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On a multidimensional Hilbert-type inequality with parameters

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Abstract

In this paper, by the use of the way of weight coefficients, the transfer formula, and the technique of real analysis, we introduce some proper parameters and obtain a multidimensional Hilbert-type inequality with the following kernel:

$$\prod_{k=1}^{s}\frac{(\min\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\gamma}{s}}}{(\max\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\lambda+\gamma}{s}}}$$

and a best possible constant factor. The equivalent form, the operator expressions with the norm, and some particular cases are also considered. The lemmas and theorems provide an extensive account of this type of inequalities.

MSC: 26D15; 47A07

Keywords: Hilbert-type inequality; weight coefficient; equivalent form; operator; norm

1 Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(y) \geq 0$, $f \in L^p(\mathbf{R}_+)$, $g \in L^q(\mathbf{R}_+)$, $\|f\|_p = (\int_0^\infty f^p(x)\,dx)^{\frac{1}{p}} > 0$, $\|g\|_q > 0$, then we have the following Hardy-Hilbert's integral inequality (*cf.* [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\|f\|_p\|g\|_q, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Assuming that $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^\infty \in l^p$, $b = \{b_n\}_{n=1}^\infty \in l^q$, $\|a\|_p = (\sum_{m=1}^\infty a_m^p)^{\frac{1}{p}} > 0$, $\|b\|_q > 0$, we have the following discrete Hardy-Hilbert's inequality with the same best constant $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q. \tag{2}$$

Inequalities (1) and (2) are important in analysis and its applications (*cf.* [1–6]).

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [7] gave an extension of (1) at $p = q = 2$ with the kernel $\frac{1}{(x+y)^\lambda}$. In recent years, Yang [3] and [4] gave some extensions of (1) and (2) as follows:

If $\lambda_1, \lambda_2 \in \mathbf{R}$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$, with

$$k(\lambda_1) = \int_0^\infty k_\lambda(t,1)t^{\lambda_1 - 1}\,dt \in \mathbf{R}_+,$$

$\phi(x) = x^{p(1-\lambda_1)-1}$, $\psi(x) = x^{q(1-\lambda_2)-1}$, $f(x), g(y) \geq 0$,

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f; \|f\|_{p,\phi} := \left(\int_0^\infty \phi(x)\left|f(x)\right|^p\,dx \right)^{\frac{1}{p}} < \infty \right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+)$, $\|f\|_{p,\phi}, \|g\|_{q,\psi} > 0$, then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x,y) f(x) g(y)\,dx\,dy < k(\lambda_1)\|f\|_{p,\phi}\|g\|_{q,\psi}, \tag{3}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing with respect to $x > 0$ $(y > 0)$, then for $a_m, b_n \geq 0$,

$$a \in l_{p,\phi} = \left\{ a; \|a\|_{p,\phi} := \left(\sum_{n=1}^\infty \phi(n)|a_n|^p \right)^{\frac{1}{p}} < \infty \right\},$$

$b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $\|a\|_{p,\phi}, \|b\|_{q,\psi} > 0$, we have the following inequality:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m,n) a_m b_n < k(\lambda_1)\|a\|_{p,\phi}\|b\|_{q,\psi}, \tag{4}$$

where the constant factor $k(\lambda_1)$ is still the best possible.

Clearly, for $\lambda = 1$, $k_1(x,y) = \frac{1}{x+y}$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, (3) reduces to (1), while (4) reduces to (2). Some other results including the multidimensional Hilbert-type integral, discrete, and half-discrete inequalities are provided by [8–26].

In this paper, by the use of the way of weight coefficients, the transfer formula and technique of real analysis, a multidimensional discrete Hilbert's inequality with parameters and a best possible constant factor is given, which is an extension of (4) for

$$k_\lambda(m,n) = \prod_{k=1}^s \frac{(\min\{m, c_k n\})^{\frac{\gamma}{s}}}{(\max\{m, c_k n\})^{\frac{\lambda+\gamma}{s}}}.$$

The equivalent form, the operator expressions with the norm, and some particular cases are also considered.

2 Some lemmas

If $i_0, j_0 \in \mathbf{N}$ ($\mathbf{N}$ is the set of positive integers), $\alpha, \beta > 0$, we put

$$\begin{aligned} \|x\|_\alpha &:= \left(\sum_{k=1}^{i_0} |x_k|^\alpha \right)^{\frac{1}{\alpha}} \quad \left(x = (x_1, \ldots, x_{i_0}) \in \mathbf{R}^{i_0}\right), \\ \|y\|_\beta &:= \left(\sum_{k=1}^{j_0} |y_k|^\beta \right)^{\frac{1}{\beta}} \quad \left(y = (y_1, \ldots, y_{j_0}) \in \mathbf{R}^{j_0}\right). \end{aligned} \tag{5}$$

