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A modified class of Ostrowski-type inequalities and error bounds of Hermite–Hadamard inequalities

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Abstract

This paper aims to extend the application of the Ostrowski inequality, a crucial tool for figuring out the error bounds of various numerical quadrature rules, including Simpson's, trapezoidal, and midpoint rules. Specifically, we develop a more comprehensive class of Ostrowski-type inequalities by utilizing the weighted version of Riemann–Liouville (RL) fractional integrals on an increasing function. We apply our findings to estimate the error bounds of Hadamard-type inequalities. Our results are more comprehensive, since we obtain the results of the existing literatures as particular cases for certain parameter values. This research motivates researchers to apply this concept to other fractional operators.

Mathematics Subject Classification: Primary 26D15; secondary 26A51; 26A33; 05A30

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1 Introduction

Fractional calculus has a rich academic history and represents a natural extension of traditional calculus. Recent advances in fractional calculus can be found in mathematical physics, biology, chemistry, engineering, signal processing, fluid mechanics, viscoelasticity, mathematical biology, and electrochemistry [1–3]. The well-known RL fractional operators include single kernels and are used to examine and evaluate memory effect phenomena in mathematical physics [4]. Fractional calculus operators with various types of kernels are important for generalizing classical mathematical inequalities. The kernels involved in other mathematical conceptions serve a critical role in their existence and applications [5, 6]. Our findings can encompass a wide range of fractional operators as the kernel includes a strictly increasing function. This allows us to extend and unify numerous previously published results in the literature. A variety of fractional integral operators that reduce to the traditional RL fractional integral operator have been developed [7–11]. Weighted fractional integral operators are revealed to be bound in the Lebesgue space, and various classical fractional integral and differential operators are found as special instances

[12]. Fractional operators are used to generalize integrals, derivatives, and in particular integrals involving inequalities [13]. We generalize the inequalities for a family of n positive functions by employing the (k,ς)-fractional integral operator [14, 15].

Inequalities are a modern mathematical analysis model that depicts the rate of advancement in the mathematical analysis competition [16]. The study of fractional partial and ordinary differential equations can greatly benefit from Hermite–Hadamard inequalities that involve fractional integral operators [17, 18]. Fractional integral inequalities have shown to be one of the most powerful and broad-reaching tools for advancing many fields of pure and applied mathematics [19, 20]. Due to their distinctive applications in numerical quadrature, transform theory, probability, and statistical issues, these inequalities have gained tremendous prominence and relevance during the last few decades [21, 22]. The Ostrowski inequality holds significant value and utility in the fields of mathematical analysis, numerical analysis, and engineering [23]. It provides an estimate of the integral mean of a function [24–26]. Such inequalities are used to derive explicit limits for the perturbed trapezoidal, midpoint, Simpson, Newton–Cotes, and left and right rectangle rules. These are also applied to various composite quadrature rules, and the analysis allows for the calculation of the partition necessary for the accuracy of the result to be within a certain error tolerance [27]. It also specifies the error boundaries of specific mean relations and numerous numerical quadrature rules of integration [28].

The Ostrowski inequality was first proposed by Ostrowski [29] in 1938 and can be expressed as follows.

Theorem 1.1 *Suppose that γ is differentiable function on J^0, $\lambda_1,\lambda_2\in J^0$, $\lambda_1<\lambda_2$, and $|\gamma'(\tau)|\leq M$ for all $\tau\in[\lambda_1,\lambda_2]$. Then we have the following inequality*:

$$\left|\gamma(\theta)-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\gamma(\tau)\,d\tau\right|\leq\left[\frac{1}{4}+\frac{(\theta-\frac{\lambda_1+\lambda_2}{2})^2}{(\lambda_2-\lambda_1)^2}\right](\lambda_2-\lambda_1)M, \tag{1.1}$$

where $\theta\in[\lambda_1,\lambda_2]$.

The Hadamard inequality provides an interesting viewpoint on convex functions in the Cartesian plane as in the following statement.

Theorem 1.2 *Suppose that γ is positive convex function defined on a real interval J. Then we have the following inequality for $\lambda_1,\lambda_2\in J$ such that $\lambda_1<\lambda_2$*:

$$\gamma\left(\frac{\lambda_1+\lambda_2}{2}\right)\leq\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\gamma(\tau)\,d\tau\leq\frac{\gamma(\lambda_1)+\gamma(\lambda_2)}{2}.$$

To begin, let us review some definitions and notions. The gamma and k-gamma functions can be represented as integrals, as outlined in [10]. Specifically, these functions are defined as follows.

Definition 1.3 If $\Re(\mu)>0$, then the k-gamma function is defined as

$$\Gamma_k(\mu)=\int_0^\infty\tau^{\mu-1}e^{-\frac{\tau^k}{k}}\,d\tau$$

with the property

$$\Gamma_k(\mu + k) = \mu\Gamma_k(\mu).$$

Setting $k = 1$ yields the classical gamma function.

Let us revisit the definitions of RL fractional integrals and their general forms [30]. The left- and right-sided RL k-fractional integrals are defined in [10]. The idea of RL fractional integrals is extended by the (k, ς)-RL fractional integrals described in [9].

Definition 1.4 The right and left RL fractional integrals of order $\mu > 0$ for a continuous function γ_1 on the finite real interval $[\lambda_1, \lambda_2]$ is given by

$$\left(\Im^{\mu}_{\lambda_1^+}\gamma_1\right)(\theta) = \frac{1}{\Gamma(\mu)}\int_{\lambda_1}^{\theta}(\theta - \tau)^{\mu-1}\gamma_1(\tau)\,d\tau, \quad \theta > \lambda_1,$$

and

$$\left(\Im^{\mu}_{\lambda_2^-}\gamma_1\right)(\theta) = \frac{1}{\Gamma(\mu)}\int_{\theta}^{\lambda_2}(\tau - \theta)^{\mu-1}\gamma_1(\tau)\,d\tau, \quad \theta < \lambda_2.$$

Definition 1.5 Let γ_1 be a continuous function on the real interval $[\lambda_1, \lambda_2]$. Then the right and left RL k-fractional integrals of order $\mu, k > 0$ are defined as

$$\left({}_k\Im^{\mu}_{\lambda_1^+}\gamma_1\right)(\theta) = \frac{1}{k\Gamma_k(\mu)}\int_{\lambda_1}^{\theta}(\theta - \tau)^{\frac{\mu}{k}-1}\gamma(\tau)\,d\tau, \quad \theta > \lambda_1,$$

and

$$\left({}_k\Im^{\mu}_{\lambda_2^-}\gamma_1\right)(\theta) = \frac{1}{k\Gamma_k(\mu)}\int_{\theta}^{\lambda_2}(\tau - \theta)^{\frac{\mu}{k}-1}\gamma_1(\tau)\,d\tau, \quad \theta < \lambda_2.$$

Definition 1.6 Let γ_1 be a continuous function on $[\lambda_1, \lambda_2]$. Then the right and left (k, ς)-RL fractional integrals of order $\mu, k > 0$ are defined as

$$\left({}^{\varsigma}_{k}\Im^{\mu}_{\lambda_1^+}\gamma_1\right)(\theta) = \frac{(\varsigma + 1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_{\lambda_1}^{\theta}\left(\theta^{\varsigma+1} - \tau^{\varsigma+1}\right)^{\frac{\mu}{k}-1}\tau^{\varsigma}\gamma_1(\tau)\,d\tau, \quad \theta > \lambda_1,$$

and

$$\left({}^{\varsigma}_{k}\Im^{\mu}_{\lambda_2^-}\gamma_1\right)(\theta) = \frac{(\varsigma + 1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_{\theta}^{\lambda_2}\left(\tau^{\varsigma+1} - \theta^{\varsigma+1}\right)^{\frac{\mu}{k}-1}\tau^{\varsigma}\gamma_1(\tau)\,d\tau, \quad \theta < \lambda_2,$$

where $\varsigma \in \mathbb{R} \setminus \{-1\}$.

The (k, ς)-weighted RL fractional integrals, which generalize the RL fractional integrals, are defined in [9].

Definition 1.7 Let γ_1 be a continuous function on $[\lambda_1,\lambda_2]$, and let σ be a nonzero increasing weight function. Then right and left weighted (k,ς)-RL fractional integrals of order $\mu, k > 0$ are defined as

$$\left({}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\gamma_1\right)(\theta) = \frac{(\varsigma+1)^{1-\frac{\mu}{k}}\sigma^{-1}(\theta)}{k\Gamma_k(\mu)}\int_{\lambda_1}^{\theta}\left(\theta^{\varsigma+1}-\tau^{\varsigma+1}\right)^{\frac{\mu}{k}-1}\tau^{\varsigma}\sigma(\tau)\gamma_1(\tau)\,d\tau, \quad \theta > \lambda_1,$$

and

$$\left({}_k^{\varsigma}\Im^{\mu}_{\lambda_2^-,\sigma}\gamma_1\right)(\theta) = \frac{(\varsigma+1)^{1-\frac{\mu}{k}}\sigma^{-1}(\theta)}{k\Gamma_k(\mu)}\int_{\theta}^{\lambda_2}\left(\tau^{\varsigma+1}-\theta^{\varsigma+1}\right)^{\frac{\mu}{k}-1}\tau^{\varsigma}\sigma(\tau)\gamma_1(\tau)\,d\tau, \quad \theta < \lambda_2,$$

where $\sigma(\theta) \neq 0$ and $\varsigma \in \mathbb{R} \setminus \{-1\}$.

An extension of RL fractional integrals for an increasing function is presented in [31], expressed in the following definition.

Definition 1.8 Let γ_1 be a continuous function on $[\lambda_1,\lambda_2]$, let γ_2 be a strictly increasing function, and let σ be a nonzero increasing weight function. Then the right and left generalized weighted RL fractional integrals of order $\mu > 0$ are defined as

$$\left({}_{\gamma_2}\Im^{\mu}_{\lambda_1^+,\sigma}\gamma_1\right)(\theta) = \frac{\sigma^{-1}(\theta)}{\Gamma(\mu)}\int_{\lambda_1}^{\theta}\left(\gamma_2(\theta)-\gamma_2(\tau)\right)^{\mu-1}\gamma_2'(\tau)\sigma(\tau)\gamma_1(\tau)\,d\tau, \quad \theta > \lambda_1,$$

and

$$\left({}_{\gamma_2}\Im^{\mu}_{\lambda_2^-,\sigma}\gamma_1\right)(\theta) = \frac{\sigma^{-1}(\theta)}{\Gamma(\mu)}\int_{\theta}^{\lambda_2}\left(\gamma_2(\tau)-\gamma_2(\theta)\right)^{\mu-1}\gamma_2'(\tau)\sigma(\tau)\gamma_1(\tau)\,d\tau, \quad \theta < \lambda_2,$$

where $\sigma(\theta) \neq 0$.

Reference [32] presents the Hadamard inequality in fractional form by utilizing RL fractional integrals. The following theorem presents a variant of the fractional Hadamard inequality for RL fractional integrals, as defined in Definition 1.4.

Theorem 1.9 *Let γ be a convex function on $[\lambda_1,\lambda_2]$, $0 \le \lambda_1 < \lambda_2$, and $\gamma \in L_1[\lambda_1,\lambda_2]$. Then the RL fractional integrals obey the following inequality*:

$$\gamma\left(\frac{\lambda_1+\lambda_2}{2}\right) \le \frac{\Gamma(\mu+1)}{2(\lambda_2-\lambda_1)^{\mu}}\left[\left(\Im^{\mu}_{\lambda_1^+}\gamma\right)(\lambda_2) + \left(\Im^{\mu}_{\lambda_2^-}\gamma\right)(\lambda_1)\right] \le \frac{\gamma(\lambda_1)+\gamma(\lambda_2)}{2}.$$

By utilizing generalized RL fractional integrals Farid et al. [33] achieved a generalization of the Hadamard inequality. The following theorem presents the generalized version of the Hadamard inequality for generalized RL fractional integrals defined in Definition 1.5.

