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Classification of stable solutions for non-homogeneous higher-order elliptic PDEs

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Abstract

Under some assumptions on the nonlinearity f, we will study the nonexistence of nontrivial stable solutions or solutions which are stable outside a compact set of $\mathbb{R}^n$ for the following semilinear higher-order problem:

$$(-\Delta)^k u = f(u) \quad \text{in } \mathbb{R}^n,$$

with $k = 1, 2, 3, 4$. The main methods used are the integral estimates and the Pohozaev identity. Many classes of nonlinearity will be considered; even the sign-changing nonlinearity, which has an adequate subcritical growth at zero as for example $f(u) = -mu + \lambda|u|^{\theta-1}u - \mu|u|^{p-1}u$, where $m \geq 0$, $\lambda > 0$, $\mu > 0$, $p, \theta > 1$. More precisely, we shall revise the nonexistence theorem of Berestycki and Lions (Arch. Ration. Mech. Anal. 82:313-345, 1983) in the class of smooth finite Morse index solutions as the well known work of Bahri and Lions (Commun. Pure Appl. Math. 45:1205-1215, 1992). Also, the case when $f(u)u$ is a nonnegative function will be studied under a large subcritical growth assumption at zero, for example $f(u) = |u|^{\theta-1}u(1 + |u|^q)$ or $f(u) = |u|^{\theta-1}ue^{|u|^q}$, $\theta > 1$ and $q > 0$. Extensions to solutions which are merely stable are discussed in the case of supercritical growth with $k = 1$.

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Keywords: higher-order equation; Liouville theorems; finite Morse index; Pohozaev identity

1 Introduction

This paper is devoted to the study of solutions, possibly unbounded and sign-changing, of the semilinear partial differential equation,

$$(-\Delta)^k u = f(u) \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $k = 1, 2, 3, 4$, $n \geq 1$ and $f \in C^1(\mathbb{R})$. Under some assumptions on the nonlinearity f, we will show that this problem does not possess a nontrivial solution with finite Morse index.

In the last decades, problems related to the nonexistence of finite Morse index solutions for second-, fourth- and sixth-order Lane-Emden equation on unbounded domains of $\mathbb{R}^n$ have received a lot of attention (see [2–12]).

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We now list some known results. We start with the second-order Lane-Emden equation

$$-\Delta u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^n, p > 1, \tag{1.2}$$

Farina [6] proved that nontrivial finite Morse index solutions of (1.2) exist if and only if $p \geq p_c(n)$ and $n \geq 11$, or $p = \frac{n+2}{n-2}$ and $n \geq 3$, where $p_c(n)$ is the so-called Joseph-Lundgren exponent. Also, in [13] several Liouville-type theorems are presented for stable solutions, where $f > 0$ is a general convex, nondecreasing function. Extensions to solutions which are merely stable outside a compact set are discussed.

For the fourth-order Lane-Emden problem

$$\Delta^2 u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^n, p > 1, \tag{1.3}$$

the subcritical case has been studied by Ramos and Rodriguez for finite Morse index sign-changing solutions (see [14]). The supercritical case is more complicated and there are several new approaches dealing with (1.3). The first approach is to use the test function $v = -\Delta u$. To this end, one has to use Souplet's inequality [15], this will give an exponent $\frac{n}{n-8} + \epsilon_n$ for some $\epsilon_n > 0$; see [16]. These results were improved in [12] by adapting Farina's approach with the restriction on the power $q < \frac{2}{3}$. The second approach was obtained by Cowan and Ghoussoub [3], Dupaigne *et al.* [17] and further exploited by Hajlaoui, Ye and one of the authors [7]. This approach improves the first upper bound $\frac{n}{n-8} + \epsilon_n$, but it again fails to catch the fourth-order Joseph-Lundgren exponent computed by Gazzola and Grunau [18]. It should be remarked that by combining these two approaches one can show that stable positive solutions to (1.3) do not exist when $n \leq 12$ and $p > 1$; see [7]. Finally in [5], Dávila *et al.* employed a monotonicity formula-based approach and gave a complete classification of stable and finite Morse index (positive or sign-changing) solutions to (1.3). A remarkable outcome of this third approach is that it gives the optimal exponent. The main tool of [5] is a monotonicity formula, used to perform a blow-down analysis and reduce the nonexistence of nontrivial entire solutions for the problem (1.3), to that of nontrivial homogeneous solutions.

Thanks to the Liouville-type theorem with finite Morse index in [8], the authors proved the nonexistence result of sign-changing solutions for the sixth-order problem

$$-\Delta^3 u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^n, p > 1. \tag{1.4}$$

Let us give a brief sketch of their method. They proved various classification theorems and Liouville-type results for C^6-solutions belonging to one of the following classes: stable solutions, solutions which are stable outside a compact set of $\mathbb{R}^n$. These results apply to the subcritical case using the Pohozaev identity. In the supercritical case, motivated by the monotonicity formula established in [19], they reduced the nonexistence of nontrivial entire solutions for the problem (1.4), to that of nontrivial homogeneous solutions. Through this approach, they gave a complete classification of stable solutions and those finite Morse indices, whether positive or sign-changing. Also, this analysis reveals the existence of a new critical exponent called the sixth-order Joseph-Lundgren exponent, also they gave the explicit value of this exponent.

In this work, we are concerned with Liouville-type theorems for the nonlinear elliptic equation (1.1) for $k = 1, 2, 3, 4$. We prove Liouville-type theorems for solutions (whether positive or sign-changing) belonging to one of the following classes: stable solutions and solutions which are stable outside a compact set. Our proof is based on a combination of the integral estimates and the Pohozaev-type identity.

The paper is organized as follows. In Section 2 we state our main results, which are then proved in Section 4. Section 3 contains some important auxiliary tools, which are used in the proofs of the main theorems.

2 Statement of the main results

In order to state our results, we present first some assumptions on the nonlinearity f:

H_1: There exists a constant $\theta > 1$ such that

$$f'(s)s^2 - \theta f(s)s \geq 0, \quad \forall s \in \mathbb{R}.$$

H_2: There exist constants $s_0 > 0$, $\theta > 1$ and $C_0 > 0$ such that

$$C_0|s|^{\theta+1} \leq f(s)s, \quad \forall |s| \leq s_0.$$

H_3: There exists a constant $0 < \alpha_0 < 1$ such that

$$\frac{2n}{n-2k}F(s) - (1+\alpha_0)f(s)s \geq 0, \quad \forall s \in \mathbb{R},$$

where $F(s) = \int_0^s f(t)\,dt$.

Remark 2.1 (1) H_1 implies H_1': There exist constants $s_0 > 0$, $\theta > 1$ and $C_0 > 0$ such that

$$C_0|s|^{\theta+1} \leq f(s)s, \quad \forall |s| \geq s_0.$$

Indeed, by H_1, we have $\frac{f}{|s|^\theta}$ is nondecreasing function for all $|s| \geq s_0$. This implies that

$$C_0|s|^{\theta+1} \leq f(s)s, \quad \forall |s| \geq s_0.$$

(2) H_1 implies the Ambrosetti-Rabinowitz condition (A-R): there exist constants $\widetilde{\theta} > 2$ and $s_0 > 0$ such that

$$f(s)s \geq \widetilde{\theta}F(s) > 0, \quad \text{for } |s| > s_0.$$

Examples We easily verify that the following functions satisfy H_1 and H_2.

1. $f(s) = C_0(1+|s|^q)|s|^{\theta-1}s$, $\theta > 1$, $q > 0$ and $C_0 > 0$.
2. $f(s) = |s|^{\theta-1}se^{|s|^q}$, $\theta > 1$ and $q > 1$.
3. $f(s) = \sum_{i=1}^{i=i_0} c_i|s|^{\theta_i-1}s$, with $\theta_i > 1$ $\forall i = 1, 2, \dots, i_0$ and $c_i > 0$ $\forall i = 1, 2, \dots, i_0$. In this example we choose $\theta = \min_{1\leq i\leq i_0}(\theta_i)$.

The examples (1) and (2) show that f can have an exponential growth at infinity. Therefore, clearly an adequate behavior of f at zero is needed to obtain the Liouville theorem. The unique and important nonexistence result for stable solutions of the nonhomogeneous second-order equation (1.1) has been recently obtained in [13]. It is shown there, among other things, that (1.1) does not admit nontrivial stable or stable outside a compact set solution provided that f is regular, positive, nondecreasing and convex function in $(0,+\infty)$. More precisely, under a mere nonnegativity assumption on the nonlinearity, the authors begin this work by stating that up to space dimension $n = 4$, bounded stable solutions of (1.1) are trivial. For the next series of results, they restricted themselves to the following class of nonlinearities:

$$f \in C^0(\mathbb{R}_+) \cap C^2(\mathbb{R}_+^*), \quad f > 0 \text{ is nondecreasing and convex in } \mathbb{R}_+^*. \tag{2.1}$$

In order to relate the nonlinearity f and the below exponents (2.3) and (2.4), they introduced a quantity q defined for $u \in \mathbb{R}_+^*$ by $q(u) = \frac{f'^2}{ff''}(u)$, whenever $ff''(u) \neq 0$ and $q(u) = +\infty$ otherwise. They assumed that $q(u)$ converges as $u \to 0^+$ and denote its limit by

$$q_0 = \lim_{u\to 0^+} q(u) \in \overline{\mathbb{R}}. \tag{2.2}$$

Define now $p_0 \in \overline{\mathbb{R}}$ the conjugate exponent of q_0, by $\frac{1}{p_0} + \frac{1}{q_0} = 1$. The exponent p_0 must be understood as a measure of the 'flatness' of f at 0. However, we establish their following theorem.

Theorem A [13] *Assume that f satisfies* (2.1) *and* (2.2). *Assume that $u \in C^2(\mathbb{R}^n)$ is stable solution of* (1.1) *with $k = 1$. Then $u \equiv 0$ if any one of the following conditions holds*:

1. $1 \le n \le 9$ *and* $1 < \underline{p_\infty}$,
2. $n = 10$, $p_0 < +\infty$ *and* $1 < \underline{p_\infty}$,
3. $n \ge 11$, $p_0 < p_c(n)$ *and* $1 < \underline{p_\infty} < p_c(n)$,

where $\underline{p_\infty} \in \overline{\mathbb{R}}$ be defined by $\overline{q_\infty} = \limsup_{u\to+\infty} q(u)$, $\frac{1}{\underline{p_\infty}} + \frac{1}{\overline{q_\infty}} = 1$.