Lemma 1 *If $g(t)$ (>0) is decreasing in $\mathbf{R}_+$ and strictly decreasing in $[n_0,\infty)\subset\mathbf{R}_+$ $(n_0\in\mathbf{N})$, satisfying $\int_0^\infty g(t)\,dt\in\mathbf{R}_+$, then we have*

$$\int_1^\infty g(t)\,dt < \sum_{n=1}^\infty g(n) < \int_0^\infty g(t)\,dt. \tag{6}$$

Proof Since by the assumption, we have

$$\int_n^{n+1} g(t)\,dt \le g(n) \le \int_{n-1}^n g(t)\,dt \quad (n=1,\dots,n_0),$$
$$\int_{n_0+1}^{n_0+2} g(t)\,dt < g(n_0+1) < \int_{n_0}^{n_0+1} g(t)\,dt,$$

it follows that

$$0 < \int_1^{n_0+2} g(t)\,dt < \sum_{n=1}^{n_0+1} g(n) < \sum_{n=1}^{n_0+1}\int_{n-1}^n g(t)\,dt = \int_0^{n_0+1} g(t)\,dt < \infty.$$

In the same way, we still have

$$0 < \int_{n_0+2}^\infty g(t)\,dt \le \sum_{n=n_0+2}^\infty g(n) \le \int_{n_0+1}^\infty g(t)\,dt < \infty.$$

Hence, choosing plus for the above two inequalities, we have (6). □

Lemma 2 *If $s\in\mathbf{N}$, $\gamma, M>0$, $\Psi(u)$ is a non-negative measurable function in $(0,1]$, and*

$$D_M := \left\{x\in\mathbf{R}_+^s; \sum_{i=1}^s x_i^\gamma \le M^\gamma\right\},$$

then we have the following transfer formula (cf. [27]):

$$\int\cdots\int_{D_M} \Psi\left(\sum_{i=1}^s\left(\frac{x_i}{M}\right)^\gamma\right) dx_1\cdots dx_s$$
$$= \frac{M^s\Gamma^s(\frac{1}{\gamma})}{\gamma^s\Gamma(\frac{s}{\gamma})}\int_0^1 \Psi(u)u^{\frac{s}{\gamma}-1}\,du. \tag{7}$$

Lemma 3 *For $s\in\mathbf{N}$, $\gamma,\varepsilon>0$, we have*

$$\sum_m \|m\|_\gamma^{-s-\varepsilon} = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma}\gamma^{s-1}\Gamma(\frac{s}{\gamma})} + O(1) \quad \bigl(\varepsilon\to 0^+\bigr), \tag{8}$$

where $\sum_m = \sum_{m_s=1}^\infty\cdots\sum_{m_1=1}^\infty$.

Proof For $M>s^{1/\gamma}$, we set

$$\Psi(u) = \begin{cases} 0, & 0<u<\frac{s}{M^\gamma},\\ (Mu^{1/\gamma})^{-s-\varepsilon}, & \frac{s}{M^\gamma}\le u\le 1.\end{cases}$$

Then by Lemma 1 and (7), it follows that

$$\begin{aligned}
\sum_m \|m\|_\gamma^{-s-\varepsilon} &\ge \int_{\{x\in\mathbf{R}_+^s;x_i\ge 1\}} \|x\|_\gamma^{-s-\varepsilon}\,dx \\
&= \lim_{M\to\infty} \int\cdots\int_{D_M} \Psi\left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma\right) dx_1\cdots dx_s \\
&= \lim_{M\to\infty} \frac{M^s\Gamma^s(\frac{1}{\gamma})}{\gamma^s\Gamma(\frac{s}{\gamma})} \int_{s/M^\gamma}^1 \left(Mu^{1/\gamma}\right)^{-s-\varepsilon} u^{\frac{s}{\gamma}-1}\,du \\
&= \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma}\gamma^{s-1}\Gamma(\frac{s}{\gamma})}.
\end{aligned}$$

By Lemma 1 and in the above way, we still find

$$0 < \sum_{\{m\in\mathbf{N}^s;m_i\ge 2\}} \|m\|_\gamma^{-s-\varepsilon} \le \int_{\{x\in\mathbf{R}_+^s;x_i\ge 1\}} \|x\|_\gamma^{-s-\varepsilon}\,dx = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma}\gamma^{s-1}\Gamma(\frac{s}{\gamma})}.$$