Theorem 1.10 *Let γ be a convex function on $[\lambda_1,\lambda_2]$, $0 \le \lambda_1 < \lambda_2$, and $\gamma \in L_1[\lambda_1,\lambda_2]$. Then the RL k-fractional integrals satisfy the following inequality*:

$$\gamma\left(\frac{\lambda_1+\lambda_2}{2}\right) \le \frac{\Gamma_k(\mu+k)}{2(\lambda_2-\lambda_1)^{\frac{\mu}{k}}}\left[\left({}_k\Im^{\mu}_{\lambda_1^+}\gamma\right)(\lambda_2) + \left({}_k\Im^{\mu}_{\lambda_2^-}\gamma\right)(\lambda_1)\right] \le \frac{\gamma(\lambda_1)+\gamma(\lambda_2)}{2}.$$

Reference [34] illustrates the fractional version of the aforementioned fractional Hadamard inequality as follows.

Theorem 1.11 *Let γ be a convex function on $[\lambda_1,\lambda_2]$, $0 \le \lambda_1 < \lambda_2$, and $\gamma \in L_1[\lambda_1,\lambda_2]$. Then the RL fractional integrals satisfy the following inequality*:

$$\gamma\left(\frac{\lambda_1+\lambda_2}{2}\right) \le \frac{2^{\mu-1}\Gamma(\mu+1)}{(\lambda_2-\lambda_1)^{\mu}}\left[\left(\Im^{\mu}_{(\frac{\lambda_1+\lambda_2}{2})^+}\gamma\right)(\lambda_2)+\left(\Im^{\mu}_{(\frac{\lambda_1+\lambda_2}{2})^-}\gamma\right)(\lambda_1)\right] \le \frac{\gamma(\lambda_1)+\gamma(\lambda_2)}{2}.$$

Reference [33] illustrates the k-fractional version of the aforementioned fractional Hadamard inequality as follows.

Theorem 1.12 *Let γ be a convex function on $[\lambda_1,\lambda_2]$, $0 \le \lambda_1 < \lambda_2$, and $\gamma \in L_1[\lambda_1,\lambda_2]$. Then the RL k-fractional integrals satisfy the following inequality*:

$$\gamma\left(\frac{\lambda_1+\lambda_2}{2}\right) \le \frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(\lambda_2-\lambda_1)^{\frac{\mu}{k}}}\left[\left({}_k\Im^{\mu}_{(\frac{\lambda_1+\lambda_2}{2})^+}\gamma\right)(\lambda_2)+\left({}_k\Im^{\mu}_{(\frac{\lambda_1+\lambda_2}{2})^-}\gamma\right)(\lambda_1)\right] \le \frac{\gamma(\lambda_1)+\gamma(\lambda_2)}{2}.$$

The Ostrowski-type inequalities for RL fractional integrals are reported in [35]. Inspired by the research discussed earlier, we intend to propose a new class of Ostrowski-type inequalities. Our approach involves the utilization of generalized weighted (k,ς)-RL fractional operators.

2 Weighted fractional integral inequalities via generalized fractional operator

In this section, we proof the Ostrowski-type inequalities by using weighted (k,ς)-RL fractional integral operator with respect to an increasing function.

In [12] an extension of weighted (k,ς)-RL fractional integrals for an increasing function is written in the following way.

Definition 2.1 Let γ_1 be a continuous function on $[\lambda_1,\lambda_2]$, let γ_2 be a strictly increasing function, and let σ be the nonzero increasing weight function. Then the right and left generalized weighted (k,ς)-RL fractional integrals of order $\mu > 0$ and type $k > 0$ are defined as

$$\left(\gamma_2{}^{\varsigma}_{k}\Im^{\mu}_{\lambda_1^+,\sigma}\gamma_1\right)(\theta) = \frac{(\varsigma+1)^{1-\frac{\mu}{k}}\sigma^{-1}(\theta)}{k\Gamma_k(\mu)}\int_{\lambda_1}^{\theta}\left(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\tau)\right)^{\frac{\mu}{k}-1}\gamma_2^{\varsigma}(\tau)\gamma_2'(\tau)\sigma(\tau)\gamma_1(\tau)\,d\tau, \quad \theta > \lambda_1,$$

and

$$\left(\gamma_2{}^{\varsigma}_{k}\Im^{\mu}_{\lambda_2^-,\sigma}\gamma_1\right)(\theta) = \frac{(\varsigma+1)^{1-\frac{\mu}{k}}\sigma^{-1}(\theta)}{k\Gamma_k(\mu)}\int_{\theta}^{\lambda_2}\left(\gamma_2^{\varsigma+1}(\tau)-\gamma_2^{\varsigma+1}(\theta)\right)^{\frac{\mu}{k}-1}\gamma_2^{\varsigma}(\tau)\gamma_2'(\tau)\sigma(\tau)\gamma_1(\tau)\,d\tau, \quad \theta < \lambda_2,$$

where $\sigma(\theta) \ne 0$, $\varsigma \in \mathbb{R}\setminus\{-1\}$, and $\gamma_2^{\varsigma+1}(\theta) = (\gamma_2(\theta))^{\varsigma+1}$.

Remark 2.2 In Definition 2.1, setting

(i) $\gamma_2(\theta) = \theta$, $\varsigma = 0$, $k = 1$, and $\sigma(\theta) = 1$, we obtain Definition 1.4;
(ii) $\gamma_2(\theta) = \theta$, $\varsigma = 0$, and $\sigma(\theta) = 1$, we obtain Definition 1.5;
(iii) $\gamma_2(\theta) = \theta$ and $\sigma(\theta) = 1$, we obtain Definition 1.6;
(iv) $\gamma_2(\theta) = \theta$, we obtain Definition 1.7;
(v) $\varsigma = 0$ and $k = 1$, we get Definition 1.8.

Theorem 2.3 *Let $\gamma_1 : J \to R$ be a differentiable function on J^0, $\lambda_1, \lambda_2 \in J^0$, and $\lambda_1 < \lambda_2$. Also, let $\gamma_2 : [\lambda_1, \lambda_2] \to R$ be a differentiable, increasing function such that $\gamma_2' \in L[\lambda_1, \lambda_2]$ and $|\gamma_1'(\tau)| \leq M$ for all $\tau \in [\lambda_1, \lambda_2]$, and let σ be a nonzero increasing weight function. Then for $\mu, \nu \geq 0$ and $k > 0$, we have the following inequality for weighted (k, ς)-RL fractional integrals*:

$$\begin{aligned}
&\bigg| \gamma_1(\theta)\Big(\big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}} + \big(\gamma_2^{\varsigma+1}(\theta) - \gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\Big) \\
&\quad - \bigg((\varsigma+1)^{\frac{\nu}{k}} \Gamma_k(\nu+k) \frac{\sigma(\theta)}{\sigma(\lambda_2)} \big(\gamma_{2,k}^{\ \varsigma} \Im^{\nu}_{\lambda_2^-,\sigma} \gamma_1\big)(\theta) \\
&\quad + (\varsigma+1)^{\frac{\mu}{k}} \Gamma_k(\mu+k) \big(\gamma_{2,k}^{\ \varsigma} \Im^{\mu}_{\lambda_1^+,\sigma} \gamma_1\big)(\theta)\bigg)\bigg| \\
&\quad \leq \frac{M}{\varsigma+1}\bigg(\theta^{\varsigma+1}\Big(\big(\gamma_2^{\varsigma+1}(\theta) - \gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}} - \big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\Big) \\
&\quad + (\varsigma+1)^{\frac{\nu}{k}} \Gamma_k(\nu+k) \frac{\sigma(\theta)}{\sigma(\lambda_2)} \big(\gamma_{2,k}^{\ \varsigma} \Im^{\nu}_{\lambda_2^-,\sigma} I\big)(\theta) \\
&\quad - (\varsigma+1)^{\frac{\mu}{k}} \Gamma_k(\mu+k) \big(\gamma_{2,k}^{\ \varsigma} \Im^{\mu}_{\lambda_1^+,\sigma} I\big)(\theta)\bigg).
\end{aligned} \tag{2.1}$$

Proof Let $\theta \in [\lambda_1, \lambda_2]$ and $\tau \in [\lambda_1, \theta]$. Since γ_2 is a strictly increasing function, $\mu \geq 0$, and $k > 0$, we have the following relation:

$$\big(\gamma_2^{\varsigma+1}(\theta) - \gamma_2^{\varsigma+1}(\tau)\big)^{\frac{\mu}{k}} \leq \big(\gamma_2^{\varsigma+1}(\theta) - \gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}. \tag{2.2}$$

Let $\tau^{\varsigma} \geq 1$. Then the following inequalities are simple consequences of (2.2) and the boundedness requirement on γ_1':

$$\begin{aligned}
&\int_{\lambda_1}^{\theta} \big(M\tau^{\varsigma} - \gamma_1'(\tau)\big)\big(\gamma_2^{\varsigma+1}(\theta) - \gamma_2^{\varsigma+1}(\tau)\big)^{\frac{\mu}{k}} \, d\tau \\
&\quad \leq \big(\gamma_2^{\varsigma+1}(\theta) - \gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}} \int_{\lambda_1}^{\theta} \big(M\tau^{\varsigma} - \gamma_1'(\tau)\big) \, d\tau
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&\int_{\lambda_1}^{\theta} \big(M\tau^{\varsigma} + \gamma_1'(\tau)\big)\big(\gamma_2^{\varsigma+1}(\theta) - \gamma_2^{\varsigma+1}(\tau)\big)^{\frac{\mu}{k}} \, d\tau \\
&\quad \leq \big(\gamma_2^{\varsigma+1}(\theta) - \gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}} \int_{\lambda_1}^{\theta} \big(M\tau^{\varsigma} + \gamma_1'(\tau)\big) \, d\tau.
\end{aligned} \tag{2.4}$$

By performing integration and straightforward calculations on (2.3) and (2.4) and utilizing Definition 2.1, we arrive at the following inequalities:

$$\begin{aligned}&\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\gamma_1(\theta)-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\gamma_1\big)(\theta)\\&\quad\le\frac{M}{\varsigma+1}\Big(\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\theta^{\varsigma+1}-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}I\big)(\theta)\Big)\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}&(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\gamma_1\big)(\theta)-\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\gamma_1(\theta)\\&\quad\le\frac{M}{\varsigma+1}\Big(\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\theta^{\varsigma+1}-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}I\big)(\theta)\Big).\end{aligned}\tag{2.6}$$

Therefore from inequality (2.5) and (2.6) we have following modulus inequality:

$$\begin{aligned}&\big|\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\gamma_1(\theta)-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\gamma_1\big)(\theta)\big|\\&\quad\le\frac{M}{\varsigma+1}\Big(\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\theta^{\varsigma+1}-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}I\big)(\theta)\Big).\end{aligned}\tag{2.7}$$

Similarly, for $\theta\in[\lambda_1,\lambda_2]$, $\tau\in[\theta,\lambda_2]$, $\nu\ge 0$, and $k>0$, we have the following inequality:

$$\big(\gamma_2^{\varsigma+1}(\tau)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\le\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}.\tag{2.8}$$

If $\tau^{\varsigma}\ge 1$, then the following inequalities are the basic consequences of (2.8) and the boundedness of γ_1':

$$\begin{aligned}&\int_{\theta}^{\lambda_2}\big(M\tau^{\varsigma}-\gamma_1'(\tau)\big)\big(\gamma_2^{\varsigma+1}(\tau)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\,d\tau\\&\quad\le\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\int_{\theta}^{\lambda_2}\big(M\tau^{\varsigma}-\gamma_1'(\tau)\big)\,d\tau\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}&\int_{\theta}^{\lambda_2}\big(M\tau^{\varsigma}+\gamma_1'(\tau)\big)\big(\gamma_2^{\varsigma+1}(\tau)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\,d\tau\\&\quad\le\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\int_{\theta}^{\lambda_2}\big(M\tau^{\varsigma}+\gamma_1'(\tau)\big)\,d\tau.\end{aligned}\tag{2.10}$$

After integrating and performing straightforward calculations on (2.9) and (2.10) and utilizing Definition 2.1, we obtain the following inequalities:

$$\begin{aligned}&(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}\gamma_1\big)(\theta)-\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\gamma_1(\theta)\\&\quad\le\frac{M}{\varsigma+1}\bigg((\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}I\big)(\theta)-\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\theta^{\varsigma+1}\bigg)\end{aligned}\tag{2.11}$$

and

$$\begin{aligned}&\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\gamma_1(\theta)-(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}\gamma_1\big)(\theta)\\&\quad\le\frac{M}{\varsigma+1}\bigg((\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}I\big)(\theta)-\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\theta^{\varsigma+1}\bigg).\end{aligned}\tag{2.12}$$