A typical example of nonlinearity function f satisfying the above conditions (2.1) and (2.2) is $f(u) = |u|^{\theta-1}u + |u|^{p-1}u$, where $p \ge \theta$. A simple calculation, we get $p_0 = \theta$ and $\underline{p_\infty} = p$. We use this nonlinearity function to establish some new Liouville-type theorems. Our method is different from (and complementary to) the one used in [13]. It exploits the attractive character of the difference between $f'(u)u^2 - \theta f(u)u \ge 0$, if $p \ge \theta$, that is, f satisfies H_1 and H_2. It will be shown in Theorem 2.1 that problem (1.1) does not possess nontrivial stable solutions if and only if $1 < \theta < p_c(n)$, $\forall p \ge \theta$. Also, we may consider nonlinearities with exponential growth at infinity, *i.e.* $\underline{p_\infty} = \infty$ satisfying H_1 and H_2, as for example $f(u) = |u|^{\theta-1}ue^{|u|^q}$, $\theta > 1$ and $q > 0$; therefore, in view again of Theorem 2.1, one has $u \equiv 0$. Furthermore, the present paper is motivated by the interesting work [1], we shall revise the nonexistence theorem of Berestycki and Lions [1] if one substitutes their assumption, which is

$$\int_{\mathbb{R}^n} |\nabla u|^2 + \int_{\mathbb{R}^n} f(u)u < +\infty,$$

by assuming that u is stable or stable outside a compact set. Therefore sign-changing nonlinearities will also be considered and we do not require that $f'(0) = 0$ as the instructive example given by Berestycki and Lions [1] is $f(u) = -mu + \lambda|u|^{\theta-1}u - \mu|u|^{p-1}u$, where λ, μ are positive constants, $m \ge 0$ and $1 < \theta, p$. Observe that the above nonlinearity satisfies (H_1), thus we shall prove that equation (1.1) does not possess a nontrivial stable solution provided $1 < p \le \frac{n+2k}{n-2k}$ and $p < \theta$, also if u is bounded solution to (1.1) and $m > 0$, then $u \equiv 0$, for any $\theta \ge p$. If $p \le \frac{n+2k}{n-2k} \le \theta$ and $m > 0$, it follows from the Pohozaev identity that there cannot exist a nontrivial solution of (1.1) which is stable outside a compact set. This result is similar to [1] for $k = 1$. To conclude, this work completes the study of Dupaigne and Farina [13] since here we do not assume that f is positive and convex function. Therefore, to be more concrete in our analysis of nonexistence, we will distinguish between stable and stable outside a compact set. We provide some elliptic decay estimates that we use frequently later in the proofs. Deriving the right decay estimates for solutions of (1.1) plays a fundamental role in most our proofs. On the other hand, we shall also consider the question of the nonexistence of stable solutions (positive or sign-changing) in the supercritical case of a second-order equation.

In order to state our results we need to recall the following.

Definition 2.1 A solution u of (1.1) belonging to $C^{2k}(\mathbb{R}^n)$

- is said to be stable if

$$Q_u(\psi) := \int_{\mathbb{R}^n} \left|D^k \psi\right|^2 dx - \int_{\mathbb{R}^n} f'(u)\psi^2 \, dx \ge 0, \quad \forall \psi \in C_c^k(\mathbb{R}^n),$$

  where

$$D^k = \begin{cases} \Delta^{\frac{k}{2}} & \text{for } k = 2, 4, \\ \nabla \Delta^{\frac{k-1}{2}} & \text{for } k = 1, 3, \end{cases}$$

- is stable outside a compact set $\mathcal{K} \subset \mathbb{R}^n$, if $Q_u(\psi) \ge 0$ for any $\psi \in C_c^k(\mathbb{R}^n \backslash \mathcal{K})$.

  More generally, the Morse index of a solution is defined as the maximal dimension of all subspaces E of $C_c^k(\mathbb{R}^n)$ such that $Q_u(\psi) < 0$ in $E \backslash \{0\}$. Clearly, a solution is stable if and only if its Morse index is equal to zero.

Remark 2.2 It is well known that any finite Morse index solution u is stable outside a compact set $\mathcal{K} \subset \mathbb{R}^n$. Indeed, there exist $K \ge 1$ and $X_K := \operatorname{Span}\{\phi_1, \ldots, \phi_K\} \subset C_c^k(\mathbb{R}^n)$ such that $Q_u(\phi) < 0$ for any $\phi \in X_K \backslash \{0\}$. Hence, $Q_u(\psi) \ge 0$ for every $\psi \in C_c^k(\mathbb{R}^n \backslash \mathcal{K})$, where $\mathcal{K} := \bigcup_{j=1}^K \operatorname{supp}(\phi_j)$.

To state the following result we need to introduce some notation. Let two critical exponents play an important role, namely the classical Sobolev exponent

$$p_s(n,k) = \begin{cases} +\infty & \text{if } n \le 2k, \\ \frac{n+2k}{n-2k} & \text{if } n > 2k, \end{cases} \tag{2.3}$$

and the Joseph-Lundgren exponent

$$p_c(n) = \begin{cases} +\infty & \text{if } n \le 10, \\ \frac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \ge 11. \end{cases} \tag{2.4}$$

Note that the exponent $p_c(n)$ is larger than the classical critical Sobolev exponent $p_s(n,1)$, $n \ge 11$.

Now we can state our main nonexistence results.

Theorem 2.1 *Let $u \in C^{2k}(\mathbb{R}^n)$ be a stable solution of* (1.1). *Assume that f satisfies H_1 and H_2. If $1 < \theta \leq p_s(n,k)$, then $u \equiv 0$.*

Theorem 2.2 *Let $u \in C^{2k}(\mathbb{R}^n)$ be a solution of* (1.1) *which is stable outside a compact set. Assume that f satisfies H_1, H_2 and H_3. If $1 < \theta < p_s(n,k)$, then $u \equiv 0$.*

The next result concerns the complete classification of entire stable solutions of the second-order equation (1.1) in the supercritical case.

Theorem 2.3 *Let $u \in C^2(\mathbb{R}^n)$ be a stable solution of* (1.1) *with $k = 1$. Assume that f satisfies H_1 and H_2. If $\frac{n+2}{n-2} < \theta < p_c(n)$, then $u \equiv 0$.*

2.1 Berestycki and Lions Liouville-type theorem

Now, we fix in this subsection

$$f(u) = -mu + \lambda|u|^{\theta-1}u - \mu|u|^{p-1}u, \tag{2.5}$$

where λ, μ are positive constants, $m \geq 0$ and $1 < \theta, p$. We will show that $u = 0$ is the unique solution of equation (1.1) under some assumptions on the parameter m, θ and p. Also, we observe that f is neither convex nor positive function in $\mathbb{R}^n$. Then we have the following.

Theorem 2.4 *Let $u \in C^{2k}(\mathbb{R}^n)$ be a stable solution of* (1.1) *with f satisfies* (2.5).

1. *If u is bounded and $m > 0$, then $u \equiv 0$, for any $\theta \geq p > 1$.*
2. *If $1 < p < \theta$ and $1 < p \leq p_s(n,k)$, then $u \equiv 0$.*

Remark 2.3 Clearly, if u is unbounded stable solution to (1.1) with $f(u) = -mu + \lambda|u|^{\theta-1}u - \mu|u|^{p-1}u$ and $m > 0$, then $u \equiv 0$, for any $\theta \geq p > 1$ and $n < 2k$.

Also, we will show, with very few restrictions, that there exists a necessary and sufficient condition for the nonexistence solutions which are stable outside a compact set of problem like (1.1).

Theorem 2.5 *Let $u \in C^{2k}(\mathbb{R}^n)$ be a solution of* (1.1) *which is stable outside a compact set with f satisfies* (2.5).

1. *If $m > 0$ and $1 < p \leq \frac{n+2k}{n-2k} \leq \theta$, then $u \equiv 0$.*
2. *If $m = 0$, $1 < p \leq \frac{n+2k}{n-2k} \leq \theta$ and $(p,\theta) \neq (\frac{n+2k}{n-2k}, \frac{n+2k}{n-2k})$, then $u \equiv 0$.*

3 Auxiliary results

In this section we prove the following lemmas and propositions, which will have a crucial role in the proof of Theorems 2.1, 2.2, 2.3, 2.4 and 2.5. Denote $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. The letter C will be used throughout to denote a generic positive constant, which may vary from line to line and only depends on arguments inside the parentheses or arguments which are otherwise clear from the context.

First, define a cut-off function $\varphi_R \in C_c^4(\mathbb{R}^n)$ such that $\varphi_R \equiv 1$ in B_R, $\varphi_R \equiv 0$ in $\mathbb{R}^n \backslash \{B_{2R}\}$, $0 \leq \varphi_R \leq 1$ in $\mathbb{R}^n$ and $|\nabla^\tau \varphi_R| \leq CR^{-\tau}$ for $\tau \leq 4$ in $A_R = \{x \in \mathbb{R}^n, R \leq |x| \leq 2R\}$.

