commonly use the form as following (see [6, 10, 19] et al.)

$$\|b\|_{\Lambda_\beta(\mathbb{G})} = \sup_{\substack{x,y\in\mathbb{G}\\ x\neq y}} \frac{|b(x) - b(y)|}{(\rho(y^{-1}x))^\beta} < \infty.$$

And $\|b\|_{\Lambda_\beta(\mathbb{G})} = 0$ if and only if b is constant.

(b) In (ii), when $p = 1$, we have

$$\|b\|_{\mathrm{Lip}_{\beta,1}(\mathbb{G})} = \sup_{B \ni x} \frac{1}{|B|^{\beta/Q}} \left(\frac{1}{|B|} \int_B \left| b(x) - b_B \right| \mathrm{d}x \right) := \|b\|_{\mathrm{Lip}_\beta(\mathbb{G})}.$$

Lemma 2.3 (see [6, 20, 22]) *Assume that $0 < \beta < 1$ and b is a locally integrable function on $\mathbb{G}$.*

(i) *When $1 \leq p < \infty$, then*

$$\|b\|_{\Lambda_\beta(\mathbb{G})} = \|b\|_{\mathrm{Lip}_\beta(\mathbb{G})} \approx \|b\|_{\mathrm{Lip}_{\beta,p}(\mathbb{G})}.$$

(ii) *Let balls $B_1 \subset B_2 \subset \mathbb{G}$ and $b \in \mathrm{Lip}_{\beta,p}(\mathbb{G})$ with $p \in [1,\infty)$. Then there is a positive constant C depending only on B_1 and B_2, such that*

$$|b_{B_1} - b_{B_2}| \leq C \|b\|_{\mathrm{Lip}_{\beta,p}(\mathbb{G})} |B_2|^{\beta/Q}.$$

(iii) *When $1 \leq p < \infty$, then there exists a positive constant C depending only on β and p, such that*

$$\left| b(x) - b(y) \right| \leq C \|b\|_{\mathrm{Lip}_{\beta,p}(\mathbb{G})} |B|^{\beta/Q}$$

holds for any ball B containing x and y.

2.3 Some pointwise estimates and auxiliary lemmas

Hereafter, for a function b defined on $\mathbb{G}$, set

$$b^-(x) := -\min\{b, 0\} = \begin{cases} 0, & \text{if } b(x) \geq 0, \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) = |b(x)| - b^-(x)$. Obviously, $b(x) = b^+(x) - b^-(x)$.

With the help of the proof of Theorem 1.3 in [30], the following characterization for nonnegative Lipschitz functions can be obtained.

Lemma 2.4 *Let $0 < \beta < 1$ and $b \in L^1_{\mathrm{loc}}(\mathbb{G})$. Then the following assertions are equivalent:*

(i) *$b \in \Lambda_\beta(\mathbb{G})$ and $b \geq 0$.*

(ii) *For all $1 \leq s < \infty$, there exists a positive constant C such that*

$$\sup_B |B|^{-\beta/Q} \left(|B|^{-1} \int_B \left| b(x) - M_B(b)(x) \right|^s \mathrm{d}x \right)^{1/s} \leq C. \tag{2.2}$$

(iii) *There is an $1 \leq s < \infty$ such that (2.2) holds.*

Proof Since the implication (ii) $\Rightarrow$ (iii) is naturally true, and the implication (iii) $\Rightarrow$ (i) has been proved in [30, Theorem 1.3], so we only need to consider (i) $\Rightarrow$ (ii).

Suppose assertion (i) is true, i.e., $b \in \Lambda_\beta(\mathbb{G})$ and $b \geq 0$, then we can derive from [30, Theorem 1.3] that (2.2) holds for arbitrary s with $Q/(Q-\beta) < s < \infty$. Using Hölder's inequality, it is not difficult to find that (2.2) holds for $1 \leq s \leq Q/(Q-\beta)$ as well.

Hence, we prove that the implication (i) $\Rightarrow$ (ii). □

The following strong type estimate of M_α can be achieved from [18, Proposition A] or [2, Theorem 1.6] when the weights are constant 1 (see [18, 23] or [2] for more details).