For $s = 1$, $0 < \sum_{m=1}^1 \|m\|_\gamma^{-1-\varepsilon} < \infty$; for $s \ge 2$,

$$\begin{aligned}
0 &< \sum_{\{m\in\mathbf{N}^s;\exists i_0,m_{i_0}=1\}} \|m\|_\gamma^{-s-\varepsilon} \le a + \sum_{\{m\in\mathbf{N}^{s-1};m_i\ge 2\}} \|m\|_\gamma^{-(s-1)-(1+\varepsilon)} \\
&\le a + \frac{\Gamma^{s-1}(\frac{1}{\gamma})}{(1+\varepsilon)(s-1)^{(1+\varepsilon)/\gamma}\gamma^{s-2}\Gamma(\frac{s-1}{\gamma})} < \infty \quad (a\in\mathbf{R}_+),
\end{aligned}$$

and then

$$\begin{aligned}
\sum_m \|m\|_\gamma^{-s-\varepsilon} &= \sum_{\{m\in\mathbf{N}^s;\exists i_0,m_{i_0}=1\}} \|m\|_\gamma^{-s-\varepsilon} + \sum_{\{m\in\mathbf{N}^s;m_i\ge 2\}} \|m\|_\gamma^{-s-\varepsilon} \\
&\le O_1(1) + \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma}\gamma^{s-1}\Gamma(\frac{s}{\gamma})} \quad \left(\varepsilon\to 0^+\right).
\end{aligned} \tag{9}$$

Then we have (8). □

Example 1 For $s \in \mathbf{N}$, $0 < c_1 \le \cdots \le c_s < \infty$, $\lambda_1, \lambda_2 > -\gamma$, $\lambda_1 + \lambda_2 = \lambda$, we set

$$k_\lambda(x,y) := \prod_{k=1}^s \frac{(\min\{x,c_k y\})^{\frac{\gamma}{s}}}{(\max\{x,c_k y\})^{\frac{\lambda+\gamma}{s}}} \quad \left((x,y)\in\mathbf{R}_+^2 = \mathbf{R}_+\times\mathbf{R}_+\right).$$

(a) We find

$$\begin{aligned}
k_s(\lambda_1) &:= \int_0^\infty k_\lambda(1,u)u^{\lambda_2-1}\,du \overset{u=1/t}{=} \int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}\,dt \\
&= \int_0^\infty \prod_{k=1}^s \frac{(\min\{t,c_k\})^{\frac{\gamma}{s}}}{(\max\{t,c_k\})^{\frac{\lambda+\gamma}{s}}} t^{\lambda_1-1}\,dt \\
&= \int_0^{c_1} \prod_{k=1}^s \frac{(\min\{t,c_k\})^{\frac{\gamma}{s}} t^{\lambda_1-1}}{(\max\{t,c_k\})^{\frac{\lambda+\gamma}{s}}}\,dt + \int_{c_s}^\infty \prod_{k=1}^s \frac{(\min\{t,c_k\})^{\frac{\gamma}{s}} t^{\lambda_1-1}}{(\max\{t,c_k\})^{\frac{\lambda+\gamma}{s}}}\,dt
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{s-1}\int_{c_i}^{c_{i+1}}\prod_{k=1}^{s}\frac{(\min\{t,c_k\})^{\frac{\gamma}{s}}t^{\lambda_1-1}}{(\max\{t,c_k\})^{\frac{\lambda+\gamma}{s}}}\,dt\\
&=\prod_{k=1}^{s}\frac{1}{c_k^{(\lambda+\gamma)/s}}\int_0^{c_1}t^{\lambda_1+\gamma-1}\,dt+\prod_{k=1}^{s}c_k^{\gamma/s}\int_{c_s}^{\infty}t^{-\lambda_2-\gamma-1}\,dt\\
&+\sum_{i=1}^{s-1}\int_{c_i}^{c_{i+1}}\prod_{k=1}^{i}\frac{c_k^{\frac{\gamma}{s}}}{t^{\frac{\lambda+\gamma}{s}}}\prod_{k=i+1}^{s}\frac{t^{\frac{\gamma}{s}}}{c_k^{\frac{\lambda+\gamma}{s}}}t^{\lambda_1-1}\,dt\\
&=\frac{c_1^{\lambda_1+\gamma}}{\lambda_1+\gamma}\frac{1}{\prod_{k=1}^{s}c_k^{\frac{\lambda+\gamma}{s}}}+\frac{1}{(\lambda_2+\gamma)c_s^{\lambda_2+\gamma}}\prod_{k=1}^{s}c_k^{\frac{\gamma}{s}}\\
&+\sum_{i=1}^{s-1}\frac{\prod_{k=1}^{i}c_k^{\frac{\gamma}{s}}}{\prod_{k=i+1}^{s}c_k^{\frac{\lambda+\gamma}{s}}}\int_{c_i}^{c_{i+1}}t^{\lambda_1-\frac{i\lambda}{s}+(1-\frac{2i}{s})\gamma-1}\,dt.
\end{aligned}
$$

