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Optimal partial regularity of very weak solutions to nonhomogeneous A-harmonic systems

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Abstract

We study partial regularity of very weak solutions to some nonhomogeneous A-harmonic systems. To obtain the reverse Hölder inequality of the gradient of a very weak solution, we construct a suitable test function by Hodge decomposition. With the aid of Gehring's lemma, we prove that these very weak solutions are weak solutions. Further, we show that these solutions are in fact optimal Hölder continuity based on A-harmonic approximation technique.

Keywords: nonhomogeneous A-harmonic systems; very weak solution; optimal partial regularity; Hodge decomposition; A-harmonic approximation technique

1 Introduction

We consider optimal partial regularity of very weak solutions to nonhomogeneous A-harmonic systems of the following type:

$$-\operatorname{div} A(x, u, Du) = f(x), \quad (1.1)$$

where $u : \Omega \rightarrow \mathbb{R}^N$ is a vector-valued function on a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$), and $Du = \{D_\alpha u^i\}$ ($1 \leq \alpha \leq n, 1 \leq i \leq N$) stands for the gradient matrix of u , $A(x, u, \xi) : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \rightarrow \mathbb{R}^{nN}$ is a measurable function, and $A_i^\alpha(x, u, \xi)$ ($1 \leq \alpha \leq n, 1 \leq i \leq N$) are of class C^1 in ξ . To define the very weak solutions to systems (1.1) and obtain the optimal partial regularity results, we need to impose certain structural and regularity conditions on A and to restrict u and f to a particular class of functions as follows: for some $p \geq 2$,

(H1) A is a bounded operator, that is, there exists a constant $\beta > 0$ such that

$$|A(x, u, \xi)| \leq \beta(1 + |\xi|^2)^{\frac{p-1}{2}} \quad \text{for all } (x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN};$$

(H2) A is differentiable with respect to $\xi \in \mathbb{R}^{nN}$, that is, there exists a constant $\alpha > 0$ such that

$$D_\xi A(x, u, \xi) \zeta \cdot \zeta \geq \alpha(1 + |\xi|^2)^{\frac{p-2}{2}} |\zeta|^2$$

for all $x \in \Omega, u \in \mathbb{R}^N$, and $\xi, \zeta \in \mathbb{R}^{nN}$;

(H3) there exist a constant $\gamma \in (0, 1)$ and a nondecreasing function $K : [0, \infty) \rightarrow [0, \infty)$ such that

$$|A(x, u, \xi) - A(\tilde{x}, \tilde{u}, \xi)| \leq K(|u|) (|x - \tilde{x}|^p + |u - \tilde{u}|^p)^{\frac{\gamma}{p}} (1 + |\xi|)^{\frac{p}{2}}$$

for all $x, \tilde{x} \in \Omega, u, \tilde{u} \in R^N$ and $\xi \in R^{nN}$; without loss of generality, we take $K \geq 1$;

(H4) f is a given vector field in R^N of class $L^{\frac{nq}{n(p-1)+q}}_{loc}(\Omega), q > p$.

Under these assumptions, we can now define very weak solutions to (1.1).

Definition 1 A mapping $u \in W^{1,r}_{loc}(\Omega), p - 1 \leq r < p$, is called a very weak solution to (1.1) if

$$\int_{\Omega} A(x, u, Du) \cdot D\phi \, dx = \int_{\Omega} f(x)\phi \, dx \tag{1.2}$$

for all $\phi \in W^{1, \frac{r}{r-p+1}}_0(\Omega)$.

In order to improve the integrability of a very weak solution to (1.1), we need to prove a suitable reverse Hölder inequality. In 1973, Gehring [1] discovered the crucial self-improving property of the reverse Hölder inequality and applied it to establish higher integrability of n -dimensional k -quasiconformal mapping. Subsequently, Meyers and Elcrat [2] generalized this inequality based on Caccioppoli’s inequality. They improved the integrability of weak solutions to nonlinear elliptic systems with the help of Gehring’s lemma. Especially, they pointed out that regularity properties remained valid in a somewhat slightly larger Sobolev space to linear elliptic systems depending on the duality. In fact, this regularity result about very weak solutions was first showed by Meyers [3] in 1963. Unfortunately, neither the method used in [2] for proving the reverse Hölder inequality nor the duality employed in [2, 3] can be applied to deal with very weak solutions to nonlinear elliptic systems. To overcome these difficulties, Lewis [4] used the technique of harmonic analysis and successfully proved that very weak solutions to nonlinear elliptic systems are indeed weak solutions. Later Iwaniec and Sbordone [5] achieved a similar result via the methods of Hodge decomposition and prior estimation.

Since then, studies on properties of very weak solutions to partial differential equations, especially for regularity of very weak solutions to A-harmonic systems, have attracted considerable attention. Following the method of Iwaniec and Sbordone [5], Giachetti, Leonetti, and Schiachi [6] obtained the partial regularity result of A-harmonic systems $\operatorname{div} A(x, u, Du) = 0$. Tong, Gu, and Xu [7] extended their result to nonhomogeneous A-harmonic systems $\operatorname{div} A(x, Du) = f(x)$ and improved the integrability of very weak solutions. Greco, Iwaniec, and Sbordone [8] even applied this method to the p -harmonic equation $\operatorname{div} |Du|^{p-2} Du = \operatorname{div} f$.

Motivated by these works, we mainly consider the optimal partial regularity to nonhomogeneous A-harmonic systems in the form of (1.1) under assumptions (H1)-(H4).