Lemma 2.5 *Let $0 < \alpha < Q$, $1 < p < Q/\alpha$ and $1/q = 1/p - \alpha/Q$. If $f \in L^p(\mathbb{G})$. then there exists a positive constant C such that*

$$\left\|M_\alpha(f)\right\|_{L^q(\mathbb{G})} \leq C\|f\|_{L^p(\mathbb{G})}.$$

Remark 5

(a) By Lemma 2.5, if $0 < \alpha < Q$, $1 < p < Q/\alpha$ and $f \in L^p(\mathbb{G})$, then $M_\alpha(f)(x) < \infty$ for almost every $x \in \mathbb{G}$.

(b) The above lemma can also refer to Theorem 3.3 in [15] when Young function $\Phi(t) = t^p$ and its complementary function $\Psi(t) = t^q$ with $1/q = 1/p - \alpha/Q$.

The following gives the pointwise estimate for $[b, M_\alpha]$ on $\mathbb{G}$ when $b \in \Lambda_\beta(\mathbb{G})$.

Lemma 2.6 *Let $0 \leq \alpha < Q$, $0 < \beta < 1$, $0 < \alpha + \beta < Q$ and $f : \mathbb{G} \to \mathbb{R}$ be a locally integrable function. If $b \in \Lambda_\beta(\mathbb{G})$ and $b \geq 0$, then*

$$\left|[b, M_\alpha](f)(x)\right| \leq \|b\|_{\Lambda_\beta(\mathbb{G})} M_{\alpha+\beta}(f)(x)$$

holds for arbitrary $x \in \mathbb{G}$ such that $M_\alpha(f)(x) < \infty$.

Proof Similar to the discussion of Lemma 2.11 in [33]. For any given $x \in \mathbb{G}$ such that $M_\alpha(f)(x) < \infty$, if $b \in \Lambda_\beta(\mathbb{G})$ and $b \geq 0$, we have

$$\begin{aligned}
\left|[b, M_\alpha](f)(x)\right| &\leq \sup_{\substack{B \ni x \\ B \subset \mathbb{G}}} \frac{1}{|B|^{1-\alpha/Q}} \int_B \left|b(x) - b(y)\right| \left|f(y)\right| \, \mathrm{d}y \\
&\leq \|b\|_{\Lambda_\beta(\mathbb{G})} \sup_{\substack{B \ni x \\ B \subset \mathbb{G}}} \frac{1}{|B|^{1-(\alpha+\beta)/Q}} \int_B \left|f(y)\right| \, \mathrm{d}y \\
&\leq \|b\|_{\Lambda_\beta(\mathbb{G})} M_{\alpha+\beta}(f)(x).
\end{aligned}$$

□

Similar to Lemma 2.3 in [34], we get the following result.

Lemma 2.7 *Let $0 \leq \alpha < Q$, $B \subset \mathbb{G}$ be a given ball, and f be a locally integrable function. Then, for any $x \in B$, one has*

$$M_\alpha(f\chi_B)(x) = M_{\alpha,B}(f)(x). \tag{2.3}$$

Proof Some ideas are taken from [1] and [34]. Reasoning as the discussion of Lemma 2.3 in [34]. For any $x \in B$, it is easy to verify that

$$M_\alpha(f\chi_B)(x) \geq M_{\alpha,B}(f)(x) \tag{2.4}$$

from the definitions of $M_\alpha(f\chi_B)(x)$ and $M_{\alpha,B}(f)(x)$.

So, in order to prove the equality (2.3) is true, we only need to prove the following relation, namely, for any $\mathbb{G}$-ball $B^* \ni x$, there exist $\mathbb{G}$-ball $B' \ni x$ and $B' \subset B$, such that

$$\frac{1}{|B^*|^{1-\alpha/Q}} \int_{B^*} \left|f(y)\chi_B(y)\right| \mathrm{d}y \le \frac{1}{|B'|^{1-\alpha/Q}} \int_{B'} \left|f(y)\right| \mathrm{d}y. \tag{2.5}$$

In fact, for the case $B^* \cap B = \emptyset$, it is clear that (2.5) is true since $f(y)\chi_B(y) = 0$ for any $y \in B^*$.