If $\lambda_1-\frac{i\lambda}{s}+(1-\frac{2i}{s})\gamma\neq 0$, then

$$
\int_{c_i}^{c_{i+1}}t^{\lambda_1-\frac{i\lambda}{s}+(1-\frac{2i}{s})\gamma-1}\,dt=\frac{c_{i+1}^{\lambda_1-\frac{i\lambda}{s}+(1-\frac{2i}{s})\gamma}-c_i^{\lambda_1-\frac{i\lambda}{s}+(1-\frac{2i}{s})\gamma}}{\lambda_1-\frac{i\lambda}{s}+(1-\frac{2i}{s})\gamma};
$$

if there exists a $i_0\in\{1,\ldots,s-1\}$, such that $\lambda_1-\frac{i_0\lambda}{s}+(1-\frac{2i_0}{s})\gamma=0$, then we find

$$
\int_{c_{i_0}}^{c_{i_0+1}}t^{\lambda_1-\frac{i_0\lambda}{s}+(1-\frac{2i_0}{s})\gamma-1}\,dt=\ln\left(\frac{c_{i_0+1}}{c_{i_0}}\right)=\lim_{i\to i_0}\int_{c_i}^{c_{i+1}}t^{\lambda_1-\frac{i\lambda}{s}+(1-\frac{2i}{s})\gamma-1}\,dt,
$$

and we still indicate $\ln(\frac{c_{i_0+1}}{c_{i_0}})$ by the following formal expression:

$$
\frac{c_{i_0+1}^{\lambda_1-\frac{i_0\lambda}{s}+(1-\frac{2i_0}{s})\gamma}-c_{i_0}^{\lambda_1-\frac{i_0\lambda}{s}+(1-\frac{2i_0}{s})\gamma}}{\lambda_1-\frac{i_0\lambda}{s}+(1-\frac{2i_0}{s})\gamma}.
$$

Hence, we may set

$$
\begin{aligned}
k_s(\lambda_1)&=\frac{c_1^{\lambda_1+\gamma}}{\lambda_1+\gamma}\frac{1}{\prod_{k=1}^{s}c_k^{\frac{\lambda+\gamma}{s}}}+\frac{1}{(\lambda_2+\gamma)c_s^{\lambda_2+\gamma}}\prod_{k=1}^{s}c_k^{\frac{\gamma}{s}}\\
&+\sum_{i=1}^{s-1}\left[\frac{c_{i+1}^{\lambda_1-\frac{i\lambda}{s}+(1-\frac{2i}{s})\gamma}-c_i^{\lambda_1-\frac{i\lambda}{s}+(1-\frac{2i}{s})\gamma}}{\lambda_1-\frac{i\lambda}{s}+(1-\frac{2i}{s})\gamma}\frac{\prod_{k=1}^{i}c_k^{\frac{\gamma}{s}}}{\prod_{k=i+1}^{s}c_k^{\frac{\lambda+\gamma}{s}}}\right].
\end{aligned}\tag{10}
$$

In particular, (i) for $s=1$ (or $c_s=\cdots=c_1$), we have $k_\lambda(x,y)=\frac{(\min\{x,c_1y\})^{\gamma}}{(\max\{x,c_1y\})^{\lambda+\gamma}}$ and

$$
k_1(\lambda_1)=\frac{\lambda+2\gamma}{(\lambda_1+\gamma)(\lambda_2+\gamma)}\frac{1}{c_1^{\lambda_2}};\tag{11}
$$

(ii) for $s=2$, we have $k_\lambda(x,y)=\frac{(\min\{x,c_1y\}\min\{x,c_2y\})^{\gamma/2}}{(\max\{x,c_1y\}\max\{x,c_2y\})^{(\lambda+\gamma)/2}}$ and

$$
k_2(\lambda_1)=\left(\frac{c_1}{c_2}\right)^{\frac{\gamma}{2}}\left[\frac{c_1^{\lambda_1-\frac{\lambda}{2}}}{(\lambda_1+\gamma)c_2^{\frac{\lambda}{2}}}+\frac{1}{(\lambda_2+\gamma)c_2^{\lambda_2}}+\frac{c_2^{\lambda_1-\frac{\lambda}{2}}-c_1^{\lambda_1-\frac{\lambda}{2}}}{(\lambda_1-\frac{\lambda}{2})c_2^{\frac{\lambda}{2}}}\right];\tag{12}
$$

(iii) for $\gamma = 0$, we have $\lambda_1, \lambda_2 > 0$, $k_\lambda(x,y) = \frac{1}{\prod_{k=1}^{s}(\max\{x,c_k y\})^{\frac{\lambda}{s}}}$ and

$$k_s(\lambda_1) = \widetilde{k}_s(\lambda_1) := \frac{c_1^{\lambda_1}}{\lambda_1}\frac{1}{\prod_{k=1}^{s} c_k^{\frac{\lambda}{s}}} + \frac{1}{\lambda_2 c_s^{\lambda_2}} + \sum_{i=1}^{s-1}\frac{c_{i+1}^{\lambda_1-\frac{i}{s}\lambda} - c_i^{\lambda_1-\frac{i}{s}\lambda}}{\lambda_1 - \frac{i}{s}\lambda}\frac{1}{\prod_{k=i+1}^{s} c_k^{\frac{\lambda}{s}}}; \tag{13}$$