For the sake of desired results, we first need to improve the exponent of integrability for the gradient of a very weak solution to an even slightly better one than the natural exponent p . The crucial difficulty is to construct an appropriate test function below the natural exponent. In this article, we follow the spirit of Iwaniec and Sbordone [5] using the Hodge

decomposition to construct it. Combining the Sobolev imbedding theorem, Young’s inequality, Poincaré’s inequality, and so on, we improve the exponents of integrability of very weak solutions to (1.1). In other words, we successfully prove that very weak solutions to (1.1) are in fact weak solutions. More precisely, we obtain the following result.

Theorem 1 *Let u be a very weak solution to systems (1.1). Assume that the structure conditions (H1), (H2), and (H4) hold. Then there exist exponents $p - 1 < r_1 = r_1(n, N, p, \alpha, \beta) < p < r_2 = r_2(n, N, p, \alpha, \beta) < \infty$ such that $u \in W_{loc}^{1,r_1}(\Omega)$ belongs to $W_{loc}^{1,r_2}(\Omega)$.*

A direct consequence of this result follows immediately.

Corollary 1 *Under the assumptions of Theorem 1, there exists $r_1 = r_1(n, N, p, \alpha, \beta) < p$ such that every very weak solution $u \in W_{loc}^{1,r}(\Omega)$ with $r_1 < r < p$ belongs to $W_{loc}^{1,p}(\Omega)$.*

Further, we establish the optimal partial regularity result of very weak solutions to (1.1). Generally speaking, we cannot expect that weak solutions to (1.1) will be C^2 -solutions even under reasonable assumptions on operator A and f . This is initially pointed out by De Giorgi [9, 10] and Giusti and Miranda [11]. Thus, our aim is to obtain the optimal Hölder continuity of very weak solutions to (1.1). Fortunately, we achieve it by means of A -harmonic approximation technique and obtain the optimal Hölder continuity $C^{1,\gamma}$ in the regular set of the following:

Theorem 2 *Let $u \in W_{loc}^{1,r}(\Omega)$, $r_1 < r < p$, be a very weak solution to (1.1). Consider r_1 as in Corollary 1. Suppose that assumptions (H1)-(H4) hold. Then there exists an open set $\Omega_0 \subset \Omega$ such that $u \in C^{1,\gamma}(\Omega_0)$ for γ is defined in (H3). We have*

$$\Omega - \Omega_0 = \Sigma_1 \cup \Sigma_2,$$

where

$$\Sigma_1 = \left\{ x_0 \in \Omega : \liminf_{R \rightarrow 0^+} \int_{B_R(x_0)} |Du - (Du)_{x_0,R}|^p dx > 0 \right\}$$

and

$$\Sigma_2 = \left\{ x_0 \in \Omega : \limsup_{R \rightarrow 0^+} (|u_{x_0,R}| + |(Du)_{x_0,R}|) = \infty \right\}.$$

Moreover, we have $|\Omega - \Omega_0| = 0$.

To close this section, we briefly summarize the notation used in this paper. As noted before, we consider a bounded domain $\Omega \subset R^n (n \geq 2)$ and mappings from Ω to R^N . We write $B_r(x_0) = \{x \in \Omega : |x - x_0| < r\}, x_0 \in \Omega$. For a given set X , we denote by $|X|$ its n -dimensional Lebesgue measure. If $|X| > 0$, then the average of a given $g \in L^1(X)$ over X is denoted by $f_X g dx$, that is, $f_X g dx = \frac{1}{|X|} \int_X g dx$. In particular, we write $g_{x_0,r} = f_{B_r(x_0)} g dx$. Let α_n denote the volume of the unit ball in R^n , that is, $\alpha_n = |B_1(0)|$, then $|B_r(x_0)| = \alpha_n r^n$.

The rest of this paper is arranged as follows. In Section 2, we provide some necessary preliminary lemmas. In Section 3, we prove the main results.

2 Preliminary lemmas

Before proving the results, we state a few useful lemmas.

The first one is a stability result of the Hodge decomposition, from which we could construct a suitable test-function concerning estimates below the natural exponent for (1.1).

Lemma 1 ([5]) *Let $\Omega \subset R^n$ be a regular domain, and $w \in W_0^{1,r}(\Omega, R^N), r > 1$, and let $-1 < \epsilon < r - 1$. Then there exist $\phi \in W_0^{1, \frac{r}{1+\epsilon}}(\Omega, R^N)$ and a divergence-free matrix field $H \in L^{\frac{r}{1+\epsilon}}(\Omega, R^{nN})$ such that*

$$|Dw|^\epsilon Dw = D\phi + H. \tag{2.1}$$

Moreover,

$$\|H\|_{\frac{r}{1+\epsilon}} \leq C_r(\Omega, N) |\epsilon| \|Dw\|_r^{1+\epsilon}. \tag{2.2}$$

The most useful case for us in Lemma 1 is where ϵ is negative. For $u \in W_{loc}^{1,r}(\Omega)$ with $p - 1 < r < p$ that is a very weak solution to (1.1), we can set $\epsilon = r - p$ ($-1 < \epsilon < 0$). Then there exists $\phi \in W_0^{1, \frac{r}{1+r-p}}(\Omega)$; thus, ϕ can be illustrated as a test-function in (1.2). In view of (2.1) and (2.2), we also can get an estimate of $D\phi$, which is similar to (2.2).

Applying Lemma 2, we can decompose the left term of the Hodge decomposition into two terms that could be controlled more easily in the proof of Theorem 1.

Lemma 2 ([12]) *For every $X, Y \in R^n, X \neq 0, Y \neq 0$, and $0 \leq \epsilon < 1$, we have the inequality*

$$||X|^{-\epsilon} X - |Y|^{-\epsilon} Y| \leq 2^\epsilon \frac{1 + \epsilon}{1 - \epsilon} |X - Y|^{1-\epsilon}.$$

In the end of this section, we shall introduce a form of Gehring’s lemma, which plays an important role in the proof of Theorem 1. It implies in particular that from it higher integrability of $g(x)$ follows.