From (2.11) and (2.12) we have following modulus inequality:

$$\begin{aligned}&\bigg|\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\gamma_1(\theta)-(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}\gamma_1\big)(\theta)\bigg|\\&\quad\le\frac{M}{\varsigma+1}\bigg((\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}I\big)(\theta)-\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\theta^{\varsigma+1}\bigg).\end{aligned}\tag{2.13}$$

Inequalities (2.7) and (2.13) involving modulus together form inequality (2.1). □

Corollary 2.4 *For $\mu=\nu$ in (2.1), we have the following generalized weighted (k,ς)-fractional integral inequality*:

$$\begin{aligned}&\bigg|\gamma_1(\theta)\Big(\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\mu}{k}}+\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\Big)\\&\qquad-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\bigg(\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_2^-,\sigma}\gamma_1\big)(\theta)+\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\gamma_1\big)(\theta)\bigg)\bigg|\\&\quad\le\frac{M}{\varsigma+1}\bigg(\theta^{\varsigma+1}\Big(\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}-\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\mu}{k}}\Big)\\&\qquad+(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_2^-,\sigma}I\big)(\theta)-\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}I\big)(\theta)\bigg).\end{aligned}$$

Corollary 2.5 *For $\gamma_2(\theta)=\theta$ and $\sigma(\theta)=1$ in (2.1), we have the following inequality for (k,ς)-RL fractional integrals*:

$$\begin{aligned}&\Big|\gamma_1(\theta)\Big(\big(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\big)^{\frac{\nu}{k}}+\big(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\big)^{\frac{\mu}{k}}\Big)\\&\qquad-\Big((\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\big({}_k^{\varsigma}\Im^{\nu}_{\lambda_2^-}\gamma_1\big)(\theta)+(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big({}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+}\gamma_1\big)(\theta)\Big)\Big|\\&\quad\le\frac{M}{\varsigma+1}\bigg(\frac{\mu}{\mu+k}\big(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\big)^{\frac{\mu}{k}+1}+\frac{\nu}{\nu+k}\big(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\big)^{\frac{\nu}{k}+1}\bigg).\end{aligned}$$

Corollary 2.6 *For $\gamma_2(\theta)=\theta$, $k=1$, and $\sigma(\theta)=1$ in (2.1), we have the following inequality for RL fractional integrals*:

$$\begin{aligned}&\Big|\gamma_1(\theta)\Big(\big(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\big)^{\nu}+\big(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\big)^{\mu}\Big)\\&\qquad-\Big((\varsigma+1)^{\nu}\Gamma(\nu+1)\big({}^{\varsigma}\Im^{\nu}_{\lambda_2^-}\gamma_1\big)(\theta)+(\varsigma+1)^{\mu}\Gamma(\mu+1)\big({}^{\varsigma}\Im^{\mu}_{\lambda_1^+}\gamma_1\big)(\theta)\Big)\Big|\\&\quad\le\frac{M}{\varsigma+1}\bigg(\frac{\mu}{\mu+1}\big(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\big)^{\mu+1}+\frac{\nu}{\nu+1}\big(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\big)^{\nu+1}\bigg).\end{aligned}$$

Corollary 2.7 *For $\mu = \nu = k = 1$, $\sigma(\theta) = 1$, and $\gamma_2(\theta) = \theta$ in (2.1), we have the following fractional integral inequality*:

$$\left|\gamma_1(\theta) - \frac{(\varsigma + 1)}{(\lambda_2^{\varsigma+1} - \lambda_1^{\varsigma+1})}\int_{\lambda_1}^{\lambda_2} \tau^{\varsigma}\gamma_1(\tau)\,d\tau\right| \leq \frac{M(\lambda_2^{\varsigma+1} - \lambda_1^{\varsigma+1})}{\varsigma + 1}\left(\frac{(\theta^{\varsigma+1} - \frac{a^{\varsigma+1}+\lambda_2^{\varsigma+1}}{2})^2}{(\lambda_2^{\varsigma+1} - \lambda_1^{\varsigma+1})^2} + \frac{1}{4}\right).$$

Remark 2.8 If we set $\gamma_2(\theta) = \theta$, $k = 1$, $\varsigma = 0$, and $\sigma(\theta) = 1$ in (2.1), then we get the following inequality for RL fractional integrals given in [36, Theorem 1.2]:

$$\left|\gamma_1(\theta)\big((\lambda_2 - \theta)^{\nu} + (\theta - \lambda_1)^{\mu}\big) - \big(\Gamma(\nu + 1)\big(\Im^{\nu}_{\lambda_2^-}\gamma_1\big)(\theta) + \Gamma(\mu + 1)\big(\Im^{\mu}_{\lambda_1^+}\gamma_1\big)(\theta)\big)\right| \leq M\left(\frac{\mu}{\mu + 1}(\theta - \lambda_1)^{\mu+1} + \frac{\nu}{\nu + 1}(\lambda_2 - \theta)^{\nu+1}\right).$$

Remark 2.9 For $\mu = \nu = k = 1$, $\sigma(\theta) = 1$, $\varsigma = 0$, and $\gamma_2(\theta) = \theta$ in (2.1), we obtain the Östrowski inequality (1.1).

Next, we present a more extended fractional Ostrowski-type inequality for the weighted (k,ς)-RL fractional integrals operators.

Theorem 2.10 *Let $\gamma_1 : J \to \mathbb{R}$ be a differentiable function on J^0, $\lambda_1, \lambda_2 \in J^0$, and $\lambda_1 < \lambda_2$. Also, let $\gamma_2 : [\lambda_1, \lambda_2] \to \mathbb{R}$ be a differentiable increasing function with $\gamma_2' \in L[\lambda_1, \lambda_2]$, and suppose $m \leq \gamma_1'(\tau) \leq M$ for all $\tau \in [\lambda_1, \lambda_2]$. Let σ be a nonzero increasing weight function. Then, for $\mu, \nu \geq 0$ and $k > 0$, we have the following inequality hold for weighted (k,ς)-RL*:

$$\begin{aligned}
&\big(\big(\gamma_2^{\varsigma+1}(\theta) - \gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}} + \big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\big)\gamma_1(\theta) \\
&\quad - \Big((\varsigma + 1)^{\frac{\mu}{k}}\Gamma_k(\mu + k)\big(\gamma_{2,k}^{\ \varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\gamma_1\big)(\theta) \\
&\quad + (\varsigma + 1)^{\frac{\nu}{k}}\Gamma_k(\nu + k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_{2,k}^{\ \varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}\gamma_1\big)(\theta)\Big) \\
&\leq \frac{M}{\varsigma + 1}\big(\theta^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\theta) - \gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}} - (\varsigma + 1)^{\frac{\mu}{k}}\Gamma_k(\mu + k)\big(\gamma_{2,k}^{\ \varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}I\big)(\theta)\big) \\
&\quad - \frac{m}{\varsigma + 1}\Big((\varsigma + 1)^{\frac{\nu}{k}}\Gamma_k(\nu + k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_{2,k}^{\ \varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}I\big)(\theta) \\
&\quad - \theta^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\Big)
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
&(\varsigma + 1)^{\frac{\mu}{k}}\Gamma_k(\mu + k)\big(\gamma_{2,k}^{\ \varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\gamma_1\big)(\theta) + (\varsigma + 1)^{\frac{\nu}{k}}\Gamma_k(\nu + k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_{2,k}^{\ \varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}\gamma_1\big)(\theta) \\
&\quad - \big(\big(\gamma_2^{\varsigma+1}(\theta) - \gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}} + \big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\gamma_1(\theta)\big) \\
&\leq \frac{M}{\varsigma + 1}\Big((\varsigma + 1)^{\frac{\nu}{k}}\Gamma_k(\nu + k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_{2,k}^{\ \varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}I\big)(\theta) - \theta^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\Big)
\end{aligned}$$

$$-\frac{m}{\varsigma+1}\big(\theta^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}$$
$$-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}I\big)(\theta)\big). \tag{2.15}$$

Proof Let $\theta\in[\lambda_1,\lambda_2]$ and $\tau\in[\lambda_1,\theta]$. Since γ_2 is a strictly increasing function, $\mu\geq 0$, and $k>0$, then we have the following inequality:

$$\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\tau)\big)^{\frac{\mu}{k}}\leq\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}. \tag{2.16}$$

Let $\tau^{\varsigma}\geq 1$. Then from (2.16) and the boundedness condition on γ_1', as a simple consequence, we get following inequalities:

$$\int_{\lambda_1}^{\theta}\big(M\tau^{\varsigma}-\gamma_1'(\tau)\big)\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\tau)\big)^{\frac{\mu}{k}}\,d\tau$$
$$\leq\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\int_{\lambda_1}^{\theta}\big(M\tau^{\varsigma}-\gamma_1'(\tau)\big)\,d\tau \tag{2.17}$$

and

$$\int_{\lambda_1}^{\theta}\big(\gamma_1'(\tau)-m\tau^{\varsigma}\big)\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\tau)\big)^{\frac{\mu}{k}}\,d\tau$$
$$\leq\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\int_{\lambda_1}^{\theta}\big(\gamma_1'(\tau)-m\tau^{\varsigma}\big)\,d\tau. \tag{2.18}$$

By performing integration and straightforward calculations on (2.17) and (2.18) and utilizing Definition 2.1 we arrive at the following inequalities:

$$\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\gamma_1(\theta)-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\gamma_1\big)(\theta)$$
$$\leq\frac{M}{\varsigma+1}\Big(\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\theta^{\varsigma+1}-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}I\big)(\theta)\Big) \tag{2.19}$$

and

$$(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\gamma_1\big)(\theta)-\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\gamma_1(\theta)$$
$$\leq-\frac{m}{\varsigma+1}\Big(\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\theta^{\varsigma+1}-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}I\big)(\theta)\Big). \tag{2.20}$$

Similarly, for $\theta\in[\lambda_1,\lambda_2]$, $\tau\in[\theta,\lambda_2]$, $\nu\geq 0$, and $k>0$, we have the following inequality:

$$\big(\gamma_2^{\varsigma+1}(\tau)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\leq\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}. \tag{2.21}$$

If $\tau^{\varsigma}\geq 1$, then the following inequalities are a simple consequence of (2.21) and the boundedness condition on γ_1':

$$\int_{\theta}^{\lambda_2}\big(M\tau^{\varsigma}-\gamma_1'(\tau)\big)\big(\gamma_2^{\varsigma+1}(\tau)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\,d\tau$$
$$\leq\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\int_{\theta}^{\lambda_2}\big(M\tau^{\varsigma}-\gamma_1'(\tau)\big)\,d\tau \tag{2.22}$$

and

$$\int_{\theta}^{\lambda_2}\big(\gamma_1'(\tau)-m\tau^{\varsigma}\big)\big(\gamma_2^{\varsigma+1}(\tau)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\,d\tau$$
$$\leq\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\int_{\theta}^{\lambda_2}\big(\gamma_1'(\tau)-m\tau^{\varsigma}\big)\,d\tau. \tag{2.23}$$

By performing integration and straightforward calculations on (2.22) and (2.23) and utilizing Definition 2.1 we arrive at the following inequalities:

$$(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big({}_{\gamma_2,k}^{\varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}\gamma_1\big)(\theta)-\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\gamma_1(\theta)$$
$$\leq\frac{M}{\varsigma+1}\bigg((\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big({}_{\gamma_2,k}^{\varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}I\big)(\theta)-\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\theta^{\varsigma+1}\bigg) \tag{2.24}$$

and

$$\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\gamma_1(\theta)-(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big({}_{\gamma_2,k}^{\varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}\gamma_1\big)(\theta)$$
$$\leq-\frac{m}{\varsigma+1}\bigg((\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big({}_{\gamma_2,k}^{\varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}I\big)(\theta)-\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\theta^{\varsigma+1}\bigg). \tag{2.25}$$

By adding (2.19) and (2.25) we get (2.14). Similarly by adding (2.20) and (2.24) we get (2.15). □