Lemma 3.1 *For any $v \in C^8(\mathbb{R}^n)$, $m > 4$ and $\epsilon > 0$ arbitrary small number, there exists a constant $C_{\epsilon,m} > 0$ such that*

1. $R^{-4}\int_{B_{2R}}|\Delta v|^2\varphi_R^{2m-4}\,dx \le \epsilon^2\int_{B_{2R}}(\Delta^2 v)^2\varphi_R^{2m}\,dx + \epsilon^2 R^{-2}\int_{B_{2R}}|\nabla(\Delta v)|^2\varphi_R^{2m-2}\,dx + C_{\epsilon,m}R^{-8}\int_{B_{2R}}v^2\varphi_R^{2m-8}\,dx$,
2. $R^{-2}\int_{B_{2R}}|\nabla(\Delta v)|^2\varphi_R^{2m-2}\,dx \le \epsilon\int_{B_{2R}}(\Delta^2 v)^2\varphi_R^{2m}\,dx + C_{\epsilon,m}R^{-8}\int_{B_{2R}}v^2\varphi_R^{2m-8}\,dx$,
3. $R^{-6}\int_{B_{2R}}|\nabla v|^2\varphi_R^{2m-6}\,dx \le \epsilon^3\int_{B_{2R}}(\Delta^2 v)^2\varphi_R^{2m}\,dx + C_{\epsilon,m}R^{-8}\int_{B_{2R}}v^2\varphi_R^{2m-8}\,dx$,
4. $R^{-4}\int_{B_{2R}}|\nabla^2 v|^2\varphi_R^{2m-4}\,dx \le \epsilon^3\int_{B_{2R}}(\Delta^2 v)^2\varphi_R^{2m}\,dx + C_{\epsilon,m}R^{-8}\int_{B_{2R}}v^2\varphi_R^{2m-8}\,dx$,
5. $R^{-2}\int_{B_{2R}}|\nabla^3 v|^2\varphi_R^{2m-2}\,dx \le \epsilon^3\int_{B_{2R}}(\Delta^2 v)^2\varphi_R^{2m}\,dx + C_{\epsilon,m}R^{-8}\int_{B_{2R}}v^2\varphi_R^{2m-8}\,dx$.

Proof Fix $m > 4$. Let $v \in C^8(\mathbb{R}^n)$ and $\varphi_R \in C_c^4(\mathbb{R}^n)$ defined as above.

Proof of 1. Integrating by parts, we get

$$\begin{aligned} &R^{-4}\int_{B_{2R}}(\Delta v)^2\varphi_R^{2m-4}\,dx \\ &\quad = R^{-4}\int_{B_{2R}} v\bigl(\Delta^2 v\varphi_R^{2m-4} + \Delta v\Delta\bigl(\varphi_R^{2m-4}\bigr) + 2\nabla(\Delta v)\nabla\bigl(\varphi_R^{2m-4}\bigr)\bigr)\,dx. \end{aligned} \tag{3.1}$$

An application of Young's inequality yields

$$\begin{aligned} &R^{-4}\int_{B_{2R}} v\bigl(\Delta^2 v\varphi_R^{2m-4} + \Delta v\Delta\bigl(\varphi_R^{2m-4}\bigr) + 2\nabla(\Delta v)\nabla\bigl(\varphi_R^{2m-4}\bigr)\bigr)\,dx \\ &\quad \le \epsilon^2\int_{B_{2R}}\bigl(\Delta^2 v\bigr)^2\varphi_R^{2m}\,dx + \epsilon^2 R^{-2}\int_{B_{2R}}\bigl|\nabla(\Delta v)\bigr|^2\varphi_R^{2m-2}\,dx \\ &\qquad + \frac{R^{-4}}{2}\int_{B_{2R}}(\Delta v)^2\varphi_R^{2m-4}\,dx + C_{\epsilon,m}R^{-8}\int_{B_{2R}}v^2\varphi_R^{2m-8}\,dx. \end{aligned}$$

Inserting the latter inequality into (3.1), we obtain

$$\begin{aligned} R^{-4}\int_{B_{2R}}(\Delta v)^2\varphi_R^{2m-4}\,dx &\le \epsilon^2\int_{B_{2R}}\bigl(\Delta^2 v\bigr)^2\varphi_R^{2m}\,dx + \epsilon^2 R^{-2}\int_{B_{2R}}\bigl|\nabla(\Delta v)\bigr|^2\varphi_R^{2m-2}\,dx \\ &\quad + C_{\epsilon,m}R^{-8}\int_{B_{2R}}v^2\varphi_R^{2m-8}\,dx. \end{aligned} \tag{3.2}$$

Proof of 2. Integrating by parts and using again Young's inequality, we obtain

$$\begin{aligned} &R^{-2}\int_{B_{2R}}\bigl|\nabla(\Delta v)\bigr|^2\varphi_R^{2m-2}\,dx \\ &\quad = -R^{-2}\int_{B_{2R}}\Delta v\Delta^2 v\varphi_R^{2m-2}\,dx - R^{-2}\int_{B_{2R}}\Delta v\nabla(\Delta v)\nabla\bigl(\varphi_R^{2m-2}\bigr)\,dx \\ &\quad \le \epsilon\int_{B_{2R}}\bigl(\Delta^2 v\bigr)^2\varphi_R^{2m}\,dx + \frac{2R^{-4}}{\epsilon}\int_{B_{2R}}(\Delta v)^2\varphi_R^{2m-4}\,dx + C\epsilon R^{-2}\int_{B_{2R}}\bigl|\nabla(\Delta v)\bigr|^2\varphi_R^{2m-2}\,dx. \end{aligned}$$

Inserting (3.2) into the latter, we derive

$$R^{-2}\int_{B_{2R}}\bigl|\nabla(\Delta v)\bigr|^2\varphi_R^{2m-2}\,dx \le \epsilon\int_{B_{2R}}\bigl(\Delta^2 v\bigr)^2\varphi_R^{2m}\,dx + C_{\epsilon,m}R^{-8}\int_{B_{2R}}v^2\varphi_R^{2m-8}\,dx. \tag{3.3}$$

Proof of 3. Integrating by parts, we obtain

$$\begin{aligned} \int_{B_{2R}}|\nabla v|^2\varphi_R^{2m-6}\,dx &= \frac{R^{-6}}{2}\int_{B_{2R}}\Delta\bigl(v^2\bigr)\varphi_R^{2m-6}\,dx - R^{-6}\int_{B_{2R}}v\Delta v\varphi_R^{2m-6}\,dx \\ &\le \epsilon R^{-4}\int_{B_{2R}}(\Delta v)^2\varphi_R^{2m-4}\,dx + C_{\epsilon,m}R^{-8}\int_{B_{2R}}v^2\varphi_R^{2m-8}\,dx. \end{aligned}$$

From (3.2) and (3.3), we deduce

$$R^{-6}\int_{B_{2R}}|\nabla v|^2\varphi_R^{2m-6}\,dx\le\epsilon^3\int_{B_{2R}}\left(\Delta^2 v\right)^2\varphi_R^{2m}\,dx+C_{\epsilon,m}R^{-8}\int_{B_{2R}}v^2\varphi_R^{2m-8}\,dx. \tag{3.4}$$

Proof of 4. Integrating by parts, we obtain

$$\begin{aligned}&R^{-4}\int_{B_{2R}}\left|\nabla^2 v\right|^2\varphi_R^{2m-4}\,dx\\&\quad=-R^{-4}\int_{B_{2R}}\nabla v\nabla(\Delta v)\varphi_R^{2m-4}\,dx+\frac{R^{-4}}{2}\int_{B_{2R}}|\nabla v|^2\Delta\left(\varphi_R^{2m-4}\right)dx.\end{aligned} \tag{3.5}$$

Using Young's inequality and from (3.3) and (3.4), we obtain

$$\begin{aligned}R^{-4}\int_{B_{2R}}\left|\nabla^2 v\right|^2\varphi_R^{2m-4}\,dx&\le R^{-2}\int_{B_{2R}}\left|\nabla(\Delta v)\right|^2\varphi_R^{2m-2}\,dx+CR^{-6}\int_{B_{2R}}|\nabla v|^2\varphi_R^{2m-6}\,dx\\&\le\epsilon^3\int_{B_{2R}}\left(\Delta^2 v\right)^2\varphi_R^{2m}+C_{\epsilon,m}R^{-8}\int_{B_{2R}}v^2\varphi_R^{2m-8}.\end{aligned}$$

Proof of 5. Integrating by parts, we get

$$\begin{aligned}&R^{-2}\int_{B_{2R}}\left|\nabla^3 v\right|^2\varphi_R^{2m-2}\,dx\\&\quad=R^{-2}\int_{B_{2R}}\left(\left|\nabla(\Delta v)\right|^2\varphi_R^{2m-2}+v_{ij}\Delta v_i\left(\varphi_R^{2m-2}\right)_j+\frac{1}{2}\left|\nabla^2 v\right|^2\Delta\left(\varphi_R^{2m-2}\right)\right)dx,\end{aligned}$$

where $f_i=\frac{\partial f}{\partial x_i}$, $f_{ij}=\frac{\partial^2 f}{\partial x_j\partial x_i}$ and $f_{ijk}=\frac{\partial^3 f}{\partial x_k\partial x_j\partial x_i}$. (Here and in the sequel, we use the Einstein summation convention: an index occurring twice in a product is to be summed from 1 up to the space dimension.)

Using Young's inequality of the above, we deduce

$$\begin{aligned}&R^{-2}\int_{B_{2R}}\left|\nabla^3 v\right|^2\varphi_R^{2m-2}\,dx\\&\quad\le CR^{-2}\int_{B_{2R}}\left|\nabla(\Delta v)\right|^2\varphi_R^{2m-2}\,dx+CR^{-4}\int_{B_{2R}}\left|\nabla^2 v\right|^2\varphi_R^{2m-4}\,dx,\end{aligned} \tag{3.6}$$

which gives the desired conclusion. □

Lemma 3.2 *For any $m>4$ and $\epsilon>0$ arbitrary small number, there exists a constant $C_{\epsilon,m}>0$ such that*

$$\left(\Delta^2\left(u\varphi_R^m\right)\right)^2\le(1+\epsilon)\left(\varphi_R^m\Delta^2 u\right)^2+C_{\epsilon,m}\mathbf{B}(u,\varphi_R,m), \tag{3.7}$$

where $\mathbf{B}(u,\varphi_R,m)=(R^{-4}|\Delta u|^2\varphi_R^{2m-4}+R^{-2}|\nabla(\Delta u)|^2\varphi_R^{2m-2}+R^{-6}|\nabla u|^2\varphi_R^{2m-6}+R^{-8}u^2\varphi_R^{2m-8}+R^{-4}|\nabla^2 u|^2\varphi_R^{2m-4})$.