Lemma 3 ([2, 13]) *Let $0 < R < R_0 \leq \text{dist}(x_0, \partial\Omega), x_0 \in \Omega$. Suppose that $g(x) \in L^p(B_R(x_0)), f(x) \in L^t(B_R(x_0)), t > p, 1 < p < \infty$, satisfy the reverse Hölder inequality*

$$\int_{B_{R/2}(x_0)} |g(x)|^p dx \leq \theta \int_{B_R(x_0)} |g(x)|^p dx + C_* \left[\int_{B_R(x_0)} |g(x)|^s dx \right]^{p/s} + \int_{B_R(x_0)} |f(x)|^p dx$$

for some $1 \leq s < p, 0 \leq \theta < 1$. Then $g \in L_{loc}^{p'}(\Omega)$ for some $p' = p'(\theta, p, n, C_*)$ ($t \geq p' > p$), and

$$\left[\int_{B_{R/2}(x_0)} |g(x)|^{p'} dx \right]^{1/p'} \leq C_* \left[\int_{B_R(x_0)} |g(x)|^p dx \right]^{1/p} + C_* \left[\int_{B_R(x_0)} |f(x)|^{p'} dx \right]^{1/p'},$$

where $C_* = C_*(n, C_*, p, \theta, R_0)$.

3 Proof of the main theorems

In this section, we give a proof of partial regularity results. Consider u solving (1.1) on $B_R(x_0) \Subset \Omega$, where we restrict $0 < R < R_0 < \min\{1, \text{dist}(x_0, \partial\Omega)\}$.

3.1 Proof of Theorem 1

Proof Fix a cut-off function $\eta \in C_0^\infty(B_R(x_0))$ satisfying $0 \leq \eta \leq 1$, $|D\eta| \leq C/R$, and $\eta \equiv 1$ on $B_{R/2}(x_0)$. Let $u \in W_{loc}^{1,r}(B_R(x_0))$ with $p - 1 < r < p$ be a very weak solution to (1.1). Denote $u - u_{x_0,R} - p_0(x - x_0)$ by v , where $p_0 \in R^{nN}$. We find that v has calculus mean-value 0 on $B_R(x_0)$, that is, $v_{x_0,R} = 0$. Notice that $\eta v \in W_0^{1,r}(B_R(x_0))$ and $-1 < r - p < 0$. Then there exist $\phi \in W_0^{1, \frac{r}{1+r-p}}(B_R(x_0))$ and $h \in L^{\frac{r}{1+r-p}}(B_R(x_0))$ such that $|D(\eta v)|^{r-p} D(\eta v) = D\phi + h$ according to the Hodge decomposition. Thus, ϕ is admissible as a test-function in the definition of very weak solutions. Set $-\varepsilon = r - p$ ($-1 < -\varepsilon < 0$) for convenience. Then $r = p - \varepsilon$, and we have

$$|D(\eta v)|^{-\varepsilon} D(\eta v) = D\phi + h, \tag{3.1}$$

where h satisfies

$$\|h\|_{\frac{p-\varepsilon}{1-\varepsilon}} \leq C_r(\Omega, N)\varepsilon \|D(\eta v)\|_{p-\varepsilon}^{1-\varepsilon}. \tag{3.2}$$

Further, applying Poincaré’s inequality with constant C_p and noting that $v_{x_0,R} = 0$, we get

$$\begin{aligned} \|D(\eta v)\|_{p-\varepsilon}^{1-\varepsilon} &\leq (\|vD\eta\|_{p-\varepsilon} + \|\eta Dv\|_{p-\varepsilon})^{1-\varepsilon} \\ &\leq \left(\frac{C}{R}\|v\|_{p-\varepsilon} + \|Dv\|_{p-\varepsilon}\right)^{1-\varepsilon} \\ &\leq (CC_p^{\frac{1}{p-\varepsilon}}\|Dv\|_{p-\varepsilon} + \|Dv\|_{p-\varepsilon})^{1-\varepsilon} \\ &< (1 + CC_p^{\frac{1}{p-\varepsilon}})\|Dv\|_{p-\varepsilon}^{1-\varepsilon}. \end{aligned} \tag{3.3}$$

In view of (3.2) and (3.3), we have

$$\|h\|_{\frac{p-\varepsilon}{1-\varepsilon}} \leq C_1\varepsilon\|Dv\|_{p-\varepsilon}^{1-\varepsilon}, \tag{3.4}$$

where $C_1 = C_r(\Omega, N)(1 + CC_p^{\frac{1}{p-\varepsilon}})$.

In particular, combining (3.1) and (3.2), we find

$$\begin{aligned} \|D\phi\|_{\frac{p-\varepsilon}{1-\varepsilon}} &= \left\| |D(\eta v)|^{-\varepsilon} D(\eta v) - h \right\|_{\frac{p-\varepsilon}{1-\varepsilon}} \\ &\leq \left\| |D(\eta v)|^{-\varepsilon} D(\eta v) \right\|_{\frac{p-\varepsilon}{1-\varepsilon}} + \|h\|_{\frac{p-\varepsilon}{1-\varepsilon}} \\ &\leq \|D(\eta v)\|_{p-\varepsilon}^{1-\varepsilon} + C_r(\Omega, N)\varepsilon \|D(\eta v)\|_{p-\varepsilon}^{1-\varepsilon} \\ &\leq (1 + C_r(\Omega, N)\varepsilon) \|D(\eta v)\|_{p-\varepsilon}^{1-\varepsilon}. \end{aligned}$$

Substituting (3.3) into this estimate, we have

$$\|D\phi\|_{\frac{p-\varepsilon}{1-\varepsilon}} \leq C_2\|Dv\|_{p-\varepsilon}^{1-\varepsilon}, \tag{3.5}$$

where $C_2 = (1 + C_r(\Omega, N)\varepsilon)(1 + CC_p^{\frac{1}{p-\varepsilon}})$.