Corollary 2.11 *For $\mu=\nu$ in (2.14) and (2.15), we have the following weighted (k,ς)-fractional integral inequalities*:

$$\Big(\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}+\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\mu}{k}}\Big)\gamma_1(\theta)$$
$$-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\bigg(\big({}_{\gamma_2,k}^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\gamma_1\big)(\theta)+\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big({}_{\gamma_2,k}^{\varsigma}\Im^{\mu}_{\lambda_2^-,\sigma}\gamma_1\big)(\theta)\bigg)$$
$$\leq\frac{M}{\varsigma+1}\Big(\theta^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big({}_{\gamma_2,k}^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}I\big)(\theta)\Big)$$
$$-\frac{m}{\varsigma+1}\bigg((\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big({}_{\gamma_2,k}^{\varsigma}\Im^{\mu}_{\lambda_2^-,\sigma}I\big)(\theta)$$
$$-\theta^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\mu}{k}}\bigg)$$

and

$$(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\bigg(\big({}_{\gamma_2,k}^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\gamma_1\big)(\theta)+\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big({}_{\gamma_2,k}^{\varsigma}\Im^{\mu}_{\lambda_2^-,\sigma}\gamma_1\big)(\theta)\bigg)$$
$$-\Big(\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}+\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\mu}{k}}\gamma_1(\theta)\Big)$$

$$\leq \frac{M}{\varsigma+1}\left((\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\left(\gamma_2,{}_k^\varsigma\Im^{\mu}_{\lambda_2^-,\sigma}I\right)(\theta)-\theta^{\varsigma+1}\left(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\right)^{\frac{\mu}{k}}\right)$$
$$-\frac{m}{\varsigma+1}\left(\theta^{\varsigma+1}\left(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\right)^{\frac{\mu}{k}}-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\left(\gamma_2,{}_k^\varsigma\Im^{\mu}_{\lambda_1^+,\sigma}I\right)(\theta)\right).$$

Corollary 2.12 *Substituting $\gamma_2(\theta)=\theta$ and $\sigma(\theta)=1$ into (2.14) and (2.15), we get the following inequalities for (k,ς)-RL fractional integrals*:

$$\left(\left(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\right)^{\frac{\mu}{k}}+\left(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\right)^{\frac{\nu}{k}}\right)\gamma_1(\theta)-\left((\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\left({}_k^\varsigma\Im^{\mu}_{\lambda_1^+}\gamma\right)(\theta)\right.$$
$$\left.+(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\left({}_k^\varsigma\Im^{\nu}_{\lambda_2^-}\gamma_1\right)(\theta)\right)$$
$$\leq \frac{M}{\varsigma+1}\left(\frac{\mu}{\mu+k}\left(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\right)^{\frac{\mu}{k}+1}\right)-\frac{m}{\varsigma+1}\left(\frac{\nu}{\nu+k}\left(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\right)^{\frac{\nu}{k}+1}\right)$$

and

$$(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\left({}_k^\varsigma\Im^{\mu}_{\lambda_1^+}\gamma_1\right)(\theta)+(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\left({}_k^\varsigma\Im^{\nu}_{\lambda_2^-}\gamma_1\right)(\theta)$$
$$-\left(\left(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\right)^{\frac{\mu}{k}}+\left(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\right)^{\frac{\nu}{k}}\gamma_1(\theta)\right)$$
$$\leq \frac{M}{\varsigma+1}\left(\frac{\nu}{\nu+k}\left(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\right)^{\frac{\nu}{k}+1}\right)-\frac{m}{\varsigma+1}\left(\frac{\mu}{\mu+k}\left(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\right)^{\frac{\mu}{k}+1}\right).$$

Corollary 2.13 *Setting $\gamma_2(\theta)=\theta$, $\sigma(\theta)=1$, and $k=1$ in (2.14) and (2.15), we get the following fractional integral inequalities for RL fractional integrals*:

$$\left(\left(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\right)^{\mu}+\left(\lambda_2^{\varsigma+1}-\gamma_2^{\varsigma+1}\right)^{\nu}\right)\gamma_1(\theta)-\left((\varsigma+1)^{\mu}\Gamma(\mu+1)\left({}^\varsigma\Im^{\mu}_{\lambda_1^+}\gamma_1\right)(\theta)\right.$$
$$\left.+(\varsigma+1)^{\nu}\Gamma(\nu+1)\left({}^\varsigma\Im^{\nu}_{\lambda_2^-}\gamma_1\right)(\theta)\right)$$
$$\leq \frac{M}{\varsigma+1}\left(\frac{\mu}{\mu+1}\left(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\right)^{\mu+1}\right)-\frac{m}{\varsigma+1}\left(\frac{\nu}{\nu+1}\left(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\right)^{\nu+1}\right)$$

and

$$(\varsigma+1)^{\mu}\Gamma(\mu+1)\left({}^\varsigma\Im^{\mu}_{\lambda_1^+}\gamma_1\right)(\theta)+(\varsigma+1)^{\nu}\Gamma(\nu+1)\left({}^\varsigma\Im^{\nu}_{\lambda_2^-}\gamma_1\right)(\theta)$$
$$-\left(\left(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\right)^{\mu}+\left(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\right)^{\nu}\gamma_1(\theta)\right)$$
$$\leq \frac{M}{\varsigma+1}\left(\frac{\nu}{\nu+1}\left(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\right)^{\nu+1}\right)-\frac{m}{\varsigma+1}\left(\frac{\mu}{\mu+1}\left(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\right)^{\mu+1}\right).$$

Corollary 2.14 *Setting $\mu=\nu=k=1$, $\sigma(\theta)=1$, and $\gamma_2(\theta)=\theta$ in (2.14) and (2.15), we get the following fractional integral inequalities for RL fractional integrals*:

$$\left(\lambda_2^{\varsigma+1}-\lambda_1^{\varsigma+1}\right)\gamma_1(\theta)-(\varsigma+1)\int_{\lambda_1}^{\lambda_2}\tau^{\varsigma}\gamma_1(\tau)\,d\tau$$
$$\leq \frac{M}{2(\varsigma+1)}\left(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\right)^2-\frac{m}{2(\varsigma+1)}\left(\theta^{\varsigma+1}-\lambda_2^{\varsigma+1}\right)^2$$

and

$$(\varsigma+1)\int_{\lambda_1}^{\lambda_2}\tau^{\varsigma}\gamma_1(\tau)\,d\tau-\left(\lambda_2^{\varsigma+1}-\lambda_1^{\varsigma+1}\right)\gamma_1(\theta)$$
$$\leq\frac{M}{2(\varsigma+1)}\left(\theta^{\varsigma+1}-\lambda_2^{\varsigma+1}\right)^2-\frac{m}{2(\varsigma+1)}\left(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\right)^2.$$

Remark 2.15 Setting $\gamma_2(\theta)=\theta$, $\sigma(\theta)=1$, $\varsigma=0$, and $k=1$ in (2.14) and (2.15), we obtain the inequalities for RL fractional integrals given in [36, Theorem 1.3]:

$$\left((\theta-\lambda_1)^{\mu}+(\lambda_2-h)^{\nu}\right)\gamma_1(\theta)-\left(\Gamma(\mu+1)\left(\Im_{\lambda_1^+}^{\mu}\gamma_1\right)(\theta)+\Gamma(\nu+1)\left(\Im_{\lambda_2^-}^{\nu}\gamma_1\right)(\theta)\right)$$
$$\leq M\left(\frac{\mu}{\mu+1}(\theta-\lambda_1)^{\mu+1}\right)-m\left(\frac{\nu}{\nu+1}(\lambda_2-\theta)^{\nu+1}\right)$$

and

$$\Gamma(\mu+1)\left(\Im_{\lambda_1^+}^{\mu}\gamma_1\right)(\theta)+\Gamma(\nu+1)\left(\Im_{\lambda_2^-}^{\nu}\gamma_1\right)(\theta)-\left((\theta-\lambda_1)^{\mu}+(\lambda_2-\theta)^{\nu}\gamma_1(\theta)\right)$$
$$\leq M\left(\frac{\nu}{\nu+1}(\lambda_2-\theta)^{\nu+1}\right)-m\left(\frac{\mu}{\mu+1}(\theta-\lambda_1)^{\mu+1}\right).$$

Remark 2.16 By setting $m=-M$ in Theorem 2.10 and making some rearrangements, we obtain Theorem 2.3.

Theorem 2.17 *Let $\gamma_1:J\to\mathbb{R}$ be a differentiable function on J^0, $\lambda_1,\lambda_2\in J^0$, and $\lambda_1<\lambda_2$. Also, let $\gamma_2:[\lambda_1,\lambda_2]\to\mathbb{R}$ be a differentiable and increasing function such that $\gamma_2'(\tau)\in L[\lambda_1,\lambda_2]$ and $|\gamma_1'(\tau)|\leq M$ for all $\tau\in[\lambda_1,\lambda_2]$, and let σ be a nonzero increasing weight function. Then for $\mu,\nu\geq 0$ and $k>0$, we have the following inequality for weighted (k,ς)-RL fractional integrals*:

$$\begin{aligned}&\Bigg|\gamma_1(\lambda_2)\left(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\right)^{\frac{\nu}{k}}+\gamma_1(\lambda_1)\left(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\right)^{\frac{\mu}{k}}\\&\quad-\left((\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\left({}_{\gamma_2,k}^{\varsigma}\Im_{\theta^+,\sigma}^{\nu}\gamma_1\right)(\lambda_2)\right.\\&\quad\left.+(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\left({}_{\gamma_2,k}^{\varsigma}\Im_{\theta^-,\sigma}^{\mu}\gamma_1\right)(\lambda_1)\right)\Bigg|\\&\leq\frac{M}{\varsigma+1}\left(\lambda_2^{\varsigma+1}\left(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\right)^{\frac{\nu}{k}}-\lambda_1^{\varsigma+1}\left(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\right)^{\frac{\mu}{k}}\right.\\&\quad+(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\left({}_{\gamma_2,k}^{\varsigma}\Im_{\theta^-,\sigma}^{\mu}\lambda_1\right)\\&\quad\left.-(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\left({}_{\gamma_2,k}^{\varsigma}\Im_{\theta^+,\sigma}^{\nu}\lambda_2\right)\right).\end{aligned}\tag{2.26}$$

Proof Let $\theta\in[\lambda_1,\lambda_2]$ and $\tau\in[\lambda_1,\theta]$. Since the function γ_2 is strictly increasing function, $\mu\geq 0$, and $k>0$, we have the following inequality:

$$\left(\gamma_2^{\varsigma+1}(\tau)-\gamma_2^{\varsigma+1}(\lambda_1)\right)^{\frac{\mu}{k}}\leq\left(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\right)^{\frac{\mu}{k}}.\tag{2.27}$$

If $\tau^{\varsigma} \geq 1$, then then the following inequalities are a simple consequence of (2.27) and boundedness condition on γ_1':

$$\begin{aligned}&\int_{\lambda_1}^{\theta}\big(M\tau^{\varsigma}-\gamma_1'(\tau)\big)\big(\gamma_2^{\varsigma+1}(\tau)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\,d\tau\\&\quad\leq\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\int_{\lambda_1}^{\theta}\big(M\tau^{\varsigma}-\gamma_1'(\tau)\big)\,d\tau\end{aligned}\tag{2.28}$$

and

$$\begin{aligned}&\int_{\lambda_1}^{\theta}\big(M\tau^{\varsigma}+\gamma_1'(\tau)\big)\big(\gamma_2^{\varsigma+1}(\tau)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\,d\tau\\&\quad\leq\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\int_{\lambda_1}^{\theta}\big(M\tau^{\varsigma}+\gamma_1'(\tau)\big)\,d\tau.\end{aligned}\tag{2.29}$$

Utilizing Definition 2.1, integrating, and performing some straightforward calculations on (2.28) and (2.29), we obtain the following inequalities:

$$\begin{aligned}&(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^-,\sigma}^{\mu}\gamma_1\big)(\lambda_1)-\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\gamma_1(\lambda_1)\\&\quad\leq\frac{M}{\varsigma+1}\Big((\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^-,\sigma}^{\mu}I\big)(\lambda_1)-\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\lambda_1^{\varsigma+1}\Big)\end{aligned}\tag{2.30}$$

and

$$\begin{aligned}&\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\gamma_1(\lambda_1)-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^-,\sigma}^{\mu}\gamma_1\big)(\lambda_1)\\&\quad\leq\frac{M}{\varsigma+1}\Big((\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^-,\sigma}^{\mu}\lambda_1\big)-\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\lambda_1^{\varsigma+1}\Big).\end{aligned}\tag{2.31}$$