Proof Let $\varphi_R\in C_c^4(\mathbb{R}^n)$ be defined as above and $m>4$. Direct calculation yields

$$\Delta^2\left(u\varphi_R^m\right)=\varphi_R^m\Delta^2 u+\mathbf{A}\left(u,\varphi_R^m\right), \tag{3.8}$$

where $\mathbf{A}(u,\varphi_R^m)=2\Delta u\Delta\varphi_R^m+4\nabla u\nabla(\Delta\varphi_R^m)+u\Delta^2\varphi_R^m+4\nabla(\Delta u)\nabla(\varphi_R^m)+4u_{ij}(\varphi_R^m)_{ij}$.

Thus,

$$\big(\Delta^2\big(u\varphi_R^m\big)\big)^2 = \big(\varphi_R^m\Delta^2 u\big)^2 + \mathbf{A}^2\big(u,\varphi_R^m\big) + 2\mathbf{A}\big(u,\varphi_R^m\big)\varphi_R^m\Delta^2 u.$$

Now by the Young inequality, for any $\epsilon > 0$, there exists C_ϵ a constant such that

$$\big(\Delta^2\big(u\varphi_R^m\big)\big)^2 \le (1+\epsilon)\big(\varphi_R^m\Delta^2 u\big)^2 + C_\epsilon\mathbf{A}^2\big(u,\varphi_R^m\big). \tag{3.9}$$

For the second term on the right hand side of inequality (3.9), one obtains

$$\begin{aligned}
\mathbf{A}^2\big(u,\varphi_R^m\big) &\le C_\epsilon\big(|\Delta u|^2\big|\Delta\varphi_R^m\big|^2 + |\nabla u|^2\big|\nabla\big(\Delta\varphi_R^m\big)\big|^2 + |u|^2\big|\Delta^2\varphi_R^m\big|^2 \\
&\quad + \big|\nabla(\Delta u)\big|^2\big|\nabla\big(\varphi_R^m\big)\big|^2 + |u_{ij}|^2\big|\big(\varphi_R^m\big)_{ij}\big|^2\big) \\
&\le C_{\epsilon,m}\big(R^{-4}|\Delta u|^2\varphi_R^{2m-4} + R^{-2}\big|\nabla(\Delta u)\big|^2\varphi_R^{2m-2} + R^{-6}|\nabla u|^2\varphi_R^{2m-6} \\
&\quad + R^{-8}u^2\varphi_R^{2m-8} + R^{-4}\big|\nabla^2 u\big|^2\varphi_R^{2m-4}\big),
\end{aligned}$$

which gives the desired inequality (3.7). □

Using the previous lemmas, we obtain the following results.

Proposition 3.1 *Let $u \in C^{2k}(\mathbb{R}^n)$ be a stable solution of* (1.1). *Assume that f satisfies H_1 and H_2. Then there exists a constant $C > 0$ such that, for any $R > 0$, we have*

$$\int_{B_R}\big(|u|^{\theta+1} + \big|D^k u\big|^2\big)\,dx \le CR^{n-2k\frac{\theta+1}{\theta-1}} \quad \textit{and} \quad \int_{B_R} f(u)u\,dx \le CR^{n-2k\frac{\theta+1}{\theta-1}}.$$

When attempting to prove the nonexistence of the nontrivial solution which is stable outside a compact set of (1.1) in the subcritical case, we need first to establish the following proposition.

Proposition 3.2 *Let $u \in C^{2k}(\mathbb{R}^n)$ be a solution of* (1.1) *which is stable outside a compact set. Assume that f satisfies H_1 and H_2. Then there exists a constant $C > 0$ such that, for any $R > 0$, we have*

$$\int_{B_R}\big(|u|^{\theta+1} + \big|D^k u\big|^2\big)\,dx \le C\big(1 + R^{n-2k\frac{\theta+1}{\theta-1}}\big) \quad \textit{and} \quad \int_{B_R} f(u)u\,dx \le C\big(1 + R^{n-2k\frac{\theta+1}{\theta-1}}\big).$$

Proof of Proposition 3.1 The proof of the case $k = 1, 2, 3$, bears resemblance to an argument found in [5, 6, 8]. For more details, please see the proof of proposition 4 in [6] for the case $k = 1$, the proof of Lemma 4.2 in [5] for the case $k = 2$ and the proof of Proposition 1.2 in [8] for the case $k = 3$. For this reason, we omit the details.

Proof of the case $k = 4$. Let $\varphi_R \in C_c^4(\mathbb{R}^n)$ defined as above, let u be a solution of equation (1.1). The function $u\varphi_R^m$ belongs to $C_c^4(\mathbb{R}^n)$, and thus it can be used as a test function in the quadratic form Q_u. Hence, the stability assumption on u gives

$$\int_{B_{2R}} f'(u)u^2\varphi_R^{2m}\,dx \le \int_{B_{2R}}\big|\Delta^2\big(u\varphi_R^m\big)\big|^2\,dx.$$

Applying Lemma 3.2, we obtain

$$\int_{B_{2R}} f'(u)u^2\varphi_R^{2m}\,dx \le (1+\epsilon)\int_{B_{2R}} \left(\varphi_R^m\Delta^2 u\right)^2 dx + C_\epsilon \int_{B_{2R}} \mathbf{B}(u,\varphi_R,m)\,dx. \tag{3.10}$$

In view of Lemma 3.1, we get

$$\int_{B_{2R}} f'(u)u^2\varphi_R^{2m}\,dx \le (1+\epsilon)\int_{B_{2R}} \left(\varphi_R^m\Delta^2 u\right)^2 + C_\epsilon R^{-8}\int_{B_{2R}} u^2\varphi_R^{2m-8}\,dx. \tag{3.11}$$

Multiplying equation (1.1) by $u\varphi_R^{2m}$ and integrating by parts, we get

$$\int_{B_{2R}} \Delta^2 u\Delta^2\left(u\varphi_R^{2m}\right)dx = \int_{B_{2R}} f(u)u\varphi_R^{2m}\,dx.$$

From (3.8), we derive

$$\begin{aligned}
&\int_{B_{2R}} \Delta^2 u\Delta^2\left(u\varphi_R^{2m}\right)dx \\
&\quad= \int_{B_{2R}} \Delta^2 u\bigl\{\left(\Delta^2 u\right)\varphi_R^{2m} + 2\Delta u\Delta\left(\varphi_R^{2m}\right) + 4u_{ij}\left(\varphi_R^{2m}\right)_{ij} \\
&\qquad + 4\nabla(\Delta u)\nabla\left(\varphi_R^{2m}\right) + 4\nabla u\nabla\left(\Delta\left(\varphi_R^{2m}\right)\right) + u\Delta^2\left(\varphi_R^{2m}\right)\bigr\}\,dx,
\end{aligned}$$

therefore

$$\int_{B_{2R}} \left(\left(\Delta^2 u\right)^2\varphi_R^{2m} - f(u)u\varphi_R^{2m}\right)dx = -\int_{B_{2R}} \Delta^2 u\mathbf{A}\left(u,\varphi_R^{2m}\right)dx. \tag{3.12}$$

Then, using Young's inequality, we derive

$$\int_{B_{2R}} \left(\left(\Delta^2 u\right)^2\varphi_R^{2m} - f(u)u\varphi_R^{2m}\right)dx \le \epsilon\int_{B_{2R}} \left(\Delta^2 u\right)^2\varphi_R^{2m}\,dx + C_{\epsilon,m}\int_{B_{2R}} \mathbf{B}(u,\varphi_R,m)\,dx.$$

Applying again Lemma 3.1, we have

$$\begin{aligned}
&\int_{B_{2R}} \left(\Delta^2 u\right)^2\varphi_R^{2m}\,dx - \int_{B_{2R}} f(u)u\varphi_R^{2m}\,dx \\
&\quad\le \epsilon\int_{B_{2R}} \left(\Delta^2 u\right)^2\varphi_R^{2m}\,dx + C_{\epsilon,m}R^{-8}\int_{B_{2R}} u^2\varphi_R^{2m-8}\,dx.
\end{aligned} \tag{3.13}$$

Multiplying (3.13) by θ and combining it with (3.11), we derive

$$\begin{aligned}
&\int_{B_{2R}} \left[f'(u)u^2 - \theta f(u)u\right]\varphi_R^{2m}\,dx + \left[\theta(1-\epsilon) - (1+\epsilon)\right]\int_{B_{2R}} \left(\Delta^2 u\right)^2\varphi_R^{2m}\,dx \\
&\quad\le CR^{-8}\int_{B_{2R}} u^2\varphi_R^{2m-8}\,dx.
\end{aligned}$$

From (H_1) and for ϵ sufficiently small such that $\epsilon < \frac{\theta-1}{\theta+1}$, we deduce

$$\int_{B_{2R}} \left(\Delta^2 u\right)^2\varphi_R^{2m}\,dx \le CR^{-8}\int_{B_{2R}} u^2\varphi_R^{2m-8}\,dx. \tag{3.14}$$

By Young's inequality, we have

$$\int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m}\,dx \le \frac{2}{\theta+1}\int_{B_{2R}} |u|^{\theta+1}\varphi_R^{(\theta+1)(m-4)}\,dx + CR^{n-8\frac{\theta+1}{\theta-1}}. \tag{3.15}$$

As above, we find from (3.13) that

$$\int_{B_{2R}} f(u)u\varphi_R^{2m}\,dx \le (1+\epsilon)\int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m}\,dx + C_{\epsilon,m}R^{-8}\int_{B_{2R}} u^2\varphi_R^{2m-8}\,dx.$$

Using (3.14) in the latter, we obtain

$$\begin{aligned}\int_{B_{2R}} f(u)u\varphi_R^{2m}\,dx &\le C_\epsilon R^{-8}\int_{B_{2R}} u^2\varphi_R^{2m-8}\,dx\\ &\le \frac{2}{\theta+1}\int_{B_{2R}} |u|^{\theta+1}\varphi_R^{(\theta+1)(m-4)}\,dx + CR^{n-8\frac{\theta+1}{\theta-1}}.\end{aligned} \tag{3.16}$$