(iv) for $\gamma = -\lambda$, we have $\lambda < \lambda_1, \lambda_2 < 0$, $k_\lambda(x,y) = \frac{1}{\prod_{k=1}^{s}(\min\{x,c_k y\})^{\frac{\lambda}{s}}}$ and

$$k_s(\lambda_1) = \widehat{k}_s(\lambda_1) := \frac{c_1^{-\lambda_2}}{(-\lambda_2)} + \frac{1}{(-\lambda_1)c_s^{-\lambda_1}}\prod_{k=1}^{s} c_k^{\frac{-\lambda}{s}} + \sum_{i=1}^{s-1}\left(\frac{c_{i+1}^{\lambda_1-\frac{s-i}{s}\lambda} - c_i^{\lambda_1-\frac{s-i}{s}\lambda}}{\lambda_1 - \frac{s-i}{s}\lambda}\prod_{k=1}^{i} c_k^{\frac{-\lambda}{s}}\right); \tag{14}$$

(v) for $\lambda = 0$, we have $\lambda_2 = -\lambda_1$, $|\lambda_1| < \gamma$ $(\gamma > 0)$,

$$k_0(x,y) = \prod_{k=1}^{s}\left(\frac{\min\{x,c_k y\}}{\max\{x,c_k y\}}\right)^{\frac{\gamma}{s}},$$

and

$$k_s(\lambda_1) = k_s^{(0)}(\lambda_1) := \frac{c_1^{\lambda_1+\gamma}}{\gamma+\lambda_1}\frac{1}{\prod_{k=1}^{s} c_k^{\frac{\gamma}{s}}} + \frac{c_s^{\lambda_1-\gamma}}{\gamma-\lambda_1}\prod_{k=1}^{s} c_k^{\frac{\gamma}{s}} + \sum_{i=1}^{s-1}\left[\frac{c_{i+1}^{\lambda_1+(1-\frac{2i}{s})\gamma} - c_i^{\lambda_1+(1-\frac{2i}{s})\gamma}}{\lambda_1+(1-\frac{2i}{s})\gamma}\frac{\prod_{k=1}^{i} c_k^{\frac{\gamma}{s}}}{\prod_{k=i+1}^{s} c_k^{\frac{\gamma}{s}}}\right]. \tag{15}$$

(b) Since for $j_0 \in \mathbf{N}$, we find

$$k_\lambda(x,y)\frac{1}{y^{j_0-\lambda_2}} = \frac{1}{y^{j_0-\lambda_2}}\prod_{k=1}^{s}\frac{(\min\{c_k^{-1}x,y\})^{\frac{\gamma}{s}}}{c_k^{\frac{\lambda}{s}}(\max\{c_k^{-1}x,y\})^{\frac{\lambda+\gamma}{s}}}$$
$$= \begin{cases} \frac{1}{y^{j_0-\lambda_2-\gamma}}\prod_{k=1}^{s}\frac{1}{c_k^{\frac{\lambda}{s}}(c_k^{-1}x)^{\frac{\lambda+\gamma}{s}}}, & 0 < y \le c_s^{-1}x,\\ \frac{1}{y^{j_0+\lambda_1+\gamma-\frac{i}{s}(\lambda+2\gamma)}}\frac{\prod_{k=i+1}^{s}(c_k^{-1}x)^{\frac{\gamma}{s}}}{\prod_{k=1}^{s} c_k^{\frac{\lambda}{s}}\prod_{k=1}^{i}(c_k^{-1}x)^{\frac{\lambda+\gamma}{s}}}, & c_{i+1}^{-1}x < y \le c_i^{-1}x\ (i = 1,\ldots,s-1),\\ \frac{1}{y^{j_0+\lambda_1+\gamma}}\prod_{k=1}^{s}\frac{(c_k^{-1}x)^{\frac{\gamma}{s}}}{c_k^{\frac{\lambda}{s}}(y)^{\frac{\lambda+\gamma}{s}}}, & c_1^{-1}x < y < \infty, \end{cases}$$

for $\lambda_2 \le j_0 - \gamma$ $(\lambda_1 > -\gamma)$, $k_\lambda(x,y)\frac{1}{y^{j_0-\lambda_2}}$ is decreasing for $y > 0$ and strictly decreasing for the large enough variable y. In the same way, for $i_0 \in \mathbf{N}$, we find