Now we divide $B^* \cap B \neq \emptyset$ into two cases to consider.

(a) When the relation between B^* and B is inclusion. Without loss of generality, let $B^* \supset B$, then (2.5) is valid when we take $B' = B = B^* \cap B$.

(b) When $B^* \not\subset B$ and $B \not\subset B^*$, we consider the relation between $|B|$ and $|B^*|$.

(i) Assume $|B| \le |B^*|$. Then we may take $B' = B \supset B^* \cap B$, so (2.5) is true.

(ii) Assume $|B| > |B^*|$. Firstly, since $B^* \cap B$ is a bounded set in $\mathbb{G}$ and $x \in B^* \cap B$, then there exists not only a minimal ball B''' containing the intersection $B^* \cap B$, but also a maximal ball $B'' \subset B^* \cap B$ containing x inscribed in the ball B at a point P, namely, $x \in B'' \subset B^* \cap B$, $\partial B'' \cap \partial B = \{P\}$, $B^* \cap B \subset B'''$, and $|B'''| \le |B^*|$. Indeed, when the spherical center of B^* belongs to $B^* \cap B$, we can take $B''' = B^*$, otherwise $|B'''| < |B^*|$.

Secondly, there is a ball $B' \subset B$ such that $x \in B'' \subset B'$, $\partial B' \cap \partial B = \{P\}$, and $|B'| = |B'''| \le |B^*|$. Let $B' = (B' \cap B^*) \cup (B' \setminus B^*)$ and $B^* \cap B = (B' \cap B^*) \cup ((B^* \cap B) \setminus B')$ satisfy $x \in B'' \subset B' \cap B^*$ and $x \notin (B' \setminus B^*) \cup ((B^* \cap B) \setminus B')$.

Furthermore, for a given ball B, $f\chi_B$ is integrable and finite almost everywhere since f is a locally integrable function. Observe the fact that $B' \setminus B^*$ is larger than $(B^* \cap B) \setminus B'$, and neither contains x. Then there is an $\Omega \subset B' \setminus B^*$ such that $\int_{(B^* \cap B)\setminus B'} |f(y)| \,\mathrm{d}y \le \int_\Omega |f(y)| \,\mathrm{d}y$.

Combined with the discussion above, it follows that

$$\begin{aligned}
&\frac{1}{|B^*|^{1-\alpha/Q}} \int_{B^*} \left|f(y)\chi_B(y)\right| \mathrm{d}y \\
&= \left(\frac{|B'|}{|B^*|}\right)^{1-\alpha/Q} \frac{1}{|B'|^{1-\alpha/Q}} \int_{B^* \cap B} \left|f(y)\chi_B(y)\right| \mathrm{d}y \\
&\le \frac{1}{|B'|^{1-\alpha/Q}} \left(\int_{B' \cap B^*} \left|f(y)\chi_B(y)\right| \mathrm{d}y + \int_{(B^* \cap B)\setminus B'} \left|f(y)\chi_B(y)\right| \mathrm{d}y \right) \\
&\le \frac{1}{|B'|^{1-\alpha/Q}} \left(\int_{B' \cap B^*} \left|f(y)\chi_B(y)\right| \mathrm{d}y + \int_{\Omega} \left|f(y)\chi_B(y)\right| \mathrm{d}y \right) \\
&\le \frac{1}{|B'|^{1-\alpha/Q}} \left(\int_{B' \cap B^*} \left|f(y)\chi_B(y)\right| \mathrm{d}y + \int_{B' \setminus B^*} \left|f(y)\chi_B(y)\right| \mathrm{d}y \right) \\
&= \frac{1}{|B'|^{1-\alpha/Q}} \int_{B'} \left|f(y)\right| \mathrm{d}y.
\end{aligned}$$

Therefore, we obtain that (2.5) is valid.

