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Weak convergence theorem for variational inequality problems with monotone mapping in Hilbert space

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Abstract
We know that variational inequality problem is very important in the nonlinear analysis. The main purpose of this paper is to propose an iterative method for finding an element of the set of solutions of a variational inequality problem with a monotone and Lipschitz continuous mapping in Hilbert space. This iterative method is based on the extragradient method. We get a weak convergence theorem. Using this result, we obtain three weak convergence theorems for the equilibrium problem, the constrained convex minimization problem, and the split feasibility problem.

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Keywords: iterative method; extragradient method; weak convergence; variational inequality; monotone mapping; equilibrium problem; constrained convex minimization problem; split feasibility problem

1 Introduction

The variational inequality problem is a generalization of the nonlinear complementarity problem. It is widely used in economics, engineering, mechanics, signal processing, image processing, and so on. The variational inequality was first derived from the mechanics problems in the early 1960s. In 1964, the existence and uniqueness of solutions of variational inequalities were presented for the first time. Subsequently, some scientists have published a series of articles. In the 1970s, the variational inequality problem had been used in many fields. In the 1990s, the variational inequality problem became more important in nonlinear analysis.

Let $\mathbb{R}$ be the set of real numbers. Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. A mapping $A : C \to H$ is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

A mapping $A : C \to H$ is called Lipschitz continuous if there exists $k \in \mathbb{R}$ with $k > 0$ such that

$$\|Ax - Ay\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

Such A is called k-Lipschitz continuous. If $k = 1$, such A is called a nonexpansive mapping. The variational inequality problem is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

We denote the set of solutions of this variational inequality problem by $\mathrm{VI}(C, A)$.

In 1976, Korpelevich [1] proposed the following so-called extragradient method for solving the variational inequality problem in the finite-dimensional Euclidean space $\mathbb{R}^n$.

Theorem 1.1 ([1]) *Let C be a nonempty closed convex subset of an n-dimensional Euclidean space $\mathbb{R}^n$. Let A be a monotone and k-Lipschitz continuous mapping of C into H. Assume that $\mathrm{VI}(C, A)$ is nonempty. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0 = x \in C$ and*

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \end{cases} \tag{1.2}$$

for every $n = 0, 1, 2, \ldots$, where $\lambda \in (0, \frac{1}{k})$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge to the same point $z \in \mathrm{VI}(C, A)$.

In this paper, based on the extragradient method, we introduce an iterative method for finding an element of the set of solutions of a variational inequality problem for a monotone and Lipschitz continuous mapping in Hilbert space. We obtain a weak convergence theorem. As applications, we can use this result to solve equilibrium problems, constrained convex minimization problems, and split feasibility problems.

2 Preliminaries

Let $\mathbb{R}$ be the set of real numbers. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $\{x_n\}$ be a sequence in H, we denote the sequence $\{x_n\}$ converging weakly to x by $x_n \rightharpoonup x$ and the sequence $\{x_n\}$ converging strongly to x by $x_n \to x$. Let C be a nonempty closed convex subset of H. For each $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.1}$$

P_C is called the metric projection of H into C. We know that P_C is nonexpansive. A set-valued mapping $T : H \to 2^H$ is called monotone if

$$\langle x - y, u - v \rangle \geq 0, \quad \forall (x, u), (y, v) \in G(T).$$

A monotone mapping $T : H \to 2^H$ is called maximal if its graph is not properly contained in the graph of any other monotone mapping on H. It is known that a monotone mapping T is maximal if and only if for $(x, u) \in H \times H$, $\langle x - y, u - v \rangle \geq 0$ for each $(y, v) \in G(T)$ implies $u \in Tx$.

Lemma 2.1 ([2]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if we have the inequality*

$$\langle x - z, z - y \rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

Lemma 2.2 ([2]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if we have the inequality*

$$\|x - y\|^2 \geq \|x - z\|^2 + \|y - z\|^2, \quad \forall y \in C. \tag{2.3}$$

Lemma 2.3 ([3]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz continuous mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$; i.e.,*

$$N_C v = w \in H: \quad \langle v - u, w\rangle \geq 0, \quad \forall u \in C.$$

Define

$$Tv = \begin{cases} Av + N_C v, & \forall v \in C, \\ \emptyset, & \forall v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$.