Thus, based on inequalities (2.30) and (2.31), we can derive the following modulus inequality:

$$\begin{aligned}&\Big|\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\gamma_1(\lambda_1)-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^-,\sigma}^{\mu}\gamma_1\big)(\lambda_1)\Big|\\&\quad\leq\frac{M}{\varsigma+1}\Big((\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^-,\sigma}^{\mu}\lambda_1\big)-\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\lambda_1^{\varsigma+1}\Big).\end{aligned}\tag{2.32}$$

Similarly, if $\theta\in[\lambda_1,\lambda_2]$, $\tau\in[\theta,\lambda_2]$, $\nu\geq 0$, and $k>0$, then we have the following inequality:

$$\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\tau)\big)^{\frac{\nu}{k}}\leq\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}.\tag{2.33}$$

Let $\tau^{\varsigma}\geq 1$. Then from (2.33) and the boundedness condition on γ_1', as a simple consequence, we get the following inequalities:

$$\begin{aligned}&\int_{\theta}^{\lambda_2}\big(M\tau^{\varsigma}-\gamma_1'(\tau)\big)\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\tau)\big)^{\frac{\nu}{k}}\,d\tau\\&\quad\leq\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\int_{\theta}^{\lambda_2}\big(M\tau^{\varsigma}-\gamma_1'(\tau)\big)\,d\tau\end{aligned}\tag{2.34}$$

and

$$\int_{\theta}^{\lambda_2}\big(M\tau^{\varsigma}+\gamma_1'(\tau)\big)\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\tau)\big)^{\frac{\nu}{k}}\,d\tau \leq \big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\int_{\theta}^{\lambda_2}\big(M\tau^{\varsigma}+\gamma_1'(\tau)\big)\,d\tau. \tag{2.35}$$

Integrating (2.34) and (2.35), performing some straightforward calculations, and utilizing Definition 2.1, we obtain the following inequalities:

$$\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\gamma_1(\lambda_2)-(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^+,\sigma}^{\mu}\gamma_1\big)(\lambda_2) \leq \frac{M}{\varsigma+1}\Big(\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\lambda_2^{\varsigma+1}-(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^+,\sigma}^{\mu}\lambda_2\big)\Big) \tag{2.36}$$

and

$$(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^+,\sigma}^{\nu}\gamma_1\big)(\lambda_2)-\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\gamma_1(\lambda_2) \leq \frac{M}{\varsigma+1}\Big(\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\lambda_2^{\varsigma+1}-(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^+,\sigma}^{\nu}\lambda_2\big)\Big). \tag{2.37}$$

From (2.36) and (2.37) we have following modulus inequality:

$$\Big|\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\gamma_1(\lambda_2)-(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^+,\sigma}^{\nu}\gamma_1\big)(\lambda_2)\Big| \leq \frac{M}{\varsigma+1}\Big(\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\lambda_2^{\varsigma+1}-(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^+,\sigma}^{\nu}\lambda_2\big)\Big). \tag{2.38}$$

Modulus inequalities (2.32) and (2.38) constitute inequality (2.26). □

Corollary 2.18 *For $\mu=\nu$ in (2.26), we have the following weighted (k,ς)-RL fractional integral inequality*:

$$\begin{aligned}&\Big|\gamma_1(\lambda_2)\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\mu}{k}}+\gamma_1(\lambda_1)\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\\&\quad-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^+,\sigma}^{\mu}\gamma_1\big)(\lambda_2)+\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^-,\sigma}^{\mu}\gamma_1\big)(\lambda_1)\Big|\\&\leq\frac{M}{\varsigma+1}\Big(\lambda_2^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\mu}{k}}-\lambda_1^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\\&\quad+(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\Big(\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^-,\sigma}^{\mu}\lambda_1\big)-\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^+,\sigma}^{\mu}\lambda_2\big)\Big)\Big).\end{aligned}$$

Corollary 2.19 *By setting $\gamma_2(\theta) = \theta$ and $\sigma(\theta) = 1$ in* (2.26) *we obtain the following inequality for (k, ς)-RL fractional integrals*:

$$\begin{aligned}&\Big|\gamma_1(\lambda_2)\big(\lambda_2^{\varsigma+1} - \theta^{\varsigma+1}\big)^{\frac{\nu}{k}} + \gamma_1(\lambda_1)\big(\theta^{\varsigma+1} - \lambda_1^{\varsigma+1}\big)^{\frac{\mu}{k}}\\ &\quad - \big((\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\big({}_k^{\varsigma}\Im_{\theta^+}^{\nu}\gamma_1\big)(\lambda_2) + (\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big({}_k^{\varsigma}\Im_{\theta^-}^{\mu}\gamma_1\big)(\lambda_1)\big)\Big|\\ &\quad\le \frac{M}{\varsigma+1}\bigg(\frac{\nu}{\nu+k}\big(\lambda_2^{\varsigma+1} - \theta^{\varsigma+1}\big)^{\frac{\nu}{k}+1} + \frac{\mu}{\mu+k}\big(\theta^{\varsigma+1} - \lambda_1^{\varsigma+1}\big)^{\frac{\mu}{k}+1}\bigg).\end{aligned}$$

Corollary 2.20 *By putting $\gamma_2(\theta) = \theta$, $\sigma(\theta) = 1$, and $k = 1$ in* (2.26) *we obtain the following inequality for RL fractional integrals*:

$$\begin{aligned}&\Big|\gamma_1(\lambda_2)\big(\lambda_2^{\varsigma+1} - \theta^{\varsigma+1}\big)^{\nu} + \gamma_1(\lambda_1)\big(\theta^{\varsigma+1} - \lambda_1^{\varsigma+1}\big)^{\mu}\\ &\quad - \big((\varsigma+1)^{\nu}\Gamma(\nu+1)\big({}^{\varsigma}\Im_{\theta^+}^{\nu}\gamma_1\big)(\lambda_2) + (\varsigma+1)^{\mu}\Gamma(\mu+1)\big({}^{\varsigma}\Im_{\theta^-}^{\mu}\gamma_1\big)(\lambda_1)\big)\Big|\\ &\quad\le \frac{M}{\varsigma+1}\bigg(\frac{\mu}{\mu+1}\big(\theta^{\varsigma+1} - \lambda_1^{\varsigma+1}\big)^{\mu+1} + \frac{\nu}{\nu+1}\big(\lambda_2^{\varsigma+1} - \theta^{\varsigma+1}\big)^{\nu+1}\bigg).\end{aligned}$$

Corollary 2.21 *By substituting $\mu = \nu = k = 1$, $\sigma(\theta) = 1$, and $\gamma_2(\theta) = \theta$ into* (2.26) *we obtain the following inequality for RL fractional integrals*:

$$\begin{aligned}&\left|\gamma_1(\lambda_2)\big(\lambda_2^{\varsigma+1} - \theta^{\varsigma+1}\big) + \gamma_1(\lambda_1)\big(\theta^{\varsigma+1} - \lambda_1^{\varsigma+1}\big) - (\varsigma+1)\int_{\lambda_1}^{\lambda_2}\tau^{\varsigma}\gamma_1(\tau)\,d\tau\right|\\ &\quad\le \frac{M}{\varsigma+1}\left(\left(\theta^{\varsigma+1} - \frac{a^{\varsigma+1} + \lambda_2^{\varsigma+1}}{2}\right)^2 + \frac{(\lambda_2^{\varsigma+1} - \lambda_1^{\varsigma+1})^2}{4}\right).\end{aligned}$$

Remark 2.22 By setting $\gamma_2(\theta) = \theta$, $\sigma(\theta) = 1$, $\varsigma = 0$, and $k = 1$ in (2.26) we obtain the inequality for RL fractional integrals given in [36, Theorem 1.4]:

$$\begin{aligned}&\big|\gamma_1(\lambda_2)(\lambda_2 - \theta)^{\nu} + \gamma_1(\lambda_1)(\theta - \lambda_1)^{\mu}\\ &\quad - \big(\Gamma(\nu+1)\big(\Im_{\theta^+}^{\nu}\gamma_1\big)(\lambda_2) + \Gamma(\mu+1)\big(\Im_{\theta^-}^{\mu}\gamma_1\big)(\lambda_1)\big)\big|\\ &\quad\le M\bigg(\frac{\mu}{\mu+1}(\theta - \lambda_1)^{\mu+1} + \frac{\nu}{\nu+1}(\lambda_2 - \theta)^{\nu+1}\bigg).\end{aligned}$$

Theorem 2.23 *Let $\gamma_1 : J \to \mathbb{R}$ be a differentiable function in J^0, $\lambda_1, \lambda_2 \in J^0$, and $\lambda_1 < \lambda_2$. Also, let $\gamma_2 : [\lambda_1, \lambda_2] \to R$ be a differentiable increasing function with $\gamma_2' \in L[\lambda_1, \lambda_2]$ and $m \le \gamma_1'(\tau) \le M$ for all $\tau \in [\lambda_1, \lambda_2]$, and let σ be a nonzero increasing weight function. Then for $\mu, \nu \ge 0$ and $k > 0$, we have the following inequalities for weighted RL k-fractional integrals*:

$$\begin{aligned}&(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^-,\sigma}^{\mu}\gamma_1\big)(\lambda_1) + (\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^+,\sigma}^{\nu}\gamma_1\big)(\lambda_2)\\ &\quad - \big(\big(\gamma_2^{\varsigma+1}(\theta) - \gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\gamma_1(\lambda_1) + \big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\gamma_1(\lambda_2)\big)\\ &\quad\le \frac{M}{\varsigma+1}\big((\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im_{\theta^-,\sigma}^{\mu}\lambda_1\big) - \lambda_1^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\theta) - \gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\big)\end{aligned}$$

$$
\begin{aligned}
&-\frac{m}{\varsigma+1}\Big(\lambda_2^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\\
&-(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2{}_{,k}^{\varsigma}\Im^{\nu}_{\theta^+,\sigma}\lambda_2\big)\Big)
\end{aligned}
\tag{2.39}
$$

and

$$
\begin{aligned}
&\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\gamma_1(\lambda_1)+\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}\gamma_1(\lambda_2)\\
&\quad-\Big((\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2{}_{,k}^{\varsigma}\Im^{\mu}_{\theta^-,\sigma}\gamma_1\big)(\lambda_1)\\
&\quad+(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2{}_{,k}^{\varsigma}\Im^{\nu}_{\theta^+,\sigma}\gamma_1\big)(\lambda_2)\Big)\\
&\quad\le\frac{M}{\varsigma+1}\Big(\lambda_2^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}-(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2{}_{,k}^{\varsigma}\Im^{\nu}_{\theta^+,\sigma}\lambda_2\big)\Big)\\
&\quad-\frac{m}{\varsigma+1}\Big((\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2{}_{,k}^{\varsigma}\Im^{\mu}_{\theta^-,\sigma}\lambda_1\big)-\lambda_1^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\Big).
\end{aligned}
\tag{2.40}
$$

Proof Since γ_2 is strictly increasing function, for $\theta\in[\lambda_1,\lambda_2]$, $\tau\in[\lambda_1,\theta]$, $\mu\ge0$, and $k>0$, we have the following inequality:

$$
\big(\gamma_2^{\varsigma+1}(\tau)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\le\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}.
\tag{2.41}
$$

If $\tau^{\varsigma}\ge1$, then the following inequalities are a simple consequence of (2.41) and the boundedness condition on γ_1':

$$
\begin{aligned}
&\int_{\lambda_1}^{\theta}\big(M\tau^{\varsigma}-\gamma_1'(\tau)\big)\big(\gamma_2^{\varsigma+1}(\tau)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\,d\tau\\
&\quad\le\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\int_{\lambda_1}^{\theta}\big(M\tau^{\varsigma}-\gamma_1'(\tau)\big)\,d\tau
\end{aligned}
\tag{2.42}
$$

and

$$
\begin{aligned}
&\int_{\lambda_1}^{\theta}\big(\gamma_1'(\tau)-m\tau^{\varsigma}\big)\big(\gamma_2^{\varsigma+1}(\tau)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\,d\tau\\
&\quad\le\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\int_{\lambda_1}^{\theta}\big(\gamma_1'(\tau)-m\tau^{\varsigma}\big)\,d\tau.
\end{aligned}
\tag{2.43}
$$