Combining (2.4) and (2.5) together yields (2.3), which completes the proof. □

Remark 6

(a) Further, by applying Lemma 2.7 and the definition of $M_{\alpha,B}(\chi_B)(x)$, we have

$$M_\alpha(\chi_B)(x) = M_{\alpha,B}(\chi_B)(x) = |B|^{\alpha/Q}.$$

(b) For the case $\alpha = 0$, the following results are also valid, namely

$$M(\chi_B)(x) = M_B(\chi_B)(x) = \chi_B(x), \qquad M(f\chi_B)(x) = M_B(f)(x).$$

Referring to [1, page 3331] or [34], through elementary calculations and derivations, it is easy to check that the following assertions are true.

Lemma 2.8 *Let b be a locally integrable function on $\mathbb{G}$ and $B \subset \mathbb{G}$ be an arbitrary given ball.*

(i) *If $E = \{x \in B : b(x) \le b_B\}$ and $F = B \setminus E = \{x \in B : b(x) > b_B\}$. Then the following equality*

$$\int_E \left|b(x) - b_B\right| \mathrm{d}x = \int_F \left|b(x) - b_B\right| \mathrm{d}x$$

is trivially true.

(ii) *Then for any $x \in B$, we have $|b_B| \le |B|^{-\alpha/Q} M_{\alpha,B}(b)(x)$.*

3 Proofs of principal results

We now give the proof of the principal results.

3.1 Proof of Theorem 1.1

In order to prove Theorem 1.1, we first consider the following lemma.

Lemma 3.1 *Let $0 < \beta < 1$ and $0 < \alpha < Q$. Assume that b is a locally integrable function on $\mathbb{G}$, which satisfies*

$$\sup_B \frac{1}{|B|^{\beta/Q}} \left(\frac{1}{|B|} \int_B \left|b(x) - |B|^{-\alpha/Q} M_{\alpha,B}(b)(x)\right|^s \mathrm{d}x \right)^{1/s} < \infty, \tag{3.1}$$

for some $s \in [1, \infty)$, then $b \in \Lambda_\beta(\mathbb{G})$.

Proof Some ideas are taken from [1, 34, 35] and [33].

For any $\mathbb{G}$-ball $B \subset \mathbb{G}$, let $E = \{x \in B : b(x) \le b_B\}$ and $F = B \setminus E = \{x \in B : b(x) > b_B\}$. Noticing from Lemma 2.8(ii) that

$$|b_B| \le |B|^{-\alpha/Q} M_{\alpha,B}(b)(x) \quad \forall x \in B.$$

Then, for any $x \in E \subset B$, we have $b(x) \le b_B \le |b_B| \le |B|^{-\alpha/Q} M_{\alpha,B}(b)(x)$. It is clear that

$$\left|b(x) - b_B\right| \le \left|b(x) - |B|^{-\alpha/Q} M_{\alpha,B}(b)(x)\right|, \quad \forall x \in E.$$

Therefore, by using Lemma 2.8(i), we get

$$\begin{aligned}\frac{1}{|B|^{1+\beta/Q}}\int_B\bigl|b(x)-b_B\bigr|\,\mathrm{d}x &= \frac{1}{|B|^{1+\beta/Q}}\int_{E\cup F}\bigl|b(x)-b_B\bigr|\,\mathrm{d}x = \frac{2}{|B|^{1+\beta/Q}}\int_E\bigl|b(x)-b_B\bigr|\,\mathrm{d}x\\ &\le \frac{2}{|B|^{1+\beta/Q}}\int_E\bigl|b(x)-|B|^{-\alpha/Q}M_{\alpha,B}(b)(x)\bigr|\,\mathrm{d}x\\ &\le \frac{2}{|B|^{1+\beta/Q}}\int_B\bigl|b(x)-|B|^{-\alpha/Q}M_{\alpha,B}(b)(x)\bigr|\,\mathrm{d}x.\end{aligned}$$