Lemma 2.4 ([4]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H satisfying the properties:*

(i) $\lim_{n\to\infty} \|x_n - u\|$ *exists for each $u \in C$;*

(ii) $\omega_w(x_n) \subset C$.

Then $\{x_n\}$ converges weakly to a point in C.

Lemma 2.5 ([5]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H. Suppose that*

$$\|x_{n+1} - u\| \leq \|x_n - u\|, \quad \forall u \in C,$$

for every $n = 0, 1, 2, \ldots$. Then the sequence $\{P_C x_n\}$ converges strongly to a point in C.

3 Main results

The main task of this article is to find an element of the set of solutions of a variational inequality problem with a monotone and Lipschitz continuous mapping in Hilbert space. We obtain a weak convergence theorem.

Theorem 3.1 *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let A be a monotone and k-Lipschitz continuous mapping of C into H. Assume that $\mathrm{VI}(C, A) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_0 = x \in C$ and*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = P_C(x_n - \lambda_n A y_n), \end{cases} \tag{3.1}$$

for every $n = 0, 1, 2, \ldots$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to the same point $z \in \mathrm{VI}(C, A)$, where $z = \lim_{n\to\infty} P_{\mathrm{VI}(C,A)} x_n$.

Proof For each $u \in \mathrm{VI}(C,A)$. From Lemma 2.2, we have

$$\begin{aligned}
\|x_{n+1}-u\|^2 &\le \|x_n-\lambda_n Ay_n-u\|^2-\|x_n-\lambda_n Ay_n-x_{n+1}\|^2\\
&= \|x_n-u\|^2-\|x_n-x_{n+1}\|^2+2\lambda_n\langle Ay_n,u-x_{n+1}\rangle\\
&= \|x_n-u\|^2-\|x_n-x_{n+1}\|^2+2\lambda_n\bigl(\langle Ay_n,u-y_n\rangle\\
&\quad+\langle Ay_n,y_n-x_{n+1}\rangle\bigr)\\
&= \|x_n-u\|^2-\|x_n-x_{n+1}\|^2+2\lambda_n\bigl(\langle Ay_n-Au,u-y_n\rangle\\
&\quad+\langle Au,u-y_n\rangle+\langle Ay_n,y_n-x_{n+1}\rangle\bigr)\\
&\le \|x_n-u\|^2-\|x_n-x_{n+1}\|^2+2\lambda_n\langle Ay_n,y_n-x_{n+1}\rangle\\
&= \|x_n-u\|^2-\|x_n-y_n\|^2-2\langle x_n-y_n,y_n-x_{n+1}\rangle\\
&\quad-\|y_n-x_{n+1}\|^2+2\lambda_n\langle Ay_n,y_n-x_{n+1}\rangle\\
&= \|x_n-u\|^2-\|x_n-y_n\|^2-\|y_n-x_{n+1}\|^2\\
&\quad+2\langle x_n-\lambda_n Ay_n-y_n,x_{n+1}-y_n\rangle.
\end{aligned}$$

Then, from Lemma 2.1, we obtain

$$\begin{aligned}
&\langle x_n-\lambda_n Ay_n-y_n,x_{n+1}-y_n\rangle\\
&\quad=\langle x_n-\lambda_n Ax_n-y_n,x_{n+1}-y_n\rangle+\langle\lambda_n Ax_n-\lambda_n Ay_n,x_{n+1}-y_n\rangle\\
&\quad\le\langle\lambda_n Ax_n-\lambda_n Ay_n,x_{n+1}-y_n\rangle\\
&\quad=\lambda_n\langle Ax_n-Ay_n,x_{n+1}-y_n\rangle\\
&\quad\le\lambda_n\|Ax_n-Ay_n\|\|x_{n+1}-y_n\|\\
&\quad\le\lambda_n k\|x_n-y_n\|\|x_{n+1}-y_n\|.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|x_{n+1}-u\|^2 &\le \|x_n-u\|^2-\|x_n-y_n\|^2-\|y_n-x_{n+1}\|^2\\
&\quad+2\lambda_n k\|x_n-y_n\|\|x_{n+1}-y_n\|\\
&\le \|x_n-u\|^2-\|x_n-y_n\|^2-\|y_n-x_{n+1}\|^2\\
&\quad+\lambda_n^2k^2\|x_n-y_n\|^2+\|x_{n+1}-y_n\|^2\\
&\le \|x_n-u\|^2+\bigl(\lambda_n^2k^2-1\bigr)\|x_n-y_n\|^2\\
&\le \|x_n-u\|^2.
\end{aligned}\tag{3.2}$$