Since it is hard to estimate $|D(\eta v)|^{-\varepsilon} D(\eta v)$ directly, we set

$$E(\eta, v) = |D(\eta v)|^{-\varepsilon} D(\eta v) - |\eta Dv|^{-\varepsilon} \eta Dv,$$

which by Lemma 2 yields

$$|E(\eta, v)| \leq 2^\varepsilon \frac{1 + \varepsilon}{1 - \varepsilon} |v D\eta|^{1-\varepsilon}.$$

Joining $E(\eta, v)$ with (3.1), we arrive at

$$D\phi = E(\eta, v) + |\eta Dv|^{-\varepsilon} \eta Dv - h.$$

Inserting $D\phi$ into equality (1.2), we get

$$\begin{aligned} & \int_{B_R(x_0)} A(x, u, Du) \cdot |\eta Dv|^{-\varepsilon} \eta Dv \, dx \\ &= \int_{B_R(x_0)} A(x, u, Du) \cdot h \, dx - \int_{B_R(x_0)} A(x, u, Du) \cdot E(\eta, v) \, dx + \int_{B_R(x_0)} f(x)\phi \, dx. \end{aligned} \tag{3.6}$$

In order to use (H2), we need to transform the left-hand side of (3.6) as follows:

$$\begin{aligned} & \int_{B_R(x_0)} A(x, u, Du) \cdot |\eta Dv|^{-\varepsilon} \eta Dv \, dx \\ &= \int_{B_R(x_0)} (A(x, u, Du) - A(x, u, p_0) + A(x, u, p_0)) \cdot |\eta Dv|^{-\varepsilon} \eta Dv \, dx \\ &= \int_{B_R(x_0)} (A(x, u, Du) - A(x, u, p_0)) \cdot |\eta Dv|^{-\varepsilon} \eta Dv \, dx \\ & \quad + \int_{B_R(x_0)} A(x, u, p_0) \cdot |\eta Dv|^{-\varepsilon} \eta Dv \, dx. \end{aligned}$$

Combining this equality with (3.6), we find

$$\begin{aligned} & \int_{B_R(x_0)} (A(x, u, Du) - A(x, u, p_0)) \cdot |\eta Dv|^{-\varepsilon} \eta Dv \, dx \\ &= - \int_{B_R(x_0)} A(x, u, p_0) \cdot |\eta Dv|^{-\varepsilon} \eta Dv \, dx + \int_{B_R(x_0)} A(x, u, Du) \cdot h \, dx \\ & \quad - \int_{B_R(x_0)} A(x, u, Du) \cdot E(\eta, v) \, dx + \int_{B_R(x_0)} f(x)\phi \, dx \\ & \leq I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} I_1 &= \left| - \int_{B_R(x_0)} A(x, u, p_0) \cdot |\eta Dv|^{-\varepsilon} \eta Dv \, dx \right|; \\ I_2 &= \left| \int_{B_R(x_0)} A(x, u, Du) \cdot h \, dx \right|; \end{aligned}$$

$$I_3 = \left| - \int_{B_R(x_0)} A(x, u, Du) \cdot E(\eta, v) \, dx \right|;$$

$$I_4 = \left| \int_{B_R(x_0)} f(x)\phi \, dx \right|.$$

Consequently, we shall derive estimate for each term of (3.7) so as to establish a reverse Hölder inequality for $|Du - p_0|^{p-\varepsilon}$.

In the case of the term on the left-hand side of (3.7), we want to derive an estimate from below in terms of $\int_{B_{R/2}(x_0)} |Du - p_0|^{p-\varepsilon} \, dx$. For this purpose, we need the inequality

$$(A(x, u, \zeta) - A(x, u, \xi)) \cdot (\zeta - \xi) \geq \alpha(1 + |\zeta|^2 + |\xi|^2)^{\frac{p-2}{2}} |\zeta - \xi|^2,$$

which can be deduced from (H2) immediately.

Then we infer that

$$\begin{aligned} & \int_{B_R(x_0)} (A(x, u, Du) - A(x, u, p_0)) \cdot |\eta Dv|^{-\varepsilon} \eta Dv \, dx \\ &= \int_{B_R(x_0)} |\eta Dv|^{-\varepsilon} \eta (A(x, u, Du) - A(x, u, p_0)) \cdot (Du - p_0) \, dx \\ &\geq \alpha \int_{B_R(x_0)} |\eta Dv|^{-\varepsilon} \eta (1 + |Du|^2 + |p_0|^2)^{\frac{p-2}{2}} |Du - p_0|^2 \, dx \\ &\geq \alpha \int_{B_{R/2}(x_0)} |Du - p_0|^{2-\varepsilon} \left(1 + \frac{|Du - p_0|^2}{2}\right)^{\frac{p-2}{2}} \, dx \\ &\geq 2^{\frac{2-p}{2}} \alpha \int_{B_{R/2}(x_0)} |Du - p_0|^{p-\varepsilon} \, dx. \end{aligned} \tag{3.8}$$

Using (H1) and Young’s inequality with exponents $\frac{p-\varepsilon}{1-\varepsilon}$ and $\frac{p-\varepsilon}{p-1}$, we find that, for $\varepsilon_1 > 0$,

$$\begin{aligned} I_1 &\leq \int_{B_R(x_0)} |A(x, u, p_0)| |Dv|^{1-\varepsilon} \, dx \\ &\leq \beta(1 + |p_0|^2)^{\frac{p-1}{2}} \int_{B_R(x_0)} |Du - p_0|^{1-\varepsilon} \, dx \\ &\leq \beta(1 + |p_0|^2)^{\frac{p-1}{2}} \int_{B_R(x_0)} (\varepsilon_1 |Du - p_0|^{p-\varepsilon} + \varepsilon_1^{-\frac{1-\varepsilon}{p-1}} 1^{\frac{p-\varepsilon}{p-1}}) \, dx \\ &\leq \beta(1 + |p_0|^2)^{\frac{p-1}{2}} \varepsilon_1 \int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} \, dx \\ &\quad + \beta(1 + |p_0|^2)^{\frac{p-1}{2}} \varepsilon_1^{-\frac{1-\varepsilon}{p-1}} \int_{B_R(x_0)} \, dx. \end{aligned} \tag{3.9}$$