By using Lemma 2.1, (3.1), and Lemma 2.2, we have

$$\begin{aligned}\frac{1}{|B|^{1+\beta/Q}}\int_B\bigl|b(x)-b_B\bigr|\,\mathrm{d}x &\le \frac{2}{|B|^{1+\beta/Q}}\int_B\bigl|b(x)-|B|^{-\alpha/Q}M_{\alpha,B}(b)(x)\bigr|\,\mathrm{d}x\\ &\le \frac{C}{|B|^{1+\beta/Q}}\left(\int_B\bigl|b(x)-|B|^{-\alpha/Q}M_{\alpha,B}(b)(x)\bigr|^s\,\mathrm{d}x\right)^{1/s}\|\chi_B\|_{L^{s'}(\mathbb{G})}\\ &\le \frac{C}{|B|^{\beta/Q}}\left(\frac{1}{|B|}\int_B\bigl|b(x)-|B|^{-\alpha/Q}M_{\alpha,B}(b)(x)\bigr|^s\,\mathrm{d}x\right)^{1/s}\le C.\end{aligned}$$

Therefore, by utilizing Lemma 2.3 and Definition 2.3, we complete the proof. □

Now, we prove the mapping properties of the nonlinear commutator $[b,M_\alpha]$ on Lebesgue spaces over some stratified Lie group $\mathbb{G}$ when the symbol b belongs to some Lipschitz space.

Proof of Theorem 1.1 Since the implications (A.2) ⇒ (A.3) and (A.5) ⇒ (A.4) are easy to obtain, we only need to verify that (A.1) ⇒ (A.2), (A.3) ⇒ (A.4), (A.4) ⇒ (A.1), and (A.2) ⇒ (A.5).

(A.1) ⇒ (A.2): Let $b\in\Lambda_\beta(\mathbb{G})$ and $b\ge 0$. We need to verify that $[b,M_\alpha]$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ for all p and q satisfy $1<p<\frac{Q}{\alpha+\beta}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha+\beta}{Q}$. For the above p and arbitrary $f\in L^p(\mathbb{G})$, it follows from Remark 5(i) that $M_\alpha(f)(x)<\infty$ for almost everywhere $x\in\mathbb{G}$. According to Lemma 2.6, we get

$$\bigl|[b,M_\alpha](f)(x)\bigr|\le\|b\|_{\Lambda_\beta(\mathbb{G})}M_{\alpha+\beta}(f)(x).$$

Then, assertion (A.2) can be obtained from Lemma 2.5.

(A.3) ⇒ (A.4): Assume that assertion (A.3) is true, that is, there exist p and q such that $[b,M_\alpha]$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$. Below we shall validate that (1.2) holds when $s=q$.

For any given $\mathbb{G}$-ball $B\subset\mathbb{G}$ and arbitrary $x\in B$, it follows immediately from Lemma 2.7 and Remark 6 that the pointwise relations

$$M_\alpha(b\chi_B)(x)=M_{\alpha,B}(b)(x)\quad\text{and}\quad M_\alpha(\chi_B)(x)=M_{\alpha,B}(\chi_B)(x)=|B|^{\alpha/Q}$$

hold. So, for arbitrary $x\in B$, we have

$$b(x)-|B|^{-\alpha/Q}M_{\alpha,B}(b)(x)=|B|^{-\alpha/Q}\bigl(b(x)|B|^{\alpha/Q}-M_{\alpha,B}(b)(x)\bigr)$$

$$= |B|^{-\alpha/Q}\big(b(x)M_\alpha(\chi_B)(x) - M_\alpha(b\chi_B)(x)\big)$$
$$= |B|^{-\alpha/Q}[b, M_\alpha](\chi_B)(x).$$

Observe that $[b, M_\alpha]$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha+\beta}{Q}$. Therefore for any ball $B \subset \mathbb{G}$, by applying Lemma 2.2, we obtain

$$\begin{aligned}\frac{1}{|B|^{\beta/Q}}\left(\frac{1}{|B|}\int_B \left|b(x) - |B|^{-\alpha/Q}M_{\alpha,B}(b)(x)\right|^q \,\mathrm{d}x\right)^{1/q} &\le |B|^{-(\alpha+\beta)/Q-1/q}\left\|[b,M_\alpha](\chi_B)\right\|_{L^q(\mathbb{G})}\\ &\le C|B|^{-(\alpha+\beta)/Q-1/q}\|\chi_B\|_{L^p(\mathbb{G})} \le C,\end{aligned}$$

which shows (1.2) holds for $s = q$, since the ball $B \subset \mathbb{G}$ is arbitrary and C does not depend on B.