Integrating (2.42) and (2.43), performing some simple calculations, and applying Definition 2.1, we obtain

$$
\begin{aligned}
&(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2{}_{,k}^{\varsigma}\Im^{\mu}_{\theta^-,\sigma}\gamma_1\big)(\lambda_1)-\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\gamma_1(\lambda_1)\\
&\quad\le\frac{M}{\varsigma+1}\Big((\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2{}_{,k}^{\varsigma}\Im^{\mu}_{\theta^-,\sigma}\lambda_1\big)-\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\lambda_1^{\varsigma+1}\Big)
\end{aligned}
\tag{2.44}
$$

and

$$
\begin{aligned}
&\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\gamma_1(\lambda_1)-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2{}_{,k}^{\varsigma}\Im^{\mu}_{\theta^-,\sigma}\gamma_1\big)(\lambda_1)\\
&\quad\le-\frac{m}{\varsigma+1}\Big((\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2{}_{,k}^{\varsigma}\Im^{\mu}_{\theta^-,\sigma}\lambda_1\big)-\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\lambda_1^{\varsigma+1}\Big).
\end{aligned}
\tag{2.45}
$$

Similarly, for $\theta \in [\lambda_1, \lambda_2]$, $\tau \in [\theta, \lambda_2]$, $\nu \geq 0$, and $k > 0$, we have

$$\big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\tau)\big)^{\frac{\nu}{k}} \leq \big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}}. \tag{2.46}$$

Let $\tau^\varsigma \geq 1$. Then the following inequalities are a simple consequence of (2.46) with boundedness condition on γ_1':

$$\begin{aligned} &\int_\theta^{\lambda_2} \big(M\tau^\varsigma - \gamma_1'(\tau)\big)\big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\tau)\big)^{\frac{\nu}{k}}\, d\tau \\ &\quad \leq \big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}} \int_\theta^{\lambda_2} \big(M\tau^\varsigma - \gamma_1'(\tau)\big)\, d\tau \end{aligned} \tag{2.47}$$

and

$$\begin{aligned} &\int_\theta^{\lambda_2} \big(\gamma_1'(\tau) - m\tau^\varsigma\big)\big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\tau)\big)^{\frac{\nu}{k}}\, d\tau \\ &\quad \leq \big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}} \int_\theta^{\lambda_2} \big(\gamma_1'(\tau) - m\tau^\varsigma\big)\, d\tau. \end{aligned} \tag{2.48}$$

Integrating (2.47) and (2.48), performing some simple calculations, and applying Definition 2.1, we obtain

$$\begin{aligned} &\big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}} \gamma_1(\lambda_2) - (\varsigma+1)^{\frac{\nu}{k}} \Gamma_k(\nu+k) \frac{\sigma(\theta)}{\sigma(\lambda_2)} \big(\gamma_2, {}_k^\varsigma \Im_{\theta^+,\sigma}^\nu \gamma_1\big)(\lambda_2) \\ &\quad \leq \frac{M}{\varsigma+1} \bigg(\big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}} \lambda_2^{\varsigma+1} - (\varsigma+1)^{\frac{\nu}{k}} \Gamma_k(\nu+k) \frac{\sigma(\theta)}{\sigma(\lambda_2)} \big(\gamma_2, {}_k^\varsigma \Im_{\theta^+,\sigma}^\nu \lambda_2\big) \bigg) \end{aligned} \tag{2.49}$$

and

$$\begin{aligned} &(\varsigma+1)^{\frac{\nu}{k}} \Gamma_k(\nu+k) \frac{\sigma(\theta)}{\sigma(\lambda_2)} \big(\gamma_2, {}_k^\varsigma \Im_{\theta^+,\sigma}^\nu \gamma_1\big)(\lambda_2) - \big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}} \gamma_1(\lambda_2) \\ &\quad \leq -\frac{m}{\varsigma+1} \bigg(\big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\nu}{k}} \lambda_2^{\varsigma+1} - (\varsigma+1)^{\frac{\nu}{k}} \Gamma_k(\nu+k) \frac{\sigma(\theta)}{\sigma(\lambda_2)} \big(\gamma_2, {}_k^\varsigma \Im_{\theta^+,\sigma}^\nu \lambda_2\big) \bigg). \end{aligned} \tag{2.50}$$

By adding (2.44) and (2.50) we get (2.39). Similarly, by adding (2.45) and (2.49) we get (2.40). □

Corollary 2.24 *For $\mu = \nu$ in (2.39) and (2.40), we have the following (k, ς)-fractional integral inequalities*:

$$\begin{aligned} &(\varsigma+1)^{\frac{\mu}{k}} \Gamma_k(\mu+k) \bigg(\big(\gamma_2, {}_k^\varsigma \Im_{\theta^-,\sigma}^\mu \gamma_1\big)(\lambda_1) + \frac{\sigma(\theta)}{\sigma(\lambda_2)} \big(\gamma_2, {}_k^\varsigma \Im_{\theta^+,\sigma}^\mu \gamma_1\big)(\lambda_2) \bigg) \\ &\quad - \Big(\big(\gamma_2^{\varsigma+1}(\theta) - \gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}} \gamma_1(\lambda_1) + \big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\mu}{k}} \gamma_1(\lambda_2) \Big) \\ &\leq \frac{M}{\varsigma+1} \Big((\varsigma+1)^{\frac{\mu}{k}} \Gamma_k(\mu+k) \big(\gamma_2, {}_k^\varsigma \Im_{\theta^-,\sigma}^\mu \lambda_1\big) - \lambda_1^{\varsigma+1} \big(\gamma_2^{\varsigma+1}(\theta) - \gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}} \Big) \\ &\quad - \frac{m}{\varsigma+1} \bigg(\lambda_2^{\varsigma+1} \big(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\mu}{k}} - (\varsigma+1)^{\frac{\mu}{k}} \Gamma_k(\mu+k) \frac{\sigma(\theta)}{\sigma(\lambda_2)} \big(\gamma_2, {}_k^\varsigma \Im_{\theta^+,\sigma}^\mu \lambda_2\big) \bigg) \end{aligned}$$

and

$$\begin{aligned}
&\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\gamma_1(\lambda_1)+\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\mu}{k}}\gamma_1(\lambda_2)\\
&\quad-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\bigg(\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\theta^-,\sigma}\gamma_1\big)(\lambda_1)+\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\theta^+,\sigma}\gamma_1\big)(\lambda_2)\bigg)\\
&\quad\le\frac{M}{\varsigma+1}\bigg(\lambda_2^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\frac{\mu}{k}}-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\theta^+,\sigma}\lambda_1\big)\bigg)\\
&\quad-\frac{m}{\varsigma+1}\big((\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\theta^-,\sigma}\lambda_1\big)-\lambda_1^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\big).
\end{aligned}$$

Corollary 2.25 *By setting $\gamma_2(\theta)=\theta$ and $\sigma(\theta)=1$ in (2.39) and (2.40) we obtain the following inequalities for (k,ς)-RL fractional integrals*:

$$\begin{aligned}
&(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big({}_k^{\varsigma}\Im^{\mu}_{\theta^-}\gamma_1\big)(\lambda_1)+(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\big({}_k^{\varsigma}\Im^{\nu}_{\theta^+}\gamma_1\big)(\lambda_2)\\
&\quad-\big(\big(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\big)^{\frac{\mu}{k}}\gamma_1(\lambda_1)+\big(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\big)^{\frac{\nu}{k}}\gamma_1(\lambda_2)\big)\\
&\quad\le\frac{M}{\varsigma+1}\bigg(\frac{\mu}{\mu+k}\big(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\big)^{\frac{\mu}{k}+1}\bigg)-\frac{m}{\varsigma+1}\bigg(\frac{\nu}{\nu+k}\big(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\big)^{\frac{\nu}{k}+1}\bigg)
\end{aligned}$$

and

$$\begin{aligned}
&\big(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\big)^{\frac{\mu}{k}}\gamma_1(\lambda_1)+\big(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\big)^{\frac{\nu}{k}}\gamma_1(\lambda_2)\\
&\quad-\big((\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big({}_k^{\varsigma}\Im^{\mu}_{\theta^-}\gamma_1\big)(\lambda_1)+(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\big({}_k^{\varsigma}\Im^{\nu}_{\theta^+}\gamma_1\big)(\lambda_2)\big)\\
&\quad\le\frac{M}{\varsigma+1}\bigg(\frac{\nu}{\nu+k}\big(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\big)^{\frac{\nu}{k}+1}\bigg)-\frac{m}{\varsigma+1}\bigg(\frac{\mu}{\mu+k}\big(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\big)^{\frac{\mu}{k}+1}\bigg).
\end{aligned}$$

Corollary 2.26 *By substituting $\gamma_2(\theta)=\theta$, $\sigma(\theta)=1$, and $k=1$ into (2.39) and (2.40) we obtain the following inequalities for RL fractional integrals*:

$$\begin{aligned}
&(\varsigma+1)^{\mu}\Gamma(\mu+1)\big({}^{\varsigma}\Im^{\mu}_{\theta^-}\gamma_1\big)(\lambda_1)+(\varsigma+1)^{\nu}\Gamma(\nu+1)\big({}^{\varsigma}\Im^{\nu}_{\theta^+}\gamma_1\big)(\lambda_2)\\
&\quad-\big(\big(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\big)\gamma_1(\lambda_1)+\big(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\big)\gamma_1(\lambda_2)\big)\\
&\quad\le\frac{M}{\varsigma+1}\bigg(\frac{\mu}{\mu+1}\big(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\big)^{\mu+1}\bigg)-\frac{m}{\varsigma+1}\bigg(\frac{\nu}{\nu+1}\big(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\big)^{\nu+1}\bigg)
\end{aligned}$$

and

$$\begin{aligned}
&\big(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\big)^{\mu}\gamma_1(\lambda_1)+\big(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\big)^{\nu}\gamma_1(\lambda_2)\\
&\quad-\big((\varsigma+1)^{\mu}\Gamma(\mu+1)\big({}^{\varsigma}\Im^{\mu}_{\theta^-}\gamma_1\big)(\lambda_1)+(\varsigma+1)^{\nu}\Gamma(\nu+1)\big({}^{\varsigma}\Im^{\nu}_{\theta^+}\gamma_1\big)(\lambda_2)\big)\\
&\quad\le\frac{M}{\varsigma+1}\bigg(\frac{\nu}{\nu+1}\big(\lambda_2^{\varsigma+1}-\theta^{\varsigma+1}\big)^{\nu+1}\bigg)-\frac{m}{\varsigma+1}\bigg(\frac{\mu}{\mu+1}\big(\theta^{\varsigma+1}-\lambda_1^{\varsigma+1}\big)^{\mu+1}\bigg).
\end{aligned}$$

Corollary 2.27 *By substituting $\mu = \nu = k = 1$, $\sigma(\theta) = 1$, and $\gamma_2(\theta) = \theta$ into* (2.39) *and* (2.40) *we obtain the following inequalities for RL fractional integrals*:

$$\begin{aligned}
&(\varsigma + 1)\int_{\lambda_1}^{\lambda_2} \tau^{\varsigma}\gamma_1(\tau)\,d\tau - \bigl(\bigl(\theta^{\varsigma+1} - \lambda_1^{\varsigma+1}\bigr)\gamma_1(\lambda_1) + \bigl(\lambda_2^{\varsigma+1} - \theta^{\varsigma+1}\bigr)\gamma_1(\lambda_2)\bigr) \\
&\quad\leq \frac{M}{2(\varsigma+1)}\bigl(\theta^{\varsigma+1} - \lambda_1^{\varsigma+1}\bigr)^2 - \frac{m}{2(\varsigma+1)}\bigl(\theta^{\varsigma+1} - \lambda_2^{\varsigma+1}\bigr)^2
\end{aligned}$$

and

$$\begin{aligned}
&\bigl(\theta^{\varsigma+1} - \lambda_1^{\varsigma+1}\bigr)\gamma_1(\lambda_1) + \bigl(\lambda_2^{\varsigma+1} - \theta^{\varsigma+1}\bigr)\gamma_1(\lambda_2) - (\varsigma + 1)\int_{\lambda_1}^{\lambda_2} \tau^{\varsigma}\gamma_1(\tau)\,d\tau \\
&\quad\leq \frac{M}{2(\varsigma+1)}\bigl(\theta^{\varsigma+1} - \lambda_2^{\varsigma+1}\bigr)^2 - \frac{m}{2(\varsigma+1)}\bigl(\theta^{\varsigma+1} - \lambda_1^{\varsigma+1}\bigr)^2.
\end{aligned}$$