From (H_1') and (H_2), we get

$$\begin{aligned}C_0\int_{B_{2R}} |u|^{\theta+1}\varphi_R^{2m}\,dx &\le \int_{B_{2R}} f(u)u\varphi_R^{2m}\,dx\\ &\le \frac{2}{\theta+1}\int_{B_{2R}} |u|^{\theta+1}\varphi_R^{(\theta+1)(m-4)}\,dx + CR^{n-8\frac{\theta+1}{\theta-1}},\end{aligned}$$

if $(\theta+1)(m-4) = 2m$, then

$$\int_{B_{2R}} |u|^{\theta+1}\varphi_R^{2m}\,dx \le CR^{n-8\frac{\theta+1}{\theta-1}}. \tag{3.17}$$

From (3.15), (3.16) and (3.17), we deduce that

$$\int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m}\,dx \le CR^{n-8\frac{\theta+1}{\theta-1}}, \quad \text{and} \quad \int_{B_{2R}} f(u)u\varphi_R^{2m}\,dx \le CR^{n-8\frac{\theta+1}{\theta-1}}.$$

Since $\varphi_R \equiv 1$ in B_R, we have

$$\int_{B_R} \left(|u|^{\theta+1} + \left(\Delta^2 u\right)^2\right)dx \le CR^{n-8\frac{\theta+1}{\theta-1}}, \quad \text{and} \quad \int_{B_R} f(u)u\,dx \le CR^{n-8\frac{\theta+1}{\theta-1}}. \qquad \square$$

Proof of Proposition 3.2 The proof of the case $k = 1, 2, 3$, bears resemblance to an argument found in [5, 6, 8]. Now, we prove the case $k = 4$. The proof is the same as the proof of Proposition 3.1. We need only to replace φ_R by $\varphi_{a,R}$, where $\varphi_{a,R} \in C_c^4(\mathbb{R}^n)$ satisfies $0 \le \varphi_{a,R} \le 1$ everywhere on $\mathbb{R}^n$ such that $\varphi_{a,R}(x) = 0$ for $|x| < a$ or $|x| > 2R$, $\varphi_{a,R}(x) = 1$ for $2a < |x| < R$ and $|\nabla^\tau \varphi_{a_0,R}| \le CR^{-\tau}$, $\tau \le 4$, for $R < |x| < 2R$. By the stability assumption on u, there exists $a_0 > 0$ such that $Q_u(u\varphi_{a_0,R}^m) \ge 0$ for any $R > 2a_0$. Hence, by the choice of the test function $\varphi_{a,R}$, the constant C_{a_0} depending on a_0, ϵ, m and u appears and the rest of the proof is unchanged. Thus Proposition 3.2 follows. $\square$

As in [20], we shall employ a cut-off function with compact support to derive a variant of the Pohozaev identity. This device allows us to avoid the spherical integrals raised in

[21], which are very difficult to control, especially for the polyharmonic situations. For $k = 1, 2, 3$, the Pohozaev identity is similar to [7, 8, 20, 22].

Proposition 3.3 *Let $u \in C^8(\mathbb{R}^n)$ be a solution of* (1.1) *and $\psi \in C_c^4(B_R)$, then*

$$\frac{n-8}{2}\int_{B_R}\left(\Delta^2 u\right)^2\psi\,dx - n\int_{B_R}F(u)\psi\,dx = \int_{B_R}B_4(u,\psi)\,dx, \tag{3.18}$$

where

$$\begin{aligned}B_4(u,\psi) = {}& F(u)\langle x,\nabla\psi\rangle - \frac{1}{2}\left(\Delta^2 u\right)^2\langle x,\nabla\psi\rangle + 2\Delta^2 u\nabla\left(\langle x,\nabla(\Delta u)\rangle\right)\nabla\psi\\ &+ \Delta^2 u\left\{\langle x,\nabla(\Delta u)\rangle\Delta\psi + 2\Delta u\Delta\psi\right\} + \Delta^2 u\left\{4\nabla(\Delta u)\nabla\psi + \Delta^2\psi\langle x,\nabla u\rangle\right\}\\ &+ \Delta^2 u\left\{\Delta\psi\Delta\left(\langle x,\nabla u\rangle\right) + 2\nabla(\Delta\psi)\nabla\left(\langle x,\nabla u\rangle\right) + 2\Delta\left[\nabla\left(\langle x,\nabla u\rangle\right)\nabla\psi\right]\right\}.\end{aligned}$$

Thanks to Propositions 3.2 and 3.3, we derive the following.

Proposition 3.4 *Let $u \in C^{2k}(\mathbb{R}^n)$ be a solution of* (1.1) *which is stable outside a compact set. Assume that f satisfies H_1 and H_2. If $1 < \theta < p_s(n,k)$, then*

$$\int_{\mathbb{R}^n}\left|D^k u\right|^2 dx = \frac{2n}{n-2k}\int_{\mathbb{R}^n}F(u)\,dx \tag{3.19}$$

and

$$\int_{\mathbb{R}^n}\left|D^k u\right|^2 dx = \int_{\mathbb{R}^n}f(u)u\,dx < \infty. \tag{3.20}$$

Proof of Proposition 3.3 Let $u \in C^8(\mathbb{R}^n)$ be a solution of (1.1) and $\psi \in C_c^4(B_R)$, we have

$$\Delta\left(\langle x,\nabla u\rangle\psi\right) = \langle x,\nabla(\Delta u)\rangle\psi + 2\Delta u\psi + \langle x,\nabla u\rangle\Delta\psi + 2\nabla\left(\langle x,\nabla u\rangle\right)\nabla\psi.$$

Multiplying equation (1.1) by $\langle x,\nabla u\rangle\psi$ and integrating by parts in B_R, we obtain

$$\int_{B_R}f(u)\langle x,\nabla u\rangle\psi\,dx = \int_{B_R}\Delta^3 u\Delta\left(\langle x,\nabla u\rangle\psi\right)dx. \tag{3.21}$$

For the right hand side of (3.21), we integrate by parts to get

$$\begin{aligned}&\int_{B_R}\Delta^3 u\Delta\left(\langle x,\nabla u\rangle\psi\right)dx\\ &\quad= \int_{B_R}\Delta^3 u\left(\langle x,\nabla(\Delta u)\rangle\psi + 2\Delta u\psi + \langle x,\nabla u\rangle\Delta\psi + 2\nabla\left(\langle x,\nabla u\rangle\right)\nabla\psi\right)dx\\ &\quad= \int_{B_R}\Delta^2 u\Delta\left[\langle x,\nabla(\Delta u)\rangle\right]\psi\,dx + 2\int_{B_R}\Delta^2 u\nabla\left[\langle x,\nabla(\Delta u)\rangle\right]\nabla\psi\,dx + 2\int_{B_R}\left(\Delta^2 u\right)^2\psi\,dx\\ &\qquad+ \int_{B_R}\Delta^2 u\left\{\langle x,\nabla(\Delta u)\rangle\Delta\psi + 2\Delta u\Delta\psi + 4\nabla(\Delta u)\nabla\psi + \langle x,\nabla u\rangle\Delta^2\psi\right\}dx\\ &\qquad+ \int_{B_R}\Delta^2 u\left\{\Delta\left[\langle x,\nabla u\rangle\right]\Delta\psi + 2\nabla\left[\langle x,\nabla u\rangle\right]\nabla(\Delta\psi)\right\}dx\\ &\qquad+ 2\int_{B_R}\Delta^2 u\Delta\left[\nabla\left(\langle x,\nabla u\rangle\right)\nabla\psi\right]dx.\end{aligned} \tag{3.22}$$

For the first term on the right hand side of (3.22), we integrate by parts to find

$$\int_{B_R} \Delta^2 u \Delta\big[\langle x, \nabla(\Delta u)\rangle\big]\psi\, dx = \frac{4-n}{2}\int_{B_R}\big(\Delta^2 u\big)^2\psi\, dx - \frac{1}{2}\int_{B_R}\big(\Delta^2 u\big)^2\langle x, \nabla\psi\rangle\, dx. \quad (3.23)$$

For the term on the left hand side of (3.22), by integrating by parts, we derive

$$\begin{aligned}\int_{B_R} f(u)\langle x, \nabla u\rangle\psi\, dx &= \int_{B_R}\langle x, \nabla\big[F(u)\big]\rangle\psi\, dx \\ &= -n\int_{B_R} F(u)\psi\, dx - \int_{B_R} F(u)\langle x, \nabla\psi\rangle\, dx. \end{aligned}\quad (3.24)$$

Therefore, the claim follows from (3.21)-(3.24). □

Here, we are concerned with the proof of Proposition 3.4.

Proof of Proposition 3.4 *To simplify the proof, we will concentrate on the case* $k = 4$ *which is the most delicate case; even we believe that the results should hold true for* $k = 1, 2, 3$, *for more details, see for example* [5, 6, 8, 23]. Let $R_0 > 0$. Assume that u is stable outside B_{R_0}. Let $0 < \alpha < \beta$. We begin by defining some smooth compactly supported functions which will be used several times in the sequel. More precisely, we choose $\phi_R \in C_c^4(\mathbb{R}^n)$ satisfies $0 \le \phi_R \le 1$ everywhere on $\mathbb{R}^n$ such that

$$\phi_R(x) = \begin{cases} 1 & \text{for } \alpha R < |x| < \beta R, \\ 0 & \text{for } |x| < \frac{\alpha}{2}R \text{ or } |x| > 2\beta R, \\ |\nabla^k\phi_R| \le CR^{-k} & \text{on } \{\frac{\alpha}{2}R < |x| < 2\beta R\}, k = 1, 2, 3, 4. \end{cases}$$

For R large enough such that $\frac{\alpha}{2}R > R_0$, then $B_{R_0} \cap \{\frac{\alpha}{2}R \le |x| \le 2\beta R\} = \emptyset$. Then u is stable in $A_{\frac{\alpha}{2}R}^{2\beta R} := \{\frac{\alpha}{2}R < |x| < 2\beta R\}$. By Proposition 3.1, there exists a constant $C > 0$ such that

$$\int_{A_{\alpha R}^{\beta R}}\big(|u|^{\theta+1} + \big(\Delta^2 u\big)^2\big)\, dx \le CR^{n-8\frac{\theta+1}{\theta-1}} \quad \text{and} \quad \int_{A_{\alpha R}^{\beta R}} f(u)u\, dx \le CR^{n-8\frac{\theta+1}{\theta-1}}. \quad (3.25)$$

Let $\psi_R \in C_c^4(\mathbb{R}^n)$ satisfies $0 \le \psi_R \le 1$ on $\mathbb{R}^n$ defined by

$$\psi_R(x) = \begin{cases} 1 & \text{for } |x| < \alpha R, \\ 0 & \text{for } |x| > \beta R, \\ |\nabla^k\psi_R| \le CR^{-k} & \text{on } \{\alpha R < |x| < \beta R\}, k = 1, 2, 3, 4. \end{cases}$$

In view of Lemma 3.1 and Proposition 3.1, we have

$$\int_{B_{\beta R}}\big(|u|^{\theta+1}\psi_R^{2m} + \big(\Delta^2 u\big)^2\psi_R^{2m}\big)\, dx \le CR^{n-8\frac{\theta+1}{\theta-1}}, \quad (3.26)$$

$$\int_{B_{\beta R}}\big(\mathbf{B}(u, \psi_R, m) + R^{-2}\big|\nabla^3 u\big|^2\psi_R^{2m-2}\big)\, dx \le CR^{n-8\frac{\theta+1}{\theta-1}}. \quad (3.27)$$

Now, we estimate all terms on the right hand side of (3.18). Take $\psi = \psi_R^{2m}$ in (3.18), $m > 4$.