$$k_\lambda(x,y)\frac{1}{x^{i_0-\lambda_1}} = \frac{1}{x^{i_0-\lambda_1}}\prod_{k=1}^{s}\frac{(\min\{x,c_k y\})^{\frac{\gamma}{s}}}{(\max\{x,c_k y\})^{\frac{\lambda+\gamma}{s}}}$$
$$= \begin{cases} \frac{1}{x^{i_0-\lambda_1-\gamma}}\prod_{k=1}^{s}\frac{1}{(c_k y)^{\frac{\lambda+\gamma}{s}}}, & 0 < x \le c_1 y,\\ \frac{1}{x^{i_0-\lambda_1-\gamma+\frac{i}{s}(\lambda+2\gamma)}}\frac{\prod_{k=1}^{i}(c_k y)^{\frac{\gamma}{s}}}{\prod_{k=i+1}^{s}(c_k y)^{\frac{\lambda+\gamma}{s}}}, & c_i y < x \le c_{i+1}y\ (i = 1,\ldots,s-1),\\ \frac{1}{x^{i_0+\lambda_2+\gamma}}\prod_{k=1}^{s}(c_k y)^{\frac{\gamma}{s}}, & c_s y < x < \infty, \end{cases}$$

then for $\lambda_1 \leq i_0 - \gamma$ ($\lambda_2 > -\gamma$), $k_\lambda(x,y)\frac{1}{x^{i_0-\lambda_1}}$ is decreasing for $x > 0$ and strictly decreasing for the large enough variable x.

In view of the above results, for $i_0, j_0 \in \mathbf{N}$, $-\gamma < \lambda_1 \leq i_0 - \gamma$, $-\gamma < \lambda_2 \leq j_0 - \gamma$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x,y)\frac{1}{y^{j_0-\lambda_2}}$ ($k_\lambda(x,y)\frac{1}{x^{i_0-\lambda_1}}$) is still decreasing for $y > 0$ ($x > 0$) and strictly decreasing for the large enough variable $y(x)$.

Definition 1 For $s, i_0, j_0 \in \mathbf{N}$, $0 < c_1 \leq \cdots \leq c_s < \infty$, $-\gamma < \lambda_1 \leq i_0 - \gamma$, $-\gamma < \lambda_2 \leq j_0 - \gamma$, $\lambda_1 + \lambda_2 = \lambda$, $m = (m_1, \ldots, m_{i_0}) \in \mathbf{N}^{i_0}$, $n = (n_1, \ldots, n_{j_0}) \in \mathbf{N}^{j_0}$, define two weight coefficients $w(\lambda_1, n)$ and $W(\lambda_2, m)$ as follows:

$$w(\lambda_1, n) := \sum_m \prod_{k=1}^{s} \frac{(\min\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\gamma}{s}}}{(\max\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\lambda+\gamma}{s}}} \frac{\|n\|_\beta^{\lambda_2}}{\|m\|_\alpha^{i_0-\lambda_1}}, \tag{16}$$

$$W(\lambda_2, m) := \sum_n \prod_{k=1}^{s} \frac{(\min\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\gamma}{s}}}{(\max\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\lambda+\gamma}{s}}} \frac{\|m\|_\alpha^{\lambda_1}}{\|n\|_\beta^{j_0-\lambda_2}}, \tag{17}$$

where $\sum_m = \sum_{m_{i_0}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty}$ and $\sum_n = \sum_{n_{j_0}=1}^{\infty} \cdots \sum_{n_1=1}^{\infty}$.

Lemma 4 *As the assumptions of Definition* 1, *then* (i) *we have*

$$w(\lambda_1, n) < K_2^{(s)} \quad \left(n \in \mathbf{N}^{j_0}\right), \tag{18}$$

$$W(\lambda_2, m) < K_1^{(s)} \quad \left(m \in \mathbf{N}^{i_0}\right), \tag{19}$$

where

$$K_1^{(s)} = \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} k_s(\lambda_1), \qquad K_2^{(s)} = \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} k_s(\lambda_1); \tag{20}$$

(ii) *for* $p > 1$, $0 < \varepsilon < \frac{p}{2}(\lambda_1 + \gamma)$, *setting* $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}$ $(\in (-\gamma, i_0 - \gamma))$, $\widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$ $(> -\gamma)$, *we have*

$$0 < \widetilde{K}_2^{(s)}\left(1 - \widetilde{\theta}_\lambda(n)\right) < w(\widetilde{\lambda}_1, n), \tag{21}$$

where

$$0 < \widetilde{\theta}_\lambda(n) = \frac{1}{k_s(\widetilde{\lambda}_1)} \int_0^{i_0^{1/\alpha}/\|n\|_\beta} \prod_{k=1}^{s} \frac{(\min\{v, c_k\})^{\frac{\gamma}{s}} v^{\widetilde{\lambda}_1-1}}{(\max\{v, c_k\})^{\frac{\lambda+\gamma}{s}}} \, dv = O\left(\frac{1}{\|n\|_\beta^{\gamma+\widetilde{\lambda}_1}}\right), \tag{22}$$

$$\widetilde{K}_2^{(s)} = \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} k_s(\widetilde{\lambda}_1). \tag{23}$$