Remark 2.28 By substituting $\gamma_2(\theta) = \theta$, $\sigma(\theta) = 1$, $\varsigma = 0$, and $k = 1$ into (2.39) and (2.40) we obtain the following inequalities for RL fractional integrals:

$$\begin{aligned}
&\Gamma(\mu + 1)\bigl(\Im^{\mu}_{\theta^-}\gamma_1\bigr)(\lambda_1) + \Gamma(\nu + 1)\bigl(\Im^{\nu}_{\theta^+}\gamma_1\bigr)(\lambda_2) \\
&\quad - \bigl((\theta - \lambda_1)\gamma_1(\lambda_1) + (\lambda_2 - \theta)\gamma_1(\lambda_2)\bigr) \\
&\quad\leq M\biggl(\frac{\mu}{\mu+1}(\theta - \lambda_1)^{\mu+1}\biggr) - m\biggl(\frac{\nu}{\nu+1}(\lambda_2 - \theta)^{\nu+1}\biggr)
\end{aligned}$$

and

$$\begin{aligned}
&(\theta - \lambda_1)^{\mu}\gamma_1(\lambda_1) + (\lambda_2 - \theta)^{\nu}\gamma_1(\lambda_2) \\
&\quad - \bigl(\Gamma(\mu + 1)\bigl(\Im^{\mu}_{\theta^-}\gamma_1\bigr)(\lambda_1) + \Gamma(\nu + 1)\bigl(\Im^{\nu}_{\theta^+}\gamma_1\bigr)(\lambda_2)\bigr) \\
&\quad\leq M\biggl(\frac{\nu}{\nu+1}(\lambda_2 - \theta)^{\nu+1}\biggr) - m\biggl(\frac{\mu}{\mu+1}(\theta - \lambda_1)^{\mu+1}\biggr).
\end{aligned}$$

Remark 2.29 If we take $m = -M$ in Theorem 2.23, then by some rearrangements we obtain Theorem 2.17.

3 General forms of weighted fractional integrals inequalities

This section is devoted to presenting general forms of the outcomes from the previous section. In this section, Theorem 2.3 takes the following specific form.

Theorem 3.1 *Under the assumptions of Theorem* 2.3, *we have*

$$\begin{aligned}
&\biggl|\gamma_1(\theta)\bigl(\bigl(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\bigr)^{\nu} + \bigl(\gamma_2^{\varsigma+1}(\theta) - \gamma_2^{\varsigma+1}(\lambda_1)\bigr)^{\mu}\bigr) \\
&\quad - \biggl((\varsigma + 1)^{\nu}\Gamma(\nu + 1)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\bigl(\gamma_2,{}^{\varsigma}\,\Im^{\nu}_{\lambda_2^-,\sigma}\,\gamma_1\bigr)(\theta) \\
&\quad + (\varsigma + 1)^{\mu}\Gamma(\mu + 1)\bigl(\gamma_2,{}^{\varsigma}\,\Im^{\mu}_{\lambda_1^+,\sigma}\,\gamma_1\bigr)(\theta)\biggr)\biggr| \\
&\quad\leq \frac{M}{\varsigma+1}\biggl(\theta^{\varsigma+1}\bigl(\bigl(\gamma_2^{\varsigma+1}(\theta) - \gamma_2^{\varsigma+1}(\lambda_1)\bigr)^{\mu} - \bigl(\gamma_2^{\varsigma+1}(\lambda_2) - \gamma_2^{\varsigma+1}(\theta)\bigr)^{\nu}\bigr)
\end{aligned}$$

$$+ (\varsigma+1)^{\nu}\Gamma(\nu+1)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\left(\gamma_2,^{\varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}\,\mathrm{id}_{[\theta,\lambda_2]}\right)(\theta)$$
$$- (\varsigma+1)^{\mu}\Gamma(\mu+1)\left(\gamma_2,^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\,\mathrm{id}_{[\lambda_1,\theta]}\right)(\theta)\bigg).$$

The notation id[·, ·] *represents the identity function on the interval* [·, ·].

Proof The result is proved in the same way as Theorem 2.3. □

In particular, Theorem 2.10 is transformed into the following form.

Theorem 3.2 *Assuming that the conditions of Theorem* 2.10 *are satisfied, we have*

$$\begin{aligned}
&\left(\left(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\right)^{\mu}+\left(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\right)^{\nu}\right)\gamma_1(\theta)\\
&\quad-\Bigg((\varsigma+1)^{\mu}\Gamma(\mu+1)\left(\gamma_2,^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\gamma_1\right)(\theta)\\
&\quad+(\varsigma+1)^{\nu}\Gamma(\nu+1)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\left(\gamma_2,^{\varsigma}_{k}\Im^{\nu}_{\lambda_2^-,\sigma}\gamma_1\right)(\theta)\Bigg)\\
&\quad\le\frac{M}{\varsigma+1}\left(\theta^{\varsigma+1}\left(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\right)^{\mu}-(\varsigma+1)^{\mu}\Gamma(\mu+1)\left(\gamma_2,^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\,\mathrm{id}_{[\lambda_1,\theta]}\right)(\theta)\right)\\
&\quad-\frac{m}{\varsigma+1}\Bigg((\varsigma+1)^{\nu}\Gamma(\nu+1)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\left(\gamma_2,^{\varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}\,\mathrm{id}_{[\theta,\lambda_2]}\right)(\theta)\\
&\quad-\theta^{\varsigma+1}\left(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\right)^{\nu}\Bigg)
\end{aligned}$$

and

$$\begin{aligned}
&(\varsigma+1)^{\mu}\Gamma(\mu+1)\left(\gamma_2,^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\gamma_1\right)(\theta)+(\varsigma+1)^{\nu}\Gamma(\nu+1)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\left(\gamma_2,^{\varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}\gamma_1\right)(\theta)\\
&\quad-\left(\left(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\right)^{\mu}+\left(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\right)^{\nu}\gamma_1(\theta)\right)\\
&\quad\le\frac{M}{\varsigma+1}\Bigg((\varsigma+1)^{\nu}\Gamma(\nu+1)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\left(\gamma_2,^{\varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}\,\mathrm{id}_{[\theta,\lambda_2]}\right)(\theta)\\
&\quad-\theta^{\varsigma+1}\left(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\right)^{\nu}\Bigg)\\
&\quad-\frac{m}{\varsigma+1}\left(\theta^{\varsigma+1}\left(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\right)^{\mu}-(\varsigma+1)^{\mu}\Gamma(\mu+1)\left(\gamma_2,^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\,\mathrm{id}_{[\lambda_1,\theta]}\right)(\theta)\right).
\end{aligned}$$

The notation id[·, ·] *represents the identity function on the interval* [·, ·].

Proof The result is proved in the same way as Theorem 2.10. □

In particular, Theorem 2.17 takes the following form.

Theorem 3.3 *Under the assumptions of Theorem* 2.17, *we have*

$$\begin{aligned}
&\Bigg|\gamma_1(\lambda_2)\left(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\right)^{\nu}+\gamma_1(\lambda_1)\left(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\right)^{\mu}\\
&\quad-\Bigg((\varsigma+1)^{\nu}\Gamma(\nu+1)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\left(\gamma_2,^{\varsigma}\Im^{\nu}_{\theta^+,\sigma}\gamma_1\right)(\lambda_2)
\end{aligned}$$

$$
\begin{aligned}
&+ (\varsigma+1)^{\mu}\Gamma(\mu+1)\big(\gamma_2,^{\varsigma}\Im^{\mu}_{\theta^-,\sigma}\gamma_1\big)(\lambda_1)\Big)\Big| \\
&\le \frac{M}{\varsigma+1}\Big(\lambda_2^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\nu}-\lambda_1^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\mu} \\
&+ (\varsigma+1)^{\mu}\Gamma(\mu+1)\big(\gamma_2,^{\varsigma}\Im^{\mu}_{\theta^-,\sigma}\lambda_1\big)-(\varsigma+1)^{\nu}\Gamma(\nu+1)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,^{\varsigma}\Im^{\nu}_{\theta^+,\sigma}\lambda_2\big)\Big).
\end{aligned}
$$

Proof The result is proved in the same way as Theorem 2.17. □

In particular, Theorem 2.23 takes the following form.

Theorem 3.4 *Under the assumptions of Theorem 2.23, we have*

$$
\begin{aligned}
&(\varsigma+1)^{\mu}\Gamma(\mu+1)\big(\gamma_2,^{\varsigma}\Im^{\mu}_{\theta^-,\sigma}\gamma_1\big)(\lambda_1)+(\varsigma+1)^{\nu}\Gamma(\nu+1)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,^{\varsigma}\Im^{\nu}_{\theta^+,\sigma}\gamma_1\big)(\lambda_2) \\
&- \Big(\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\mu}\gamma_1(\lambda_1)+\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\nu}\gamma_1(\lambda_2)\Big) \\
&\le \frac{M}{\varsigma+1}\Big((\varsigma+1)^{\mu}\Gamma(\mu+1)\big(\gamma_2,^{\varsigma}\Im^{\mu}_{\theta^-,\sigma}\lambda_1\big)-\lambda_1^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\mu}\Big) \\
&- \frac{m}{\varsigma+1}\Big(\lambda_2^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\nu}-(\varsigma+1)^{\nu}\Gamma(\nu+1)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,^{\varsigma}\Im^{\nu}_{\theta^+,\sigma}\lambda_2\big)\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\mu}\gamma_1(\lambda_1)+\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\nu}\gamma_1(\lambda_2) \\
&- \Big((\varsigma+1)^{\mu}\Gamma(\mu+1)\big(\gamma_2,^{\varsigma}\Im^{\mu}_{\theta^-,\sigma}\gamma_1\big)(\lambda_1) \\
&+ (\varsigma+1)^{\nu}\Gamma(\nu+1)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,^{\varsigma}_{k}\Im^{\nu}_{\theta^+,\sigma}\gamma_1\big)(\lambda_2)\Big) \\
&\le \frac{M}{\varsigma+1}\Big(\lambda_2^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\theta)\big)^{\nu}-(\varsigma+1)^{\nu}\Gamma(\nu+1)\frac{\sigma(\theta)}{\sigma(\lambda_2)}\big(\gamma_2,^{\varsigma}\Im^{\nu}_{\theta^+,\sigma}\lambda_2\big)\Big) \\
&- \frac{m}{\varsigma+1}\Big((\varsigma+1)^{\mu}\Gamma(\mu+1)\big(\gamma_2,^{\varsigma}\Im^{\mu}_{\theta^-,\sigma}\lambda_1\big)-\lambda_1^{\varsigma+1}\big(\gamma_2^{\varsigma+1}(\theta)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\mu}\Big).
\end{aligned}
$$

Proof The result is proved in the same way as Theorem 2.23. □

The results presented in this section also provide the fractional integral inequalities for weighted RL fractional integrals, specifically, when the function γ_2 behaves like the identity function.

4 Applications to main results

This section pertains to the applications of our main results. The first application is presented for Theorem 2.3.