The second term on the right hand side of (3.18) can be estimated as

$$\left|-\frac{1}{2}\int_{B_{\beta R}}(\Delta^2 u)^2\langle x,\nabla\psi_R^{2m}\rangle\,dx\right| = \left|-\frac{1}{2}\int_{A_{\alpha R}^{\beta R}}(\Delta^2 u)^2\langle x,\nabla\psi_R^{2m}\rangle\,dx\right|$$
$$\le C_m\int_{A_{\alpha R}^{\beta R}}(\Delta^2 u)^2\psi_R^{2m-1}\,dx \le CR^{n-8\frac{\theta+1}{\theta-1}}. \tag{3.28}$$

Next

$$\left|\int_{B_{\beta R}}\left[\Delta^2 u\left(\langle x,\nabla(\Delta u)\rangle\Delta\psi_R^{2m} + 2\Delta u\Delta\psi_R^{2m} + 4\nabla(\Delta u)\nabla\psi_R^{2m} + \Delta^2\psi_R^{2m}\langle x,\nabla u\rangle\right)\right]dx\right|$$
$$= \left|\int_{A_{\alpha R}^{\beta R}}\left[\Delta^2 u\left(\langle x,\nabla(\Delta u)\rangle\Delta\psi_R^{2m} + 2\Delta u\Delta\psi_R^{2m} + 4\nabla(\Delta u)\nabla\psi_R^{2m} + \Delta^2\psi_R^{2m}\langle x,\nabla u\rangle\right)\right]dx\right|$$
$$\le C_m\int_{A_{\alpha R}^{\beta R}}\left|\Delta^2 u\right|\left(R^{-1}\left|\nabla(\Delta u)\right|\psi_R^{2m-2} + R^{-2}|\Delta u|\psi_R^{2m-2} + R^{-3}|\nabla u|\psi_R^{2m-4}\right)dx$$
$$\le C_m\int_{A_{\alpha R}^{\beta R}}\left|\Delta^2 u\right|\left(R^{-1}\left|\nabla(\Delta u)\right|\psi_R^{m-1} + R^{-2}|\Delta u|\psi_R^{m-2} + R^{-3}|\nabla u|\psi_R^{m-3}\right)dx, \tag{3.29}$$

the last line comes from the fact that $0 \le \psi_R \le 1$, hence $\psi_R^s \le \psi_R^t$, for any $t \le s$.

By applying the Hölder inequality and the Young inequality to (3.29), we have

$$\left|\int_{B_{\beta R}}\left[\Delta^2 u\left(\langle x,\nabla(\Delta u)\rangle\Delta\psi_R^{2m} + 2\Delta u\Delta\psi_R^{2m} + 4\nabla(\Delta u)\nabla\psi_R^{2m} + \Delta^2\psi_R^{2m}\langle x,\nabla u\rangle\right)\right]dx\right|$$
$$\le \int_{A_{\alpha R}^{\beta R}}\left|\Delta^2 u\right|\left(R^{-1}\left|\nabla(\Delta u)\right|\psi_R^{m-1} + R^{-2}|\Delta u|\psi_R^{m-2} + R^{-3}|\nabla u|\psi_R^{m-3}\right)dx$$
$$\le \left(\int_{A_{\alpha R}^{\beta R}}(\Delta^2 u)^2\,dx\right)^{\frac{1}{2}}\left(\int_{A_{\alpha R}^{\beta R}}\left(R^{-1}\left|\nabla(\Delta u)\right|\psi_R^{m-1} + R^{-2}|\Delta u|\psi_R^{m-2}\right.\right.$$
$$\left.\left. + R^{-3}|\nabla u|\psi_R^{m-3}\right)^2 dx\right)^{\frac{1}{2}}$$
$$\le C\left(\int_{A_{\alpha R}^{\beta R}}(\Delta^2 u)^2\,dx\right)^{\frac{1}{2}}\left(\int_{A_{\alpha R}^{\beta R}}\left(R^{-2}\left|\nabla(\Delta u)\right|^2\psi_R^{2m-2} + R^{-4}|\Delta u|^2\psi_R^{2m-4}\right.\right.$$
$$\left.\left. + R^{-6}|\nabla u|^2\psi_R^{2m-6}\right)dx\right)^{\frac{1}{2}}. \tag{3.30}$$

Similarly, we also obtain

$$\left|\int_{B_{\beta R}}\Delta^2 u\nabla\left(\langle x,\nabla(\Delta u)\rangle\right)\nabla\psi_R^{2m}\,dx\right|$$
$$= \left|\int_{B_{\beta R}}\Delta^2 u\left(\nabla(\Delta u)\nabla\psi_R^{2m} + x_i(\Delta u)_{ij}\left(\psi_R^{2m}\right)_j\right)dx\right|$$
$$\le C_m\int_{A_{\alpha R}^{\beta R}}\left|\Delta^2 u\right|\left(R^{-1}\left|\nabla(\Delta u)\right|\psi_R^{2m-1} + \left|(\Delta u)_{ij}\right|\psi_R^{2m-1}\right)dx$$
$$\le C\left(\int_{A_{\alpha R}^{\beta R}}(\Delta^2 u)^2\,dx\right)^{\frac{1}{2}}\left(\int_{A_{\alpha R}^{\beta R}}\left(R^{-1}\left|\nabla(\Delta u)\right|\psi_R^{2m-1} + \left|(\Delta u)_{ij}\right|\psi_R^{2m-1}\right)^2 dx\right)^{\frac{1}{2}}$$
$$\le C\left(\int_{A_{\alpha R}^{\beta R}}(\Delta^2 u)^2\,dx\right)^{\frac{1}{2}}\left(\int_{A_{\alpha R}^{\beta R}}\left(R^{-2}\left|\nabla(\Delta u)\right|^2\psi_R^{4m-2} + \left((\Delta u)_{ij}\right)^2\psi_R^{4m-2}\right)dx\right)^{\frac{1}{2}}. \tag{3.31}$$

Integrating by parts and using Young's inequality, we obtain

$$\begin{aligned}
&\int_{A_{\alpha R}^{\beta R}} \bigl[R^{-2}\bigl|\nabla(\Delta u)\bigr|^2\psi_R^{4m-2} + \bigl((\Delta u)_{ij}\bigr)^2\psi_R^{4m-2}\bigr]\,dx \\
&\quad\le \int_{B_{\beta R}} \bigl[R^{-2}\bigl|\nabla(\Delta u)\bigr|^2\psi_R^{4m-2} + \bigl((\Delta u)_{ij}\bigr)^2\psi_R^{4m-2}\bigr]\,dx \\
&\quad= \int_{B_{\beta R}} \bigl(\Delta^2 u\bigr)^2\psi_R^{4m-2}\,dx + \int_{B_{\beta R}} \Delta^2 u\nabla(\Delta u)\nabla\bigl(\psi_R^{4m-2}\bigr)\,dx \\
&\qquad+ \int_{B_{\beta R}} \bigl|\nabla(\Delta u)\bigr|^2\biggl[R^{-2}\psi_R^{4m-2} + \frac{1}{2}\Delta\bigl(\psi_R^{4m-2}\bigr)\biggr]\,dx \\
&\quad\le C_m\int_{B_{\beta R}} \bigl(\Delta^2 u\bigr)^2\psi_R^{2m}\,dx + C_m R^{-2}\int_{B_{\beta R}} \bigl|\nabla(\Delta u)\bigr|^2\psi_R^{2m-2}\,dx. \qquad (3.32)
\end{aligned}$$

From (3.31) and (3.32), we obtain

$$\begin{aligned}
&\biggl|\int_{B_{\beta R}} \Delta^2 u\nabla\bigl(\bigl\langle x,\nabla(\Delta u)\bigr\rangle\bigr)\nabla\psi_R^{2m}\,dx\biggr| \\
&\quad\le C\biggl(\int_{A_{\alpha R}^{\beta R}} \bigl(\Delta^2 u\bigr)^2\,dx\biggr)^{\frac{1}{2}}\biggl(\int_{B_{\beta R}} \bigl(\bigl(\Delta^2 u\bigr)^2\psi_R^{2m} + R^{-2}\bigl|\nabla(\Delta u)\bigr|^2\psi_R^{2m-2}\bigr)\,dx\biggr)^{\frac{1}{2}}. \qquad (3.33)
\end{aligned}$$