(A.4) $\Rightarrow$ (A.1): By Lemma 2.4, it sufficiently shows that

$$\sup_B \frac{1}{|B|^{1+\beta/Q}}\int_B \left|b(x) - M_B(b)(x)\right| \,\mathrm{d}x < \infty. \tag{3.2}$$

For arbitrary fixed ball $B \subset \mathbb{G}$, we have

$$\begin{aligned}&\frac{1}{|B|^{1+\beta/Q}}\int_B \left|b(x) - M_B(b)(x)\right| \,\mathrm{d}x\\ &\quad\le \frac{1}{|B|^{1+\beta/Q}}\int_B \left|b(x) - |B|^{-\alpha/Q}M_{\alpha,B}(b)(x)\right| \,\mathrm{d}x\\ &\qquad + \frac{1}{|B|^{1+\beta/Q}}\int_B \left||B|^{-\alpha/Q}M_{\alpha,B}(b)(x) - M_B(b)(x)\right| \,\mathrm{d}x\\ &\quad := I_1 + I_2.\end{aligned} \tag{3.3}$$

For I_1, by applying statement (A.4), Lemma 2.1 (Hölder's inequality), and Lemma 2.2, we get

$$\begin{aligned}I_1 &\le \frac{1}{|B|^{1+\beta/Q}}\left(\int_B \left|b(x) - |B|^{-\alpha/Q}M_{\alpha,B}(b)(x)\right|^s \,\mathrm{d}x\right)^{1/s}\|\chi_B\|_{L^{s'}(\mathbb{G})}\\ &\le \frac{C}{|B|^{\beta/Q}}\left(\frac{1}{|B|}\int_B \left|b(x) - |B|^{-\alpha/Q}M_{\alpha,B}(b)(x)\right|^s \,\mathrm{d}x\right)^{1/s}\\ &\le C,\end{aligned}$$

where the constant C is not dependent on ball B.

Now we estimate I_2. For all $x \in B$, the following pointwise relations can be obtained immediately from Lemma 2.7 and Remark 6, i.e.,

$$M_\alpha(\chi_B)(x) = |B|^{\alpha/Q} \quad \text{and} \quad M_\alpha(b\chi_B)(x) = M_{\alpha,B}(b)(x),$$

and

$$M(\chi_B)(x) = \chi_B(x) = 1 \quad \text{and} \quad M(b\chi_B)(x) = M_B(b)(x).$$

Thus, for arbitrary $x \in B$, we get

$$\begin{aligned}
&\left||B|^{-\alpha/Q}M_{\alpha,B}(b)(x) - M_B(b)(x)\right| \\
&\quad \le |B|^{-\alpha/Q}\left|M_{\alpha,B}(b)(x) - |B|^{\alpha/Q}\left|b(x)\right|\right| + \left|\left|b(x)\right| - M_B(b)(x)\right| \\
&\quad \le |B|^{-\alpha/Q}\left|M_\alpha(b\chi_B)(x) - \left|b(x)\right|M_\alpha(\chi_B)(x)\right| \\
&\qquad + \left|\left|b(x)\right|M(\chi_B)(x) - M(b\chi_B)(x)\right| \\
&\quad \le |B|^{-\alpha/Q}\left|\left[|b|, M_\alpha\right](\chi_B)(x)\right| + \left|\left[|b|, M\right](\chi_B)(x)\right|.
\end{aligned} \tag{3.4}$$

Since statement (A.4) together with Lemma 3.1 yields $b \in \Lambda_\beta(\mathbb{G})$, which implies $|b| \in \Lambda_\beta(\mathbb{G})$. Hence, we may apply Lemma 2.6 to $[|b|, M_\alpha]$ and $[|b|, M]$ since $|b| \in \Lambda_\beta(\mathbb{G})$ and $|b| \ge 0$.