Therefore, there exists

$$c=\lim_{n\to\infty}\|x_n-u\|\tag{3.3}$$

and the sequence $\{x_n\}$ is bounded. From (3.2), we also get

$$\bigl(1-\lambda_n^2k^2\bigr)\|x_n-y_n\|^2\le\|x_n-u\|^2-\|x_{n+1}-u\|^2.$$

So, we obtain

$$\|x_n - y_n\|^2 \leq \frac{1}{1-\lambda_n^2 k^2}\left(\|x_n - u\|^2 - \|x_{n+1} - u\|^2\right). \tag{3.4}$$

Hence

$$x_n - y_n \to 0, \quad n \to \infty. \tag{3.5}$$

On the other hand, we have

$$\begin{aligned}
\|x_{n+1} - y_n\| &= \left\|P_C(x_n - \lambda_n Ay_n) - P_C(x_n - \lambda_n Ax_n)\right\| \\
&\leq \left\|(x_n - \lambda_n Ay_n) - (x_n - \lambda_n Ax_n)\right\| \\
&= \|\lambda_n Ax_n - \lambda_n Ay_n\| \\
&= \lambda_n \|Ax_n - Ay_n\| \\
&\leq \lambda_n k \|x_n - y_n\|.
\end{aligned} \tag{3.6}$$

Hence

$$x_{n+1} - y_n \to 0, \quad n \to \infty. \tag{3.7}$$

Since A is Lipschitz continuous, we get

$$Ax_{n+1} - Ay_n \to 0, \quad n \to \infty. \tag{3.8}$$

From

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - y_n\| + \|y_n - x_n\|,$$

we have

$$x_{n+1} - x_n \to 0, \quad n \to \infty. \tag{3.9}$$

Since $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to a point z. We prove that $z \in \mathrm{VI}(C,A)$. From (3.5) and (3.9), we have $y_{n_i} \rightharpoonup z$ and $x_{n_i+1} \rightharpoonup z$.

Let

$$Tv = \begin{cases} Av + N_C v, & \forall v \in C, \\ \emptyset, & \forall v \notin C. \end{cases}$$

From Lemma 2.3, we know that T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C,A)$.

For each $(v,w) \in G(T)$, we have

$$w \in Tv = Av + N_C v.$$

Hence

$$w - Av \in N_C v.$$

So, we obtain

$$\langle v - p, w - Av\rangle \geq 0, \quad \forall p \in C. \tag{3.10}$$

On the other hand, from $v \in C$ and

$$x_{n+1} = P_C(x_n - \lambda_n Ay_n),$$

we get

$$\langle x_n - \lambda_n Ay_n - x_{n+1}, x_{n+1} - v\rangle \geq 0$$

and hence

$$\left\langle v - x_{n+1}, \frac{x_{n+1} - x_n}{\lambda_n} + Ay_n\right\rangle \geq 0. \tag{3.11}$$