The sixth term on the right hand side of (3.18) yields

$$\begin{aligned}
&\biggl|\int_{B_{\beta R}} \Delta^2 u\bigl(\Delta\bigl(\psi_R^{2m}\bigr)\Delta\bigl(\langle x,\nabla u\rangle\bigr) + 2\nabla\bigl(\Delta\bigl(\psi_R^{2m}\bigr)\bigr)\nabla\bigl(\langle x,\nabla u\rangle\bigr)\bigr)\,dx\biggr| \\
&\quad= \biggl|\int_{B_{\beta R}} \Delta^2 u\bigl(\bigl\langle x,\nabla(\Delta u)\bigr\rangle\Delta\bigl(\psi_R^{2m}\bigr) + 2\Delta u\Delta\bigl(\psi_R^{2m}\bigr) \\
&\qquad+ 2\nabla u\nabla\bigl(\Delta\bigl(\psi_R^{2m}\bigr)\bigr) + 2x_i u_{ij}\bigl(\Delta\bigl(\psi_R^{2m}\bigr)\bigr)_j\bigr)\,dx\biggr| \\
&\quad\le C\biggl(\int_{A_{\alpha R}^{\beta R}} \bigl(\Delta^2 u\bigr)^2\,dx\biggr)^{\frac{1}{2}}\biggl(\int_{A_{\alpha R}^{\beta R}} \bigl(R^{-2}\bigl|\nabla(\Delta u)\bigr|^2\psi_R^{2m-2} + R^{-4}(\Delta u)^2\psi_R^{2m-4} \\
&\qquad+ R^{-6}|\nabla u|^2\psi_R^{2m-6} + R^{-4}\bigl|\nabla^2 u\bigr|^2\psi_R^{4m-6}\bigr)\,dx\biggr)^{\frac{1}{2}}. \qquad (3.34)
\end{aligned}$$

The last term on the right hand side of (3.18) can be estimated as

$$\begin{aligned}
&\int_{B_{\beta R}} \Delta^2 u\Delta\bigl(\nabla\bigl(\langle x,\nabla u\rangle\bigr)\nabla\bigl(\psi_R^{2m}\bigr)\bigr)\,dx \\
&\quad= \int_{B_{\beta R}} \Delta^2 u\bigl(3\nabla(\Delta u)\nabla\bigl(\psi_R^{2m}\bigr) + \nabla u\nabla\bigl(\Delta\bigl(\psi_R^{2m}\bigr)\bigr) + 2\times\nabla(u_i)\times\nabla\bigl(\psi_R^{2m}\bigr)_i\bigr)\,dx \\
&\qquad+ \int_{B_{\beta R}} \Delta^2 u\bigl(x_i\times(\Delta u)_{ij}\times\bigl(\psi_R^{2m}\bigr)_j + u_{ij}\times\bigl\{x_i\bigl(\Delta\bigl(\psi_R^{2m}\bigr)\bigr)_j + 2\bigl(\psi_R^{2m}\bigr)_{ij}\bigr\}\bigr)\,dx \\
&\qquad+ 2\int_{B_{\beta R}} x_i\Delta^2 u\times u_{ijk}\times\bigl(\psi_R^{2m}\bigr)_{jk}\,dx.
\end{aligned}$$

By Hölder's inequality and Young's inequality, we get

$$\begin{aligned}
&\left|\int_{B_{\beta R}} \Delta^2 u \Delta\big(\nabla\big(\langle x, \nabla u\rangle\big)\nabla\big(\psi_R^{2m}\big)\big)\,dx\right| \\
&\quad\le C\left(\int_{A_{\alpha R}^{\beta R}} \big(\Delta^2 u\big)^2\,dx\right)^{\frac{1}{2}} \\
&\quad\quad\times\left(\int_{B_{\beta R}} \big(R^{-2}\big|\nabla(\Delta u)\big|^2\psi_R^{4m-2} + R^{-6}|\nabla u|^2\psi_R^{4m-6} + (\Delta u)_{ij}^2 \times \psi_R^{4m-2}\right. \\
&\quad\quad\left. + R^{-4}\big|\nabla^2 u\big|^2 \times \psi_R^{4m-6} + R^{-2}\big|\nabla^3 u\big|^2 \times \psi_R^{4m-4}\big)\,dx\right)^{\frac{1}{2}} \\
&\quad\le C\left(\int_{A_{\alpha R}^{\beta R}} \big(\Delta^2 u\big)^2\,dx\right)^{\frac{1}{2}}\left(\int_{B_{\beta R}} \big(\big(\Delta^2 u\big)^2\psi_R^{2m} + \mathbf{B}(u, \psi_R, m)\right. \\
&\quad\quad\left. + R^{-2}\big|\nabla^3 u\big|^2\psi_R^{2m-2}\big)\,dx\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.35}$$

From hypothesis H_1, one has $(\theta + 1)F(s) \le f(s)s, \forall s \in \mathbb{R}$. Using the latter inequality, (3.25) and $1 < \theta < p_s(n, 4)$, we get

$$\int_{B_{\beta R}} F(u)\big\langle\nabla\psi_R^{2m}, x\big\rangle\,dx = o(1) \quad \text{as } R \to +\infty. \tag{3.36}$$

From (3.18), (3.25)-(3.36), and $1 < \theta < p_s(n, 4)$, we obtain

$$\int_{\mathbb{R}^n} \big(\Delta^2 u\big)^2\,dx = \frac{2n}{n-8}\int_{\mathbb{R}^n} F(u)\,dx.$$

Now, multiplying equation (1.1) by $u\psi_R^{2m}$ and integrating by parts, we get

$$\begin{aligned}
&\int_{B_{2R}} \big(\big(\Delta^2 u\big)^2\psi_R^{2m} - f(u)u\psi_R^{2m}\big)\,dx \\
&\quad= -\int_{B_{2R}} \Delta^2 u\big(2\Delta u\Delta\big(\psi_R^{2m}\big) + 4u_{ij}\big(\psi_R^{2m}\big)_{ij} + 4\nabla(\Delta u)\nabla\big(\psi_R^{2m}\big) \\
&\quad\quad + 4\nabla u\nabla\big(\Delta\big(\psi_R^2 m\big)\big) + u\Delta^2\big(\psi_R^{2m}\big)\big)\,dx.
\end{aligned}$$

By the same reasoning as above, we find

$$\int_{\mathbb{R}^n} \big(\Delta^2 u\big)^2\,dx = \int_{\mathbb{R}^n} f(u)u\,dx < \infty. \qquad \square$$

4 Proof of Theorems 2.1, 2.2, 2.3, 2.4 and 2.5

Proof of Theorem 2.1 The proof of Theorem 2.1 for the case $k = 1, 2, 3$ is exactly the same as in [5, 6, 8]. Now, we prove the case $k = 4$. Let u be a stable solution to (1.1).

Subcritical case: $1 < \theta < p_s(n, 4)$. By Proposition 3.1, there exists $C > 0$ such that

$$\int_{B_R} |u|^{\theta+1}\,dx \le CR^{n-8\frac{\theta+1}{\theta-1}}, \quad \forall R > 0.$$

Note that

$$n-8\frac{\theta+1}{\theta-1}=n-8-\frac{16}{\theta-1}<0,\quad \forall\theta\in\big(1,p_s(n,4)\big).$$

Then, if $1<\theta<p_s(n,4)$, after sending $R\to\infty$, we get $u\equiv 0$ in $\mathbb{R}^n$.

Critical case: $\theta=\frac{n+8}{n-8}$. By Proposition 3.1, we have

$$\int_{\mathbb{R}^n}\big((\Delta^2u)^2+|u|^{\theta+1}\big)\,dx<+\infty.$$

So,

$$\lim_{R\to+\infty}\int_{A_R}\big((\Delta^2u)^2+|u|^{\theta+1}\big)\,dx=0. \tag{4.1}$$

Moreover, if we come back to the proof of Proposition 3.1, we may improve the following integral estimates:

$$\int_{B_R}\big((\Delta^2u)^2+|u|^{\theta+1}\big)\,dx\le C\int_{A_R}\big(\Delta^2u\big)^2\,dx+CR^{-8}\int_{A_R}u^2\,dx.$$

By Hölder's inequality, we have

$$\begin{aligned}\int_{B_R}\big((\Delta^2u)^2+|u|^{\theta+1}\big)\,dx&\le C\int_{A_R}\big(\Delta^2u\big)^2\,dx+CR^{n\frac{\theta-1}{\theta+1}-8}\times\bigg(\int_{A_R}|u|^{\theta+1}\,dx\bigg)^{\frac{2}{\theta+1}}\\&\le C\int_{A_R}\big(\Delta^2u\big)^2\,dx+C\bigg(\int_{A_R}|u|^{\theta+1}\,dx\bigg)^{\frac{2}{\theta+1}}.\end{aligned}$$

Using (4.1), we get

$$\int_{\mathbb{R}^n}|u|^{\theta+1}\,dx=0.$$

This implies that $u\equiv 0$ in $\mathbb{R}^n$. □

Proof of Theorem 2.2 We now collect (3.19) and (3.20). By assumption H_3, if u is not identically zero, then

$$\begin{aligned}\int_{\mathbb{R}^n}\big|D^ku\big|^2\,dx&=\frac{2n}{n-2k}\int_{\mathbb{R}^n}F(u)\,dx\ge(1+\alpha_0)\int_{\mathbb{R}^n}f(u)u\,dx\\&>\int_{\mathbb{R}^n}f(u)u\,dx=\int_{\mathbb{R}^n}\big|D^ku\big|^2\,dx.\end{aligned}$$

This is a contradiction. Then $u\equiv 0$. The proof of Theorem 2.2 is thus completed. □

Proof of Theorem 2.3 The proof of Theorem 2.3 is similar to proof of Proposition 4 in [6]. Let $\gamma\in[1,2\theta-1+2\sqrt{\theta(\theta-1)})$. Multiply equation (1.1) by $|u|^{\gamma-1}u\varphi_R^2$ and integrate by parts to find

$$\begin{aligned}&\int_{B_{2R}}f(u)u|u|^{\gamma-1}\varphi_R^2\,dx\\&\quad=\frac{4\gamma}{(\gamma+1)^2}\int_{B_{2R}}\big|\nabla\big(|u|^{\frac{\gamma-1}{2}}u\big)\big|^2\varphi_R^2\,dx-\frac{1}{\gamma+1}\int_{B_{2R}}|u|^{\gamma+1}\Delta\big(\varphi_R^2\big)\,dx.\end{aligned} \tag{4.2}$$