By using Lemma 2.6, Lemma 2.7, and Remark 6, for all $x \in B$, we have

$$\left|\left[|b|, M_\alpha\right](\chi_B)(x)\right| \le \|b\|_{\Lambda_\beta(\mathbb{G})}M_{\alpha+\beta}(\chi_B)(x) \le C\|b\|_{\Lambda_\beta(\mathbb{G})}|B|^{(\alpha+\beta)/Q}$$

and

$$\left|\left[|b|, M\right](\chi_B)(x)\right| \le \|b\|_{\Lambda_\beta(\mathbb{G})}M_\beta(\chi_B)(x) \le C\|b\|_{\Lambda_\beta(\mathbb{G})}|B|^{\beta/Q}.$$

Thus, it follows from (3.4) that

$$\begin{aligned}
I_2 &\le \frac{C}{|B|^{1+(\alpha+\beta)/Q}}\int_B \left|\left[|b|, M_\alpha\right](\chi_B)(x)\right| \,\mathrm{d}x + \frac{C}{|B|^{1+\beta/Q}}\int_B \left|\left[|b|, M\right](\chi_B)(x)\right| \,\mathrm{d}x \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{G})}.
\end{aligned}$$

Substituting the above estimates for I_1 and I_2 into (3.3), we derive (3.2).

(A.2) $\Rightarrow$ (A.5): Suppose that assertion (A.2) is valid. Inference as in the proof of (A.3) $\Rightarrow$ (A.4), we know that

$$\sup_B \frac{1}{|B|^{\beta/Q}}\left(\frac{1}{|B|}\int_B \left|b(x) - |B|^{-\alpha/Q}M_{\alpha,B}(b)(x)\right|^q \,\mathrm{d}x\right)^{1/q} < \infty \tag{3.5}$$

holds for arbitrary q for which there is a p satisfying $\frac{1}{q} = \frac{1}{p} - \frac{\alpha+\beta}{Q}$.

For each $s \in [1, \infty)$, selecting an $r > Q/(Q-\beta) > 1$, we get $1 < rs(Q-\beta)/Q < rs$. Let $q = rs$ and define p by $\frac{1}{p} = \frac{1}{q} + \frac{\alpha+\beta}{Q}$. Noticing that

$$\frac{1}{s} = \frac{1}{rs} + \frac{1}{r's} = \frac{1}{q} + \frac{1}{r's},$$

by using Lemma 2.1, (3.5), and Lemma 2.2, we have

$$\begin{aligned}
&\frac{1}{|B|^{\beta/Q}}\left(\frac{1}{|B|}\int_B \left|b(x) - |B|^{-\alpha/Q}M_{\alpha,B}(b)(x)\right|^s \,\mathrm{d}x\right)^{1/s} \\
&\quad \le \frac{C}{|B|^{1/s+\beta/Q}}\left(\int_B \left|b(x) - |B|^{-\alpha/Q}M_{\alpha,B}(b)(x)\right|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \|\chi_B\|_{L^{r's}(\mathbb{G})}
\end{aligned}$$

$$\begin{aligned}
&\le \frac{C}{|B|^{1/s-1/q-\frac{1}{r's}}}\frac{1}{|B|^{\beta/Q}}\left(\frac{1}{|B|}\int_B\left|b(x)-|B|^{-\alpha/Q}M_{\alpha,B}(b)(x)\right|^q\,dx\right)^{\frac{1}{q}}\\
&\le C,
\end{aligned}$$

which is exactly what we desire.

The proof is completed. □

3.2 Proof of Theorem 1.2

Below we prove the mapping properties of $M_{\alpha,b}$ on Lebesgue spaces over some stratified Lie group $\mathbb{G}$, where the symbol b belongs to a Lipschitz space.

Proof of Theorem 1.2 Since the implications (B.2) ⇒ (B.3) and (B.5) ⇒ (B.4) are obvious, we only need to verify that (B.1) ⇒ (B.2), (B.3) ⇒ (B.4), (B.4) ⇒ (B.1), and (B.2) ⇒ (B.5).