By (H1) we have

$$\begin{aligned} I_2 &\leq \int_{B_R(x_0)} |A(x, u, Du)| |h| \, dx \\ &\leq \beta \int_{B_R(x_0)} (1 + |Du|^2)^{\frac{p-1}{2}} |h| \, dx \end{aligned}$$

$$\begin{aligned} &\leq \beta \int_{B_R(x_0)} (2 + |Du - p_0 + p_0|^2)^{\frac{p-1}{2}} |h| dx \\ &\leq \beta \int_{B_R(x_0)} (2(1 + |p_0|^2) + 2|Du - p_0|^2)^{\frac{p-1}{2}} |h| dx \\ &\leq 2^{p-1} \beta \int_{B_R(x_0)} |Du - p_0|^{p-1} |h| dx + 2^{p-1} \beta (1 + |p_0|^2)^{\frac{p-1}{2}} \int_{B_R(x_0)} |h| dx. \end{aligned}$$

Applying both Hölder’s inequality and Young’s inequality with exponents $\frac{p-\varepsilon}{p-1}$ and $\frac{p-\varepsilon}{1-\varepsilon}$, by (3.4) we further have that, for $\varepsilon_2 > 0$,

$$\begin{aligned} I_2 &\leq 2^{p-1} \beta \left(\int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx \right)^{\frac{p-1}{p-\varepsilon}} \left(\int_{B_R(x_0)} |h|^{\frac{p-\varepsilon}{1-\varepsilon}} dx \right)^{\frac{1-\varepsilon}{p-\varepsilon}} \\ &\quad + 2^{p-1} \beta (1 + |p_0|^2)^{\frac{p-1}{2}} \left(\int_{B_R(x_0)} 1^{p-\varepsilon} dx \right)^{\frac{p-1}{p-\varepsilon}} \left(\int_{B_R(x_0)} |h|^{\frac{p-\varepsilon}{1-\varepsilon}} dx \right)^{\frac{1-\varepsilon}{p-\varepsilon}} \\ &\leq 2^{p-1} \beta C_1 \varepsilon \left(\int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx \right)^{\frac{p-1}{p-\varepsilon}} \left(\int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx \right)^{\frac{1-\varepsilon}{p-\varepsilon}} \\ &\quad + 2^{p-1} \beta (1 + |p_0|^2)^{\frac{p-1}{2}} C_1 \varepsilon \left(\int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx \right)^{\frac{1-\varepsilon}{p-\varepsilon}} \left(\int_{B_R(x_0)} dx \right)^{\frac{p-1}{p-\varepsilon}} \\ &\leq 2^{p-1} \beta C_1 \varepsilon \int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx \\ &\quad + 2^{p-1} \beta (1 + |p_0|^2)^{\frac{p-1}{2}} C_1 \varepsilon \left(\varepsilon_2 \int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx + \varepsilon_2^{-\frac{1-\varepsilon}{p-1}} \int_{B_R(x_0)} dx \right) \\ &\leq 2^{p-1} \beta C_1 \varepsilon \int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx + 2^{p-1} \beta (1 + |p_0|^2)^{\frac{p-1}{2}} C_1 \varepsilon \varepsilon_2^{-\frac{1-\varepsilon}{p-1}} \int_{B_R(x_0)} dx \\ &\quad + 2^{p-1} \beta (1 + |p_0|^2)^{\frac{p-1}{2}} C_1 \varepsilon \varepsilon_2 \int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx. \tag{3.10} \end{aligned}$$

Combining (H1) and the estimate of $E(\eta, \nu)$, we find that

$$\begin{aligned} I_3 &\leq \int_{B_R(x_0)} |A(x, u, Du)| |E(\eta, \nu)| dx \\ &\leq \int_{B_R(x_0)} \beta (1 + |Du|^2)^{\frac{p-1}{2}} 2^\varepsilon \frac{1 + \varepsilon}{1 - \varepsilon} |\nu D\eta|^{1-\varepsilon} dx \\ &\leq \beta 2^\varepsilon \frac{1 + \varepsilon}{1 - \varepsilon} \left(\frac{C}{R} \right)^{1-\varepsilon} \int_{B_R(x_0)} (1 + |Du|^2)^{\frac{p-1}{2}} |\nu|^{1-\varepsilon} dx \\ &\leq \beta 2^\varepsilon \frac{1 + \varepsilon}{1 - \varepsilon} \left(\frac{C}{R} \right)^{1-\varepsilon} \int_{B_R(x_0)} (2(1 + |p_0|^2) + 2|Du - p_0|^2)^{\frac{p-1}{2}} |\nu|^{1-\varepsilon} dx \\ &\leq \beta 2^\varepsilon \frac{1 + \varepsilon}{1 - \varepsilon} \left(\frac{C}{R} \right)^{1-\varepsilon} 2^{p-1} \int_{B_R(x_0)} ((1 + |p_0|^2)^{\frac{p-1}{2}} + |Du - p_0|^{p-1}) |\nu|^{1-\varepsilon} dx. \end{aligned}$$