Proof By Lemma 1, Example 1, and (7), it follows that

$$\begin{aligned} w(\lambda_1, n) &< \int_{\mathbf{R}_+^{i_0}} \prod_{k=1}^{s} \frac{(\min\{\|x\|_\alpha, c_k\|n\|_\beta\})^{\frac{\gamma}{s}}}{(\max\{\|x\|_\alpha, c_k\|n\|_\beta\})^{\frac{\lambda+\gamma}{s}}} \frac{\|n\|_\beta^{\lambda_2}}{\|x\|_\alpha^{i_0-\lambda_1}} \, dx \\ &= \lim_{M\to\infty} \int_{\mathbf{D}_M} \prod_{k=1}^{s} \frac{(\min\{M[\sum_{i=1}^{i_0}(\frac{x_i}{M})^\alpha]^{\frac{1}{\alpha}}, c_k\|n\|_\beta\})^{\frac{\gamma}{s}}}{(\max\{M[\sum_{i=1}^{i_0}(\frac{x_i}{M})^\alpha]^{\frac{1}{\alpha}}, c_k\|n\|_\beta\})^{\frac{\lambda+\gamma}{s}}} \frac{M^{\lambda_1-i_0}\|n\|_\beta^{\lambda_2} \, dx}{[\sum_{i=1}^{i_0}(\frac{x_i}{M})^\alpha]^{\frac{i_0-\lambda_1}{\alpha}}} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{M\to\infty} \frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_0^1 \prod_{k=1}^{s} \frac{(\min\{Mu^{1/\alpha}, c_k\|n\|_\beta\})^{\frac{\gamma}{s}}}{(\max\{Mu^{1/\alpha}, c_k\|n\|_\beta\})^{\frac{\lambda+\gamma}{s}}} \frac{\|n\|_\beta^{\lambda_2} u^{\frac{i_0}{\alpha}-1}\,du}{M^{i_0-\lambda_1}u^{(i_0-\lambda_1)/\alpha}} \\
&= \lim_{M\to\infty} \frac{M^{\lambda_1}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_0^1 \prod_{k=1}^{s} \frac{(\min\{Mu^{1/\alpha}, c_k\|n\|_\beta\})^{\frac{\gamma}{s}}\|n\|_\beta^{\lambda_2}}{(\max\{Mu^{1/\alpha}, c_k\|n\|_\beta\})^{\frac{\lambda+\gamma}{s}}} u^{\frac{\lambda_1}{\alpha}-1}\,du \\
&\overset{u=\|n\|_\beta^{\alpha}M^{-\alpha}v^{\alpha}}{=} \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \int_0^\infty \prod_{k=1}^{s} \frac{(\min\{v, c_k\})^{\frac{\gamma}{s}} v^{\lambda_1-1}}{(\max\{v, c_k\})^{\frac{\lambda+\gamma}{s}}}\,dv \\
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} k_s(\lambda_1) = K_2^{(s)}.
\end{aligned}
$$

Hence, we have (18). In the same way, we have (19).

By Lemma 1, Example 1, and in the same way as obtaining (8), we have

$$
\begin{aligned}
w(\widetilde{\lambda}_1, n) &> \int_{\{x\in\mathbf{R}_+^{i_0};x_i\geq 1\}} \prod_{k=1}^{s} \frac{(\min\{\|x\|_\alpha, c_k\|n\|_\beta\})^{\frac{\gamma}{s}}}{(\max\{\|x\|_\alpha, c_k\|n\|_\beta\})^{\frac{\lambda+\gamma}{s}}} \frac{\|n\|_\beta^{\widetilde{\lambda}_2}\,dx}{\|x\|_\alpha^{i_0-\widetilde{\lambda}_1}} \\
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \int_{i_0^{1/\alpha}/\|n\|_\beta}^{\infty} \prod_{k=1}^{s} \frac{(\min\{v, c_k\})^{\frac{\gamma}{s}} v^{\widetilde{\lambda}_1-1}}{(\max\{v, c_k\})^{\frac{\lambda+\gamma}{s}}}\,dv = \widetilde{K}_2^{(s)}\bigl(1-\widetilde{\theta}_\lambda(n)\bigr) > 0,
\end{aligned}
$$

$$
0 < \widetilde{\theta}_\lambda(n) = \frac{1}{k_s(\widetilde{\lambda}_1)} \int_0^{i_0^{1/\alpha}/\|n\|_\beta} \prod_{k=1}^{s} \frac{(\min\{v, c_k\})^{\frac{\gamma}{s}} v^{\widetilde{\lambda}_1-1}}{(\max\{v, c_k\})^{\frac{\lambda+\gamma}{s}}}\,dv.
$$