Theorem 4.1 *Under the conditions of Theorem 2.3, we have*

$$\begin{aligned}
&\Big|\gamma_1(\lambda_1)\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\nu}{k}}+\gamma_1(\lambda_2)\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}} \\
&\quad-\bigg((\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\lambda_1)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}\gamma_1\big)(\lambda_1) \\
&\quad+(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\gamma_1\big)(\lambda_2)\bigg)\Big| \\
&\leq\frac{M}{\varsigma+1}\bigg(\big(\lambda_2^{\varsigma+1}-\lambda_1^{\varsigma+1}\big)\big(\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\nu}{k}}+\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}}\big) \\
&\quad+(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\lambda_1)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im^{\nu}_{\lambda_2^-,\sigma}\lambda_1\big) \\
&\quad-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\lambda_2\big)\bigg).
\end{aligned}\tag{4.1}$$

Proof Substituting first $\theta=\lambda_1$ and secondly $\theta=\lambda_2$ into (2.1) and then adding the obtained results, we get (4.1). □

Corollary 4.2 *Under the conditions of Theorem 2.3, for $\mu=\nu$ in (4.1), we obtain*

$$\begin{aligned}
&\Big|\big(\gamma_1(\lambda_1)+\gamma_1(\lambda_2)\big)\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}} \\
&\quad-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\bigg(\frac{\sigma(\lambda_1)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_2^-,\sigma}\gamma_1\big)(\lambda_1)+\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\gamma_1\big)(\lambda_2)\bigg)\Big| \\
&\leq\frac{M}{\varsigma+1}\bigg(2\big(\lambda_2^{\varsigma+1}-\lambda_1^{\varsigma+1}\big)\big(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}(\lambda_1)\big)^{\frac{\mu}{k}} \\
&\quad+(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\bigg(\frac{\sigma(\lambda_1)}{\sigma(\lambda_2)}\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_2^-,\sigma}\lambda_1\big)-\big(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+,\sigma}\lambda_2\big)\bigg)\bigg).
\end{aligned}\tag{4.2}$$

Corollary 4.3 *Under the conditions of Theorem 2.3, for $\gamma_2(\theta)=\theta$ and $\sigma(\theta)=1$ in (4.2), we obtain*

$$\begin{aligned}
&\bigg|\frac{(\gamma_1(\lambda_1)+\gamma_1(\lambda_2))}{2}-\frac{(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)}{2(\lambda_2^{\varsigma+1}-\lambda_1^{\varsigma+1})^{\frac{\mu}{k}}}\big(\big({}_k^{\varsigma}\Im^{\mu}_{\lambda_2^-}\gamma_1\big)(\lambda_1)+\big({}_k^{\varsigma}\Im^{\mu}_{\lambda_1^+}\gamma_1\big)(\lambda_2)\big)\bigg| \\
&\leq\frac{M(3\mu+k)}{2(\varsigma+1)(\mu+k)}\big(\lambda_2^{\varsigma+1}-\lambda_1^{\varsigma+1}\big).
\end{aligned}$$

Next, we derive an approximation for the Hadamard inequality for RL k-fractional integrals given in [37, Theorem 2.1].

Corollary 4.4 *Under the conditions of Theorem 2.3, for $\gamma_2(\theta)=\theta$, $\varsigma=0$, and $\sigma(\theta)=1$ in (4.2), we obtain*

$$\begin{aligned}
&\bigg|\frac{\gamma_1(\lambda_1)+\gamma_1(\lambda_2)}{2}-\frac{\Gamma_k(\mu+k)}{2(\lambda_2-\lambda_1)^{\frac{\mu}{k}}}\big[\big({}_k\Im^{\mu}_{\lambda_2^-}\gamma_1\big)(\lambda_1)+\big({}_k\Im^{\mu}_{\lambda_1^+}\gamma_1\big)(\lambda_2)\big]\bigg| \\
&\leq\frac{M(3\mu+k)}{2(\mu+k)}(\lambda_2-\lambda_1).
\end{aligned}$$

Corollary 4.5 *Under the conditions of Theorem 2.3, for $\gamma_2(\theta)=\theta$, $\sigma(\theta)=1$, and $k=1$ in (4.2), we obtain*

$$\begin{aligned}&\left|\frac{(\gamma_1(\lambda_1)+\gamma_1(\lambda_2))}{2}-\frac{(\varsigma+1)^{\mu}\Gamma(\mu+1)}{2(\lambda_2^{\varsigma+1}-\lambda_1^{\varsigma+1})^{\mu}}\left(\left({}^{\varsigma}\Im^{\mu}_{\lambda_2^-}\gamma_1\right)(\lambda_1)+\left({}^{\varsigma}\Im^{\mu}_{\lambda_1^+}\gamma_1\right)(\lambda_2)\right)\right|\\&\quad\le\frac{M(3\mu+1)}{2(\varsigma+1)(\mu+1)}\left(\lambda_2^{\varsigma+1}-\lambda_1^{\varsigma+1}\right).\end{aligned}$$

Subsequently, we derive an approximation for the Hadamard inequality for RL fractional integrals [38, Theorem 2].

Corollary 4.6 *Under the conditions of Theorem 2.3, for $\gamma_2(\theta)=\theta$, $\sigma(\theta)=1$, $\varsigma=0$, and $k=1$ in (4.2), we have*

$$\begin{aligned}&\left|\frac{\gamma_1(\lambda_1)+\gamma_1(\lambda_2)}{2}-\frac{\Gamma(\mu+1)}{2(\lambda_2-\lambda_1)^{\mu}}\left[\left(\Im^{\mu}_{\lambda_2^-}\gamma_1\right)(\lambda_1)+\left(\Im^{\mu}_{\lambda_1^+}\gamma_1\right)(\lambda_2)\right]\right|\\&\quad\le\frac{M(3\mu+1)}{2(\mu+k)}(\lambda_2-\lambda_1).\end{aligned}$$

Theorem 4.7 *Under the conditions of Theorem 2.17, we have*

$$\begin{aligned}&\Bigg|\gamma_1(\lambda_2)\left(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}\left(\frac{\lambda_1+\lambda_2}{2}\right)\right)^{\frac{\nu}{k}}+\gamma_1(\lambda_1)\left(\gamma_2^{\varsigma+1}\left(\frac{\lambda_1+\lambda_2}{2}\right)-\gamma_2^{\varsigma+1}(\lambda_1)\right)^{\frac{\mu}{k}}\\&\quad-\left((\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\frac{\lambda_1+\lambda_2}{2})}{\sigma(\lambda_2)}\left(\gamma_2,{}^{\varsigma}_{k}\Im^{\nu}_{\frac{\lambda_1+\lambda_2}{2}^+,\sigma}\gamma_1\right)(\lambda_2)\right.\\&\quad\left.+(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\left(\gamma_2,{}^{\varsigma}_{k}\Im^{\mu}_{\frac{\lambda_1+\lambda_2}{2}^-,\sigma}\gamma_1\right)(\lambda_1)\right)\Bigg|\\&\quad\le\frac{M}{\varsigma+1}\left(\lambda_2^{\varsigma+1}\left(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}\left(\frac{\lambda_1+\lambda_2}{2}\right)\right)^{\frac{\nu}{k}}\right.\\&\quad-\lambda_1^{\varsigma+1}\left(\gamma_2^{\varsigma+1}\left(\frac{\lambda_1+\lambda_2}{2}\right)-\gamma_2^{\varsigma+1}(\lambda_1)\right)^{\frac{\mu}{k}}+(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\left(\gamma_2,{}^{\varsigma}_{k}\Im^{\mu}_{\frac{\lambda_1+\lambda_2}{2}^-,\sigma}\lambda_1\right)\\&\quad\left.-(\varsigma+1)^{\frac{\nu}{k}}\Gamma_k(\nu+k)\frac{\sigma(\frac{\lambda_1+\lambda_2}{2})}{\sigma(\lambda_2)}\left(\gamma_2,{}^{\varsigma}_{k}\Im^{\nu}_{\frac{\lambda_1+\lambda_2}{2}^+,\sigma}\lambda_2\right)\right).\end{aligned}\tag{4.3}$$

Proof Inequality (4.3) can be obtained by substituting $\theta=\frac{\lambda_1+\lambda_2}{2}$ into (2.26). □

Corollary 4.8 *Under the assumptions of Theorem 2.17 with $\mu=\nu$ in (4.3), we have*

$$\begin{aligned}&\Bigg|\gamma_1(\lambda_2)\left(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}\left(\frac{\lambda_1+\lambda_2}{2}\right)\right)^{\frac{\mu}{k}}+\gamma_1(\lambda_1)\left(\gamma_2^{\varsigma+1}\left(\frac{\lambda_1+\lambda_2}{2}\right)-\gamma_2^{\varsigma+1}(\lambda_1)\right)^{\frac{\mu}{k}}\\&\quad-(\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\left(\frac{\sigma(\frac{\lambda_1+\lambda_2}{2})}{\sigma(\lambda_2)}\left(\gamma_2,{}^{\varsigma}_{k}\Im^{\nu}_{\frac{\lambda_1+\lambda_2}{2}^+,\sigma}\gamma_1\right)(\lambda_2)+\left(\gamma_2,{}^{\varsigma}_{k}\Im^{\mu}_{\frac{\lambda_1+\lambda_2}{2}^-,\sigma}\gamma_1\right)(\lambda_1)\right)\Bigg|\\&\quad\le\frac{M}{\varsigma+1}\left(\lambda_2^{\varsigma+1}\left(\gamma_2^{\varsigma+1}(\lambda_2)-\gamma_2^{\varsigma+1}\left(\frac{\lambda_1+\lambda_2}{2}\right)\right)^{\frac{\mu}{k}}\right.\end{aligned}$$

$$- \lambda_1^{\varsigma+1}\left(\gamma_2^{\varsigma+1}\left(\frac{\lambda_1+\lambda_2}{2}\right) - \gamma_2^{\varsigma+1}(\lambda_1)\right)^{\mu}$$
$$+ (\varsigma+1)^{\frac{\mu}{k}}\Gamma_k(\mu+k)\left(\left(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\frac{\lambda_1+\lambda_2}{2}^-,\sigma}\lambda_1\right) - \frac{\sigma(\frac{\lambda_1+\lambda_2}{2})}{\sigma(\lambda_2)}\left(\gamma_2,{}_k^{\varsigma}\Im^{\mu}_{\frac{\lambda_1+\lambda_2}{2}^+,\sigma}\lambda_2\right)\right)\right). \quad (4.4)$$

Next, we derive an estimation of the Hadamard inequality for RL k-fractional integrals given in [39, Theorem 2.1].

Corollary 4.9 *Under the assumptions of Theorem 2.17, for $\gamma_2(\theta) = \theta$, $\sigma(\theta) = 1$, and $\varsigma = 0$ in (4.4), we have the following inequality for RL k-fractional integrals*:

$$\left|\frac{\gamma_1(\lambda_1)+\gamma_1(\lambda_2)}{2} - \frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(\lambda_2-\lambda_1)^{\frac{\mu}{k}}}\left(\left({}_k\Im^{\nu}_{\frac{\lambda_1+\lambda_2}{2}^+}\gamma_1\right)(\lambda_2) + \left({}_k\Im^{\mu}_{\frac{\lambda_1+\lambda_2}{2}^-}\gamma_1\right)(\lambda_1)\right)\right|$$
$$\leq \frac{M\mu(\lambda_1+\lambda_2)}{2(\mu+k)}.$$

The following result provides an estimate for the Hadamard inequality for RL fractional integrals [40, Theorem 4].

Corollary 4.10 *Under the conditions of Theorem 2.17, for $\gamma_2(\theta) = \theta$, $\sigma(\theta) = 1$, $\varsigma = 0$, and $k = 1$ in (4.4), we have*

$$\left|\frac{\gamma_1(\lambda_1)+\gamma_1(\lambda_2)}{2} - \frac{2^{\mu-1}\Gamma(\mu+1)}{(\lambda_2-\lambda_1)^{\mu}}\left(\left(\Im^{\nu}_{\frac{\lambda_1+\lambda_2}{2}^+}\gamma_1\right)(\lambda_2) + \left(\Im^{\mu}_{\frac{\lambda_1+\lambda_2}{2}^-}\gamma_1\right)(\lambda_1)\right)\right|$$
$$\leq \frac{M\mu(\lambda_1+\lambda_2)}{2(\mu+1)}.$$

5 Conclusions

Inequalities are a crucial concept in mathematics that is used extensively in various fields of study. It allows us to compare and contrast the relative values of different mathematical expressions, leading to a deeper understanding of the relationships between them. Inequalities are not only essential for theoretical purposes, but also for practical applications, such as optimization problems and statistical data analysis. Understanding inequalities is a critical component of mathematical literacy, enabling individuals to evaluate and interpret quantitative information and make informed decisions in various aspects of their lives. The presented work includes generalized fractional integral inequalities for the family of generalized weighted RL fractional integrals. These operators have been extensively studied and utilized by researchers across different fields. We specifically investigate the weighted RL k-fractional integral operators and extend the established in the direction of weighted version. Our study provides a simple method for proving Östrowski-type inequalities using the weighted fractional integral operators. We explored a more comprehensive form of the Östrowski-type inequalities that is more inclusive than the current ones in the literature. These inequalities have various applications in numerical analysis, specifically in numerical integration. Furthermore, we determine the best possible error bounds for Hadamard-type inequalities. Our results, which are related to the existing literature, are obtained through the application of Theorems 2.3, 2.10, 2.17, and 2.23. These findings have significant implications in establishing error bounds for Hadamard inequalities in fractional calculus.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
All authors take equal part in the preparation of this manuscipt. All author read and approved it for submission in Journal of Ineaualities ans Applications.

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