Denoting $\beta 2^\varepsilon \frac{1+\varepsilon}{1-\varepsilon} \left(\frac{C}{R}\right)^{1-\varepsilon} 2^{p-1}$ by C_3 , we have

$$I_3 \leq C_3 (1 + |p_0|^2)^{\frac{p-1}{2}} K_1 + C_3 K_2, \tag{3.11}$$

where $K_1 = \int_{B_R(x_0)} |v|^{1-\varepsilon} dx$ and $K_2 = \int_{B_R(x_0)} |Du - p_0|^{p-1} |v|^{1-\varepsilon} dx$. Let us estimate K_1 and K_2 . Using Young's inequality with exponents $\frac{p-\varepsilon}{1-\varepsilon}$ and $\frac{p-\varepsilon}{p-1}$ and Poincaré's inequality with constant C_p , we find that, for $\varepsilon_3 > 0$,

$$\begin{aligned} K_1 &\leq \int_{B_R(x_0)} (\varepsilon_3 |v|^{p-\varepsilon} + \varepsilon_3^{-\frac{1-\varepsilon}{p-1}} 1^{\frac{p-\varepsilon}{p-1}}) dx \\ &\leq \varepsilon_3 \int_{B_R(x_0)} |v|^{p-\varepsilon} dx + \varepsilon_3^{-\frac{1-\varepsilon}{p-1}} \int_{B_R(x_0)} dx \\ &\leq \varepsilon_3 C_p \int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx + \varepsilon_3^{-\frac{1-\varepsilon}{p-1}} \int_{B_R(x_0)} dx. \end{aligned} \tag{3.12}$$

Letting $p' = \frac{n(p-\varepsilon)}{(n+1-\varepsilon)(p-1)}$ and $q' = \frac{n(p-\varepsilon)}{(n-p+1)(1-\varepsilon)}$, we see that $1 < p' < \infty$, $1 < q' < \infty$, and $\frac{1}{p'} + \frac{1}{q'} = 1$. With the aid of Hölder's inequality, we can estimate

$$\begin{aligned} K_2 &\leq \left(\int_{B_R(x_0)} |Du - p_0|^{(p-1)p'} dx \right)^{\frac{1}{p'}} \left(\int_{B_R(x_0)} |v|^{(1-\varepsilon)q'} dx \right)^{\frac{1}{q'}} \\ &\leq \left(\int_{B_R(x_0)} |Du - p_0|^{\frac{n(p-\varepsilon)}{n+1-\varepsilon}} dx \right)^{\frac{(n+1-\varepsilon)(p-1)}{n(p-\varepsilon)}} \left(\int_{B_R(x_0)} |v|^{\frac{n(p-\varepsilon)}{n-p+1}} dx \right)^{\frac{(n-p+1)(1-\varepsilon)}{n(p-\varepsilon)}}. \end{aligned}$$

Now we set $p'' = \frac{n(p-\varepsilon)}{n+1-\varepsilon}$. Then $\frac{np''}{n-p''} = \frac{n(p-\varepsilon)}{n-p+1}$. Using the Sobolev-Poincaré inequality with constant C_s , we get

$$\begin{aligned} K_2 &\leq C_s^{1-\varepsilon} \left(\int_{B_R(x_0)} |Du - p_0|^{\frac{n(p-\varepsilon)}{n+1-\varepsilon}} dx \right)^{\frac{(n+1-\varepsilon)(p-1)}{n(p-\varepsilon)}} \left(\int_{B_R(x_0)} |Du - p_0|^{\frac{n(p-\varepsilon)}{n+1-\varepsilon}} dx \right)^{\frac{(n+1-\varepsilon)(1-\varepsilon)}{n(p-\varepsilon)}} \\ &\leq C_s^{1-\varepsilon} \left(\int_{B_R(x_0)} |Du - p_0|^{\frac{n(p-\varepsilon)}{n+1-\varepsilon}} dx \right)^{\frac{n+1-\varepsilon}{n}}. \end{aligned} \tag{3.13}$$

Combining (3.11) with (3.12) and (3.13), we obtain the estimate for I_3 :

$$\begin{aligned} I_3 &\leq C_3 (1 + |p_0|^2)^{\frac{p-1}{2}} \varepsilon_3 C_p \int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx \\ &\quad + C_3 (1 + |p_0|^2)^{\frac{p-1}{2}} \varepsilon_3^{-\frac{1-\varepsilon}{p-1}} \int_{B_R(x_0)} dx \\ &\quad + C_3 C_s^{1-\varepsilon} \left(\int_{B_R(x_0)} |Du - p_0|^{\frac{n(p-\varepsilon)}{n+1-\varepsilon}} dx \right)^{\frac{n+1-\varepsilon}{n}}. \end{aligned} \tag{3.14}$$

Finally, we estimate I_4 . Using Hölder's inequality with exponents $\frac{n(p-\varepsilon)}{n(p-1)+p-\varepsilon}$ and $\frac{n(p-\varepsilon)}{n(1-\varepsilon)-p+\varepsilon}$, we have

$$\begin{aligned} I_4 &\leq \int_{B_R(x_0)} |f(x)| |\phi| dx \\ &\leq \left(\int_{B_R(x_0)} |f|^{\frac{n(p-\varepsilon)}{n(p-1)+p-\varepsilon}} dx \right)^{\frac{n(p-1)+p-\varepsilon}{n(p-\varepsilon)}} \left(\int_{B_R(x_0)} |\phi|^{\frac{n(p-\varepsilon)}{n(1-\varepsilon)-p+\varepsilon}} dx \right)^{\frac{n(1-\varepsilon)-p+\varepsilon}{n(p-\varepsilon)}}. \end{aligned}$$