The function $|u|^{\frac{\gamma-1}{2}}u\varphi_R \in C_c^1(\mathbb{R}^n)$, and thus it can be used as a test function in the quadratic form Q_u. Hence, the stability assumption on u gives

$$\begin{aligned}&\int_{B_{2R}} f'(u)|u|^{\gamma+1}\varphi_R^2\,dx\\&\quad\le \int_{B_{2R}} \bigl|\nabla\bigl(|u|^{\frac{\gamma-1}{2}}u\bigr)\bigr|^2\varphi_R^2\,dx + \int_{B_{2R}} |u|^{\gamma+1}|\nabla\varphi_R|^2\,dx - \frac{1}{2}\int_{B_{2R}} |u|^{\gamma+1}\Delta\bigl(\varphi_R^2\bigr)\,dx.\end{aligned}$$

Using (4.2) in the latter, we obtain

$$\begin{aligned}&\int_{B_{2R}} \biggl\{\bigl(f'(u)u^2-\theta f(u)u\bigr)|u|^{\gamma-1} + \biggl(\frac{4\gamma\theta}{(\gamma+1)^2}-1\biggr)\bigl|\nabla\bigl(|u|^{\frac{\gamma-1}{2}}u\bigr)\bigr|^2\biggr\}\varphi_R^2\,dx\\&\quad\le C_1(\gamma,\theta)\int_{B_{2R}} |u|^{\gamma+1}\Delta\bigl(\varphi_R^2\bigr)\,dx + \int_{B_{2R}} |u|^{\gamma+1}|\nabla\varphi_R|^2\,dx,\end{aligned}$$

where $C_1(\gamma,\theta) = (\frac{\theta}{\gamma+1}-\frac{1}{2})$. By hypothesis H_1, we obtain

$$\begin{aligned}&\biggl(\frac{4\gamma\theta}{(\gamma+1)^2}-1\biggr)\int_{B_{2R}} \bigl|\nabla\bigl(|u|^{\frac{\gamma-1}{2}}u\bigr)\bigr|^2\varphi_R^2\,dx\\&\quad\le C_1(\gamma,\theta)\int_{B_{2R}} |u|^{\gamma+1}\Delta\bigl(\varphi_R^2\bigr)\,dx + \int_{B_{2R}} |u|^{\gamma+1}|\nabla\varphi_R|^2\,dx.\end{aligned}$$

Since $\theta > 1$ and $\gamma \in [1, 2\theta-1+2\sqrt{\theta(\theta-1)})$, we have $\frac{4\gamma\theta}{(\gamma+1)^2}-1 > 0$ and

$$\int_{B_{2R}} \bigl|\nabla\bigl(|u|^{\frac{\gamma-1}{2}}u\bigr)\bigr|^2\varphi_R^2\,dx \le C(\gamma,\theta)\int_{B_{2R}} |u|^{\gamma+1}\bigl(\bigl|\Delta\bigl(\varphi_R^2\bigr)\bigr| + |\nabla\varphi_R|^2\bigr)\,dx.$$

Using again (4.2), we get

$$\int_{B_{2R}} f(u)u|u|^{\gamma-1}\varphi_R^2\,dx \le C'(\gamma,\theta)\int_{B_{2R}} |u|^{\gamma+1}\bigl(|\nabla\varphi_R|^2 + \bigl|\Delta\bigl(\varphi_R^2\bigr)\bigr|\bigr)\,dx.$$

First, we replace φ_R by φ_R^m in the latter inequality, for any $m > 2$, we derive

$$\begin{aligned}\int_{B_{2R}} f(u)u|u|^{\gamma-1}\varphi_R^{2m}\,dx &\le C(\gamma,\theta,m)\int_{B_{2R}} |u|^{\gamma+1}\varphi_R^{2m-2}\bigl(|\nabla\varphi_R|^2 + |\Delta\varphi_R|\bigr)\,dx\\&\le \frac{C}{R^2}\int_{B_{2R}} |u|^{\gamma+1}\varphi_R^{2m-2}\,dx.\end{aligned}$$

By H_1 and H_2, we get

$$\int_{B_{2R}} |u|^{\theta+\gamma}\varphi_R^{2m}\,dx \le \frac{C}{R^2}\int_{B_{2R}} |u|^{\gamma+1}\varphi_R^{2m-2}\,dx.$$

An application of Young's inequality yields

$$\int_{B_{2R}} |u|^{\theta+\gamma}\varphi_R^{2m}\,dx \le CR^{n-2\frac{\theta+\gamma}{\theta-1}} + \frac{\gamma+1}{\gamma+\theta}\int_{B_{2R}} |u|^{\gamma+\theta}\varphi_R^{(2m-2)\frac{\gamma+\theta}{\gamma+1}}\,dx.$$

Thus

$$\int_{B_R} |u|^{\theta+\gamma}\,dx \le C' R^{n-2\frac{\theta+\gamma}{\theta-1}}.$$

As in Farina's work we readily deduce, by letting $R \to +\infty$, that there is no nontrivial stable solution of (1.1), in the special case $1 < \theta < p_c(n)$. □

Proof of Theorem 2.4 We proceed as in the proof of Proposition 2.1. From (3.11) and (3.13), we deduce by replacing $f(u)$ by $-mu + \lambda|u|^{\theta-1}u - \mu|u|^{p-1}u$ that

$$\begin{aligned} &(1-\epsilon)\int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m}\,dx - \int_{B_{2R}} \left(-mu^2 + \lambda|u|^{\theta+1} - \mu|u|^{p+1}\right)\varphi_R^{2m}\,dx \\ &\quad \le C_\epsilon R^{-8}\int_{B_{2R}} u^2\varphi_R^{2m-8}\,dx \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &\int_{B_{2R}} \left(-mu^2 + \theta\lambda|u|^{\theta+1} - p\mu|u|^{p+1}\right)\varphi_R^{2m}\,dx \\ &\quad \le (1+\epsilon)\int_{B_{2R}} \left(\varphi_R^m\Delta^2 u\right)^2 + C_\epsilon R^{-8}\int_{B_{2R}} u^2\varphi_R^{2m-8}\,dx. \end{aligned} \tag{4.4}$$

Multiplying (4.3) by θ and combining it with (4.4), we derive

$$\begin{aligned} &m(\theta-1)\int_{B_{2R}} u^2\varphi_R^{2m}\,dx + \mu(\theta-p)\int_{B_{2R}} |u|^{p+1}\varphi_R^{2m}\,dx \\ &\qquad + \left[\theta(1-\epsilon)-(1+\epsilon)\right]\int_{B_{2R}} \left(\Delta^2 u\right)^2\varphi_R^{2m}\,dx \\ &\quad \le CR^{-8}\int_{B_{2R}} u^2\varphi_R^{2m-8}\,dx. \end{aligned}$$

For ϵ sufficiently small, we deduce

$$\begin{aligned} &m(\theta-1)\int_{B_{2R}} u^2\varphi_R^{2m}\,dx + \mu(\theta-p)\int_{B_{2R}} |u|^{p+1}\varphi_R^{2m}\,dx + \int_{B_{2R}} \left(\Delta^2 u\right)^2\varphi_R^{2m}\,dx \\ &\quad \le CR^{-8}\int_{B_{2R}} u^2\varphi_R^{2m-8}\,dx. \end{aligned} \tag{4.5}$$

Proof of 1. If $m > 0$ and $\theta \ge p$, then from (4.5), we deduce that

$$\int_{B_R} u^2\,dx \le CR^{-8}\int_{B_{2R}} u^2\,dx.$$

Let $J(R) := \int_{B_R} u^2\,dx$. If we iterate the above inequality, then we get

$$J(R) \le CR^{-8(k+1)} J\left(2^{k+1}R\right). \tag{4.6}$$

We deduce from the boundedness of u that the right hand side of (4.6) is of order R^M with $M = -8(k+1) + n \to 0$ as $k \to +\infty$. Hence, we can choose k large enough such that $M < 0$.

Then it follows from (4.6) that $J(R) \to 0, \text{as} R \to +\infty$. So we get

$$\int_{\mathbb{R}^n} u^2\,dx = 0.$$

Then $u \equiv 0$.

Proof of 2. If $\theta > p$, $m \geq 0$, then from (4.5) and by Young's inequality, we get

$$\int_{B_{2R}} |u|^{p+1}\varphi_R^{2m}\,dx + \int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m}\,dx \leq \frac{2}{p+1}\int_{B_{2R}} |u|^{p+1}\varphi_R^{(p+1)(m-4)}\,dx + CR^{n-8\frac{p+1}{p-1}}.$$

Choosing $2m = (p+1)(m-4)$, thus

$$\int_{B_{2R}} |u|^{p+1}\varphi_R^{2m}\,dx + \int_{B_{2R}} \left(\Delta^2 u\right)^2 \varphi_R^{2m}\,dx \leq CR^{n-8\frac{p+1}{p-1}}.$$

Consequently

$$\int_{B_R} |u|^{p+1}\,dx + \int_{B_R} \left(\Delta^2 u\right)^2\,dx \leq CR^{n-8\frac{p+1}{p-1}}.$$

The result then follows in a similar way to that in the proof of Theorem 2.1. This completes the proof of Theorem 2.4. □

Proof of Theorem 2.5 We can proceed as in the proof of Proposition 3.4, we get

$$\int_{\mathbb{R}^n} \left|D^k u\right|^2 = \frac{2n}{n-2k}\int_{\mathbb{R}^n} \left(-\frac{m}{2}u^2 + \frac{\lambda}{\theta+1}|u|^{\theta+1} - \frac{\mu}{p+1}|u|^{p+1}\right)$$

and

$$\int_{\mathbb{R}^n} \left|D^k u\right|^2 = \int_{\mathbb{R}^n} \left(-mu^2 + \lambda|u|^{\theta+1} - \mu|u|^{p+1}\right).$$

Thus

$$\begin{aligned} &\frac{2mk}{n-2k}\int_{\mathbb{R}^n} u^2\,dx + \lambda\left(1 - \frac{2n}{(n-2k)(\theta+1)}\right)\int_{\mathbb{R}^n} |u|^{\theta+1}\,dx \\ &\quad + \mu\left(\frac{2n}{(n-2k)(p+1)} - 1\right)\int_{\mathbb{R}^n} |u|^{p+1}\,dx = 0. \end{aligned}$$

This concludes the proof of Theorem 2.5. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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