Therefore from (3.10) and (3.11), we obtain

$$\begin{aligned}
&\langle v - x_{n_i+1}, w\rangle \\
&\quad \geq \langle v - x_{n_i+1}, Av\rangle \\
&\quad \geq \langle v - x_{n_i+1}, Av\rangle - \left\langle v - x_{n_i+1}, \frac{x_{n_i+1} - x_{n_i}}{\lambda_{n_i}} + Ay_{n_i}\right\rangle \\
&\quad = \langle v - x_{n_i+1}, Av - Ay_{n_i}\rangle - \left\langle v - x_{n_i+1}, \frac{x_{n_i+1} - x_{n_i}}{\lambda_{n_i}}\right\rangle \\
&\quad = \langle v - x_{n_i+1}, Av - Ax_{n_i+1}\rangle + \langle v - x_{n_i+1}, Ax_{n_i+1} - Ay_{n_i}\rangle \\
&\qquad - \left\langle v - x_{n_i+1}, \frac{x_{n_i+1} - x_{n_i}}{\lambda_{n_i}}\right\rangle \\
&\quad \geq \langle v - x_{n_i+1}, Ax_{n_i+1} - Ay_{n_i}\rangle - \left\langle v - x_{n_i+1}, \frac{x_{n_i+1} - x_{n_i}}{\lambda_{n_i}}\right\rangle.
\end{aligned} \tag{3.12}$$

As $i \to \infty$, we have

$$\langle v - z, w\rangle \geq 0. \tag{3.13}$$

Since T is maximal monotone, we have $0 \in Tz$ and hence $z \in \mathrm{VI}(C, A)$.

From Lemma 2.4, we get

$$x_n \rightharpoonup z \in \mathrm{VI}(C, A). \tag{3.14}$$

Since $x_n - y_n \to 0$, we also have

$$y_n \rightharpoonup z \in \mathrm{VI}(C, A). \tag{3.15}$$

From Lemma 2.5, we obtain

$$z = \lim_{n\to\infty} P_{\mathrm{VI}(C,A)} x_n. \tag{3.16}$$

□

4 Application

In the applications of this method, they are useful in nonlinear analysis and optimization problems in Hilbert space. This section is concerned with three weak convergence theorems for the equilibrium problem, the constrained convex minimization problem, and the split feasibility problem by Theorem 3.1.

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$. The equilibrium problem [6] is to find x^* such that

$$F\big(x^*, y\big) \geq 0, \quad \forall y \in C. \tag{4.1}$$

The set of solutions of problem (4.1) is denoted by $\mathrm{EP}(F)$.

Lemma 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying the properties*:

(A1) $F(x,x)=0$ *for all* $x \in C$;

(A2) *for each* $x \in C$, $y \mapsto F(x,y)$ *is convex and differentiable*.

Then $z \in \mathrm{EP}(F)$ *if and only if* $z \in \mathrm{VI}(C,S)$, *where* $Sx = \nabla F_y(x,y)|_{y=x}$.

Proof Let $z \in \mathrm{EP}(F)$. For each $y \in C$, $z + \lambda(y-z) = \lambda y + (1-\lambda)z \in C$, $\forall \lambda \in (0,1)$. Since for each $x \in C$, $y \mapsto F(x,y)$ is differentiable. Then

$$\langle Sz, y-z\rangle = \lim_{\lambda\to 0^+} \frac{F(z, z+\lambda(y-z)) - F(z,z)}{\lambda} \geq 0.$$

Conversely. If $z \in \mathrm{VI}(C,S)$; *i.e.*, $\langle \nabla F_y(z,y)|_{y=z}, y-z\rangle \geq 0$, $\forall y \in C$. Since for each $x \in C$, $y \mapsto F(x,y)$ is convex. Then $F(z,y) \geq F(z,z) = 0$. □

Applying Theorem 3.1 and Lemma 4.1, we obtain the following result.

Theorem 4.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1) *and* (A2). *Assume that S is monotone and k-Lipschitz continuous and* $\mathrm{EP}(F) \neq \emptyset$. *Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0 = x \in C$ and*

$$\begin{cases} y_n = P_C(x_n - \lambda_n S x_n), \\ x_{n+1} = P_C(x_n - \lambda_n S y_n), \end{cases} \tag{4.2}$$

for every $n = 0,1,2,\ldots$, where $S(x) = \nabla F_y(x,y)|_{y=x}$ and $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0, \frac{1}{k})$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to the same point $z \in \mathrm{EP}(F)$, where $z = \lim_{n\to\infty} P_{\mathrm{EP}(F)} x_n$.