Setting $p''' = \frac{p-\varepsilon}{1-\varepsilon}$, we have $\frac{np'''}{n-p'''} = \frac{n(p-\varepsilon)}{n(1-\varepsilon)-p+\varepsilon}$. Notice that $\phi \in W_0^{1, \frac{p-\varepsilon}{1-\varepsilon}}(B_R(x_0))$. Therefore, we can apply the Sobolev-Poincaré inequality to get

$$I_4 \leq C_s \left(\int_{B_R(x_0)} |f|^{\frac{n(p-\varepsilon)}{n(p-1)+p-\varepsilon}} dx \right)^{\frac{n(p-1)+p-\varepsilon}{n(p-\varepsilon)}} \left(\int_{B_R(x_0)} |D\phi|^{\frac{p-\varepsilon}{1-\varepsilon}} dx \right)^{\frac{1-\varepsilon}{p-\varepsilon}}.$$

Combining this with (3.5), with the aid of Young’s inequality, we obtain, for $\varepsilon_4 > 0$,

$$\begin{aligned} I_4 &\leq C_s C_2 \left(\int_{B_R(x_0)} |f|^{\frac{n(p-\varepsilon)}{n(p-1)+p-\varepsilon}} dx \right)^{\frac{n(p-1)+p-\varepsilon}{n(p-\varepsilon)}} \left(\int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx \right)^{\frac{1-\varepsilon}{p-\varepsilon}} \\ &\leq C_s C_2 \left(\varepsilon_4 \int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx + \varepsilon_4^{-\frac{1-\varepsilon}{p-1}} \left(\int_{B_R(x_0)} |f|^{\frac{n(p-\varepsilon)}{n(p-1)+p-\varepsilon}} dx \right)^{\frac{n(p-1)+p-\varepsilon}{n(p-1)}} \right). \end{aligned} \tag{3.15}$$

Substituting (3.8), (3.9), (3.10), (3.14), and (3.15) into (3.7), we finally have

$$\begin{aligned} &2^{\frac{2-p}{2}} \alpha \int_{B_{R/2}(x_0)} |Du - p_0|^{p-\varepsilon} dx \\ &\leq \beta(1 + |p_0|^2)^{\frac{p-1}{2}} \varepsilon_1 \int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx + 2^{p-1} \beta C_1 \varepsilon \int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx \\ &\quad + 2^{p-1} \beta(1 + |p_0|^2)^{\frac{p-1}{2}} C_1 \varepsilon \varepsilon_2 \int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx \\ &\quad + C_3(1 + |p_0|^2)^{\frac{p-1}{2}} \varepsilon_3 C_p \int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx + C_s C_2 \varepsilon_4 \int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx \\ &\quad + C_3 C_s^{1-\varepsilon} \left(\int_{B_R(x_0)} |Du - p_0|^{\frac{n(p-\varepsilon)}{n+1-\varepsilon}} dx \right)^{\frac{n+1-\varepsilon}{n}} \\ &\quad + C_s C_2 \varepsilon_4^{-\frac{1-\varepsilon}{p-1}} \left(\int_{B_R(x_0)} |f|^{\frac{n(p-\varepsilon)}{n(p-1)+p-\varepsilon}} dx \right)^{\frac{n(p-1)+p-\varepsilon}{n(p-1)}} \\ &\quad + \beta(1 + |p_0|^2)^{\frac{p-1}{2}} \varepsilon_1^{-\frac{1-\varepsilon}{p-1}} \int_{B_R(x_0)} dx + 2^{p-1} \beta(1 + |p_0|^2)^{\frac{p-1}{2}} C_1 \varepsilon \varepsilon_2^{-\frac{1-\varepsilon}{p-1}} \int_{B_R(x_0)} dx \\ &\quad + C_3(1 + |p_0|^2)^{\frac{p-1}{2}} \varepsilon_3^{-\frac{1-\varepsilon}{p-1}} \int_{B_R(x_0)} dx. \end{aligned}$$

Rearranging this inequality, we have

$$\begin{aligned} &\int_{B_{R/2}(x_0)} |Du - p_0|^{p-\varepsilon} dx \\ &\leq \theta \int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx + \frac{1}{\alpha} 2^{\frac{p-2}{2}} C_3 C_s^{1-\varepsilon} \left(\int_{B_R(x_0)} |Du - p_0|^\tau dx \right)^{\frac{p-\varepsilon}{\tau}} \\ &\quad + \frac{1}{\alpha} 2^{\frac{p-2}{2}} C_s C_2 \varepsilon_4^{-\frac{1-\varepsilon}{p-1}} \left(\int_{B_R(x_0)} |f|^{\frac{n(p-\varepsilon)}{n(p-1)+p-\varepsilon}} dx \right)^{\frac{n(p-1)+p-\varepsilon}{n(p-1)}} + C_4 \int_{B_R(x_0)} dx, \end{aligned}$$

where $\theta = \frac{1}{\alpha} 2^{\frac{p-2}{2}} (\beta(1 + |p_0|^2)^{\frac{p-1}{2}} \varepsilon_1 + 2^{p-1} \beta C_1 \varepsilon + 2^{p-1} \beta(1 + |p_0|^2)^{\frac{p-1}{2}} C_1 \varepsilon \varepsilon_2 + C_3(1 + |p_0|^2)^{\frac{p-1}{2}} \times \varepsilon_3 C_p + C_s C_2 \varepsilon_4)$, $\tau = \frac{n(p-\varepsilon)}{n+1-\varepsilon}$, and $C_4 = \frac{1}{\alpha} 2^{\frac{p-2}{2}} (\beta(1 + |p_0|^2)^{\frac{p-1}{2}} \varepsilon_1^{-\frac{1-\varepsilon}{p-1}} + 2^{p-1} \beta(1 + |p_0|^2)^{\frac{p-1}{2}} C_1 \varepsilon \times \varepsilon_2^{-\frac{1-\varepsilon}{p-1}} + C_3(1 + |p_0|^2)^{\frac{p-1}{2}} \varepsilon_3^{-\frac{1-\varepsilon}{p-1}})$.