(B.1) ⇒ (B.2): Let $b \in \Lambda_\beta(\mathbb{G})$, then, employing Definition 2.3 (i), we get

$$\begin{aligned}
M_{\alpha,b}(f)(x) &\le C\|b\|_{\Lambda_\beta(\mathbb{G})}\sup_{B\ni x}\frac{1}{|B|^{1-\alpha/Q}}\int_B\left|\rho\left(y^{-1}x\right)\right|^\beta\left|f(y)\right|\,dy\\
&\le C\|b\|_{\Lambda_\beta(\mathbb{G})}\sup_{B\ni x}\frac{1}{|B|^{1-(\alpha+\beta)/Q}}\int_B\left|f(y)\right|\,dy\\
&\le C\|b\|_{\Lambda_\beta(\mathbb{G})}M_{\alpha+\beta}(f)(x).
\end{aligned}\tag{3.6}$$

Therefore, assertion (B.2) follows from Lemma 2.5 and (3.6).

(B.3) ⇒ (B.4): For arbitrary fixed ball $B \subset \mathbb{G}$, one has

$$\begin{aligned}
\left|b(x)-b_B\right| &\le \frac{1}{|B|}\int_B\left|b(x)-b(y)\right|\,dy = \frac{1}{|B|}\int_B\left|b(x)-b(y)\right|\chi_B(y)\,dy\\
&\le \frac{1}{|B|^{\alpha/Q}}M_{\alpha,b}(\chi_B)(x)
\end{aligned}$$

for all $x \in B$. Then, for any $x \in \mathbb{G}$, we get

$$\left|\left(b(x)-b_B\right)\chi_B(x)\right| \le \frac{1}{|B|^{\alpha/Q}}M_{\alpha,b}(\chi_B)(x).$$

By using assertion (B.3) and Lemma 2.2, for any ball $B \subset \mathbb{G}$, one obtains

$$\begin{aligned}
\frac{1}{|B|^{\beta/Q}}\left(\frac{1}{|B|}\int_B\left|b(x)-b_B\right|^q\,dx\right)^{1/q} &\le \frac{1}{|B|^{(\alpha+\beta)/Q}}\left(\frac{1}{|B|}\int_B\left(M_{\alpha,b}(\chi_B)(x)\right)^q\,dx\right)^{1/q}\\
&\le \frac{C}{|B|^{1/q+(\alpha+\beta)/Q}}\|\chi_B\|_{L^p(\mathbb{G})}\\
&\le C,
\end{aligned}$$

which implies that (1.4) holds for $s = q$ since B is arbitrary and C does not depend on B.

(B.4) ⇒ (B.1): For arbitrary ball $B \subset \mathbb{G}$, using Hölder's inequality (see Lemma 2.1), Lemma 2.2, and assertion (B.4), we get

$$\frac{1}{|B|^{1+\beta/Q}}\int_B\left|b(x)-b_B\right|\,dx \le \frac{C}{|B|^{1+\beta/Q}}\left(\int_B\left|b(x)-b_B\right|^q\,dx\right)^{1/q}\left(\int_B\chi_B(x)\,dx\right)^{1/q'}$$

$$\leq \frac{C}{|B|^{\beta/Q}}\left(\frac{1}{|B|}\int_B \left|b(x)-b_B\right|^q \,\mathrm{d}x\right)^{1/q}$$
$$\leq C.$$

Furthermore, it follows from Lemma 2.3 and Definition 2.3 that $b \in \Lambda_\beta(\mathbb{G})$ since B is an arbitrary ball in $\mathbb{G}$.

(B.2) ⇒ (B.5): Similar to the course of the proof of (A.2) ⇒ (A.5), thus, we omit it.

The proof of Theorem 1.2 is completed. □

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Availability of data and materials

All data generated or analyzed during this study are included in this published article.

Declarations

Competing interests

The authors declare no competing interests.

Author contributions

All authors contributed equally to this work. All authors read the final manuscript and approved its submission.

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