Proof Putting $A = S$ in Theorem 3.1, we get the desired result by Lemma 4.1. □

Consider the following constrained convex minimization problem [7]: Find $x^* \in C$ such that

$$f(x^*) = \min_{x \in C} f(x), \tag{4.3}$$

where C is a nonempty closed convex subset of a real Hilbert space H and f is a real-valued convex function.

Lemma 4.3 *Let H is a real Hilbert space and let C be a nonempty closed convex subset of H. Let f be a convex function of H into $\mathbb{R}$. If f is differentiable, then z is a solution of* (4.3) *if and only if $z \in \mathrm{VI}(C, \nabla f)$.*

Proof Let z be a solution of (4.3). For each $x \in C$, $z + \lambda(x - z) \in C$, $\forall \lambda \in (0,1)$. Since f is differentiable, we have

$$\langle \nabla f(z), x - z \rangle = \lim_{\lambda \to 0^+} \frac{f(z + \lambda(x - z)) - f(z)}{\lambda} \geq 0.$$

Conversely, if $z \in \mathrm{VI}(C, S)$, $\langle \nabla f(z), x - z \rangle \geq 0$, $\forall x \in C$. Since f is convex, we have

$$f(x) \geq f(z) + \langle \nabla f(z), x - z \rangle \geq f(z).$$

Hence z is a solution of (4.3). □

Applying Theorem 3.1 and Lemma 4.3, we obtain the following result.

Theorem 4.4 *Let H is a real Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of H into $\mathbb{R}$. Assume f is differentiable and we assume that the set of solutions of* (4.3) *is nonempty and ∇f is k-Lipschitz continuous. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0 = x \in C$ and*

$$\begin{cases} y_n = P_C(x_n - \lambda_n \nabla f(x_n)), \\ x_{n+1} = P_C(x_n - \lambda_n \nabla f(y_n)), \end{cases} \tag{4.4}$$

for every $n = 0, 1, 2, \ldots$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to the same point z, where z is a solution of (4.3).

Proof Since f is convex, we see that ∇f is monotone. Putting $A = \nabla f$ in Theorem 3.1, we obtain the desired result by Lemma 4.3. □

Very recently, the split feasibility problem (SFP) [8–11] has been proposed. It is very important in nonlinear analysis and optimization problems. The SFP is to find a point x^* such that

$$x^* \in C \quad \text{and} \quad Bx^* \in Q, \tag{4.5}$$

where C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 and B is a bounded linear operator of H_1 into H_2.

Lemma 4.5 ([8]) *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty closed convex subsets of H_1 and H_2. Let B be a bounded linear operator of H_1 into H_2. Assume that $C \cap B^{-1}Q$ is nonempty. Let $\lambda \geq 0$. Then the following propositions are equivalent*:

(i) $z \in \mathrm{VI}(C, B^*(I - P_Q)B)$;

(ii) $z = P_C(I - \lambda B^*(I - P_Q)B)z$;

(iii) $z \in C \cap B^{-1}Q$,

where B^* is the adjoint operator of B.

Lemma 4.6 *Let H_1 and H_2 be real Hilbert spaces. Let B be a bounded linear operator of H_1 into H_2 such that $B \neq 0$. Let Q be a nonempty closed convex subset of H_2. Then $B^*(I - P_Q)B$ is monotone and $\|B\|^2$-Lipschitz continuous.*