By (H4) we get $\frac{n(p-\varepsilon)}{n(p-1)+p-\varepsilon} < \frac{nq}{n(p-1)+q}$ and $\int_{B_R(x_0)} |f|^{\frac{n(p-\varepsilon)}{n(p-1)+p-\varepsilon}} dx < M$.

Moreover, we have $(\int_{B_R(x_0)} |f|^{\frac{n(p-\varepsilon)}{n(p-1)+p-\varepsilon}} dx)^{\frac{n(p-1)}{n(p-1)+p-\varepsilon}} \leq M^{\frac{p-\varepsilon}{n(p-1)}} \int_{B_R(x_0)} |f|^{\frac{n(p-\varepsilon)}{n(p-1)+p-\varepsilon}} dx$.

Setting $C_5 = \max\{\frac{1}{\alpha} 2^{\frac{p-2}{2}} C_3 C_2 \varepsilon_4^{-\frac{1-\varepsilon}{p-1}} M^{\frac{p-\varepsilon}{n(p-1)}}, C_4\}$, we have

$$\begin{aligned} & \int_{B_{R/2}(x_0)} |Du - p_0|^{p-\varepsilon} dx \\ & \leq \theta \int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx + \frac{1}{\alpha} 2^{\frac{p-2}{2}} C_3 C_s^{1-\varepsilon} \left(\int_{B_R(x_0)} |Du - p_0|^\tau dx \right)^{\frac{p-\varepsilon}{\tau}} \\ & \quad + C_5 \left(\int_{B_R(x_0)} |f|^{\frac{n}{n(p-1)+p-\varepsilon}(p-\varepsilon)} dx + \int_{B_R(x_0)} dx \right) \\ & \leq \theta \int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx + \frac{1}{\alpha} 2^{\frac{p-2}{2}} C_3 C_s^{1-\varepsilon} \left(\int_{B_R(x_0)} |Du - p_0|^\tau dx \right)^{\frac{p-\varepsilon}{\tau}} \\ & \quad + C_5 \int_{B_R(x_0)} (|f|^{\frac{n}{n(p-1)+p-\varepsilon}} + 1)^{p-\varepsilon} dx. \end{aligned}$$

Dividing both sides by $|B_R(x_0)| = \alpha_n R^n$ yields

$$\begin{aligned} & \int_{B_{R/2}(x_0)} |Du - p_0|^{p-\varepsilon} dx \\ & \leq 2^n \theta - \int_{B_R(x_0)} |Du - p_0|^{p-\varepsilon} dx + C_6 \left(\int_{B_R(x_0)} |Du - p_0|^\tau dx \right)^{\frac{p-\varepsilon}{\tau}} \\ & \quad + 2^n C_5 \int_{B_R(x_0)} (|f|^{\frac{n}{n(p-1)+p-\varepsilon}} + 1)^{p-\varepsilon} dx, \end{aligned} \tag{3.16}$$

where $C_6 = \frac{2^n}{\alpha} 2^{\frac{p-2}{2}} C_3 C_s^{1-\varepsilon} (\alpha_n R^n)^{\frac{p-\tau-\varepsilon}{\tau}}$.

Taking $\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ sufficiently small such that $2^n \theta < 1$ and $\tau > 1$, we obtain the reverse Hölder inequality for $|Du - p_0|^{p-\varepsilon}$. Accordingly, by Lemma 2 we can derive that u belongs to $W_{loc}^{1,r'}(\Omega)$ with $r' > r$. Since $f \in L_{loc}^{\frac{nq}{n(p-1)+q}}(\Omega)$, $q > p$, reasoning as before, we get a new estimate analogous to (3.16) with exponents r' and τ' in place of r , that is, $p - \varepsilon$ and τ , respectively:

$$\begin{aligned} & \int_{B_{R/2}(x_0)} |Du - p_0|^{r'} dx \\ & \leq 2^n \theta' \int_{B_R(x_0)} |Du - p_0|^{r'} dx + C'_6 \left(\int_{B_R(x_0)} |Du - p_0|^{\tau'} dx \right)^{\frac{r'}{\tau'}} \\ & \quad + 2^n C'_5 \int_{B_R(x_0)} (|f|^{\frac{n}{n(p-1)+r'}} + 1)^{r'} dx. \end{aligned}$$

Therefore, we get $u \in W_{loc}^{1,r''}(\Omega)$ with $r'' > r'$. Repeating this process, we can improve the degree of integrability of Du again and again. Thus, it is clear that $u \in W_{loc}^{1,t}(\Omega)$ with any $t \in (r_1, r_2)$.

This completes the proof of Theorem 1. □

3.2 Proof of Theorem 2

Proof The aim of Theorem 2 is to prove that very weak solutions to systems (1.1) are not only weak solutions to (1.1) but also the optimal Hölder continuity. In fact, under the assumptions of Theorem 1, we get that $u \in W_{\text{loc}}^{1,r}(\Omega)$ with $r_1 < r < p$ are weak solutions $u \in W_{\text{loc}}^{1,p}(\Omega)$ to systems (1.1) by Corollary 1. Then we can safely infer $u \in C^{1,\gamma}(\Omega_0)$ based on A-harmonic approximation technique. The proving method is standard, so we omit the process of derivation in this paper. For more details, we refer the reader to Theorem 1.1 of [14] and the related literature. So the proof of Theorem 2 is complete. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally to writing of this paper. Both authors read and approved the final manuscript.

Acknowledgements

The authors would like to thank the anonymous referee for careful reading the manuscript and valuable comments. This work was supported by the National Natural Science foundation of China under Grant No. 11571159.

Received: 26 November 2016 Accepted: 3 January 2017 Published online: 18 January 2017

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