Proof Let $x, y \in H_1$,

$$\begin{aligned}
&\bigl\langle B^*(I - P_Q)Bx - B^*(I - P_Q)By, x - y\bigr\rangle \\
&\qquad - \frac{1}{\|B\|^2}\bigl\|B^*(I - P_Q)Bx - B^*(I - P_Q)By\bigr\|^2 \\
&\quad = \bigl\langle B^*\bigl[(I - P_Q)Bx - (I - P_Q)By\bigr], x - y\bigr\rangle \\
&\qquad - \frac{1}{\|B\|^2}\bigl\|B^*\bigl[(I - P_Q)Bx - (I - P_Q)By\bigr]\bigr\|^2 \\
&\quad = \bigl\langle (I - P_Q)Bx - (I - P_Q)By, Bx - By\bigr\rangle \\
&\qquad - \frac{1}{\|B\|^2}\bigl\|B^*\bigl[(I - P_Q)Bx - (I - P_Q)By\bigr]\bigr\|^2 \\
&\quad \geq \bigl\|(I - P_Q)Bx - (I - P_Q)By\bigr\|^2 \\
&\qquad - \bigl\|(I - P_Q)Bx - (I - P_Q)By\bigr\|^2 \\
&\quad = 0.
\end{aligned}$$

Hence

$$\begin{aligned}
&\bigl\|B^*(I - P_Q)Bx - B^*(I - P_Q)By\bigr\|^2 \\
&\quad \leq \|B\|^2\bigl\langle B^*(I - P_Q)Bx - B^*(I - P_Q)By, x - y\bigr\rangle \\
&\quad \leq \|B\|^2\bigl\|B^*(I - P_Q)Bx - B^*(I - P_Q)By\bigr\|\|x - y\|.
\end{aligned}$$

So, we obtain

$$\bigl\|B^*(I - P_Q)Bx - B^*(I - P_Q)By\bigr\| \leq \|B\|^2\|x - y\|.$$

On the other hand, we have

$$\begin{aligned}
&\bigl\langle B^*(I - P_Q)Bx - B^*(I - P_Q)By, x - y\bigr\rangle \\
&\quad \geq \frac{1}{\|B\|^2}\bigl\|B^*(I - P_Q)Bx - B^*(I - P_Q)By\bigr\|^2 \\
&\quad \geq 0.
\end{aligned}$$

Then $B^*(I-P_Q)B$ is monotone and $\|B\|^2$-Lipschitz continuous. □

Applying Theorem 3.1 and Lemma 4.5, we obtain the following result.

Theorem 4.7 *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty closed convex subsets of H_1 and H_2. Let $B : H_1 \to H_2$ be a bounded linear operator such that $B \neq 0$. Assume that $C \cap B^{-1}Q$ is nonempty. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0 = x \in C$ and*

$$\begin{cases} y_n = P_C(x_n - \lambda_n B^*(I-P_Q)Bx_n), \\ x_{n+1} = P_C(x_n - \lambda_n B^*(I-P_Q)By_n), \end{cases} \tag{4.6}$$

for every $n = 0,1,2,\ldots$, where $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0, \frac{1}{\|B\|^2})$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to the same point $z \in C \cap B^{-1}Q$, where $z = \lim_{n\to\infty} P_{C\cap B^{-1}Q}x_n$.

Proof By Lemma 4.6, we get $B^*(I-P_Q)B$ is monotone and $\|B\|^2$-Lipschitz continuous. Putting $A = B^*(I-P_Q)B$ and $k = \|B\|^2$ in Theorem 3.1, we get the desired result by Lemma 4.5. □

5 Numerical result

In this section, we use our iterative method to solve some specific practical numerical calculation problems. By using the algorithm in Theorem 4.4 and Theorem 4.7, we illustrate its convergence in solving constrained convex minimization problem and linear system of equations.

The first example is the constrained convex minimization problem of a function of one variable, which uses the algorithm in Theorem 4.4.

Example 1 In Theorem 4.4, we suppose that $H = \mathbb{R}$ and $C = [0,2]$. Consider the constrained convex minimization problem (4.3) and let the function

$$f(x) = x^3 - 3x, \quad \forall x \in [0,2]. \tag{5.1}$$

Then the problem (4.3) can be written as

$$\min_{x\in[0,2]} \left(x^3 - 3x\right). \tag{5.2}$$

It is easy to find a point $x^* = 1$ solving the problem (5.2). We can know that ∇f is monotone and 12-Lipschitz continuous. Take $k = 12$ and $\lambda_n = \frac{1}{36(n+1)} + \frac{1}{36}$.

Then by Theorem 4.4, the sequences $\{x_n\}$ and $\{y_n\}$ are generated by

$$\begin{cases} y_n = P_C[x_n - (\frac{1}{36(n+1)} + \frac{1}{36})(3x_n^2 - 3)], \\ x_{n+1} = P_C[x_n - (\frac{1}{36(n+1)} + \frac{1}{36})(3y_n^2 - 3)]. \end{cases} \tag{5.3}$$

As $n \to \infty$, we have $x_n \to x^* = 1$.

From Table 1, we can see that with the increase of the number of iterations, $\{x_n\}$ approaches the solution x^* and the errors gradually approach zero.

The second example is a 3×3 linear system of equations, which use the algorithm in Theorem 4.7.

Table 1 Numerical results as regards Example 1

n	x_n	E_n
0	0.5000	5.00E–01
10	0.9214	7.86E–02
50	0.9998	1.69E–04
100	1.0000	8.75E–08
500	1.0000	4.44E–16

Table 2 Numerical results as regards Example 2

n	x_n^1	x_n^2	x_n^3	E_n
0	1.0000	1.0000	1.0000	7.00E+00
10	2.9932	3.9399	–2.3981	2.60E+00
50	2.7238	4.3377	–4.7600	4.98E–01
100	2.7501	4.2636	–4.9161	3.73E–01
500	2.9237	4.0805	–4.9746	1.14E–01

Example 2 In Theorem 4.7, we suppose that $H_1 = H_2 = \mathbb{R}^3$. Take

$$A = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}, \tag{5.4}$$

$$b = \begin{pmatrix} 24 \\ 30 \\ -24 \end{pmatrix}. \tag{5.5}$$

Let $B = A$, $C = \mathbb{R}^3$ and $Q = \{b\}$. Then the SFP (4.5) is transformed into the problem of system of linear equations. That is to say, x_* is the solution of linear system of equations $Ax = b$ and

$$x^* = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}. \tag{5.6}$$

Then by Theorem 4.7, the sequences $\{x_n\}$ and $\{y_n\}$ are generated by

$$\begin{cases} y_n = x_n - (\frac{1}{300(n+1)} + \frac{1}{300})(A^*Ax_n - A^*b), \\ x_{n+1} = x_n - (\frac{1}{300(n+1)} + \frac{1}{300})(A^*Ay_n - A^*b). \end{cases} \tag{5.7}$$

As $n \to \infty$, we have $x_n \to x^*$.

From Table 2, we can also see that with the increase of iterative number, x_n approaches the exact solution x^* and the errors gradually approach zero.

6 Conclusion

The variational inequality problem is a very important field of study in mathematics. It is not only playing an important role in optimization problems and nonlinear analysis, but also widely used in many fields, such as economics, mechanics, signal processing, *etc.* So, more and more scientists devote their efforts to the study of variational inequalities. For a variational inequalities, we mainly study the algorithm and its convergence, existence and uniqueness of the solutions. In a real Hilbert space, The gradient-projection method for solving the variational inequality problem for an inverse-strongly monotone mapping has been studied. But this method will not be used if the inverse-strongly monotone is changed to monotone in the condition. So we propose a new iterative method to solve it. In this paper, we introduce an iterative method for finding an element of the set of solutions of a variational inequality problem with a monotone and Lipschitz continuous mapping in Hilbert space. In particular, under certain conditions, equilibrium problem, constrained convex minimization problem and split feasibility problem are, respectively, equivalent to a variational inequality problem. Then the new weak convergence theorem are obtained. The algorithm in Theorem 3.1 improves and extends Korpelevich's method [1] in the following ways:

(i) The finite-dimensional Euclidean space $\mathbb{R}^n$ is extended to the case of an infinite-dimensional Hilbert space H.
(ii) The fixed coefficient λ is extended to the case of a sequence $\{\lambda_n\}$.

Recently, the variational inequality problem has been further developed. This will attract more scholars interested in the study of the variational inequality problem. Many scholars will devote their efforts to its study. Then the variational inequality problem can be better developed in the future.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors read and approved the final manuscript.

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