For $\|n\|_\beta \geq c_1^{-1}i_0^{1/\alpha}$, we find $v \leq i_0^{1/\alpha}/\|n\|_\beta \leq c_1 \leq c_k$ ($k = 1,\dots,s$) and

$$
\widetilde{\theta}_\lambda(n) = \frac{1}{k_s(\widetilde{\lambda}_1)} \int_0^{i_0^{1/\alpha}/\|n\|_\beta} \frac{v^{\widetilde{\lambda}_1+\gamma-1}\,dv}{\prod_{k=1}^{s} c_k^{\frac{\lambda+\gamma}{s}}} = \frac{(\prod_{k=1}^{s} c_k^{\frac{\lambda+\gamma}{s}})^{-1}}{(\widetilde{\lambda}_1+\gamma)k_s(\widetilde{\lambda}_1)} \biggl(\frac{i_0^{1/\alpha}}{\|n\|_\beta}\biggr)^{\widetilde{\lambda}_1+\gamma},
$$

and then (22) follows. □

3 Main results

Setting $\Phi(m) := \|m\|_\alpha^{p(i_0-\lambda_1)-i_0}$ ($m \in \mathbf{N}^{i_0}$) and $\Psi(n) := \|n\|_\beta^{q(j_0-\lambda_2)-j_0}$ ($n \in \mathbf{N}^{j_0}$), we have the following.

Theorem 1 *If* $s, i_0, j_0 \in \mathbf{N}$, $0 < c_1 \leq \cdots \leq c_s < \infty$, $-\gamma < \lambda_1 \leq i_0-\gamma$, $-\gamma < \lambda_2 \leq j_0-\gamma$, $\lambda_1+\lambda_2 = \lambda$, $k_s(\lambda_1)$ *is indicated by* (10), *then for* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $a_m, b_n \geq 0$, $0 < \|a\|_{p,\Phi}, \|b\|_{q,\Psi} < \infty$, *we have the following inequality*:

$$
\begin{aligned}
I &:= \sum_n \sum_m \prod_{k=1}^{s} \frac{(\min\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\gamma}{s}}}{(\max\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\lambda+\gamma}{s}}} a_m b_n \\
&< \bigl(K_1^{(s)}\bigr)^{\frac{1}{p}} \bigl(K_2^{(s)}\bigr)^{\frac{1}{q}} \|a\|_{p,\Phi} \|b\|_{q,\Psi},
\end{aligned} \tag{24}
$$

where the constant factor

$$
\bigl(K_1^{(s)}\bigr)^{\frac{1}{p}} \bigl(K_2^{(s)}\bigr)^{\frac{1}{q}} = \biggl[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\biggr]^{\frac{1}{p}} \biggl[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\beta^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\biggr]^{\frac{1}{q}} k_s(\lambda_1) \tag{25}
$$

is the best possible. In particular, for $s = 1$ (*or* $c_s = \cdots = c_1$), *we have the following inequality*:

$$\sum_n \sum_m \frac{(\min\{\|m\|_\alpha, c_1\|n\|_\beta\})^\gamma a_m b_n}{(\max\{\|m\|_\alpha, c_1\|n\|_\beta\})^{\lambda+\gamma}} < \left(K_1^{(1)}\right)^{\frac{1}{p}} \left(K_2^{(1)}\right)^{\frac{1}{q}} \|a\|_{p,\Phi} \|b\|_{q,\Psi}, \tag{26}$$

where

$$\begin{aligned} &\left(K_1^{(1)}\right)^{\frac{1}{p}} \left(K_2^{(1)}\right)^{\frac{1}{q}} \\ &\quad = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\beta^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} \frac{(\lambda+2\gamma)c_1^{-\lambda_2}}{(\lambda_1+\gamma)(\lambda_2+\gamma)}. \end{aligned} \tag{27}$$

Proof By Hölder's inequality (*cf.* [28]), we have

$$\begin{aligned} I &= \sum_n \sum_m \prod_{k=1}^s \frac{(\min\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\gamma}{s}}}{(\max\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\lambda+\gamma}{s}}} \left[\frac{\|m\|_\alpha^{(i_0-\lambda_1)/q}}{\|n\|_\beta^{(j_0-\lambda_2)/p}} a_m\right] \left[\frac{\|n\|_\beta^{(j_0-\lambda_2)/p}}{\|m\|_\alpha^{(i_0-\lambda_1)/q}} b_n\right] \\ &\le \left\{\sum_m W(\lambda_2, m)\|m\|_\alpha^{p(i_0-\lambda_1)-i_0} a_m^p\right\}^{\frac{1}{p}} \left\{\sum_n w(\lambda_1, n)\|n\|_\beta^{q(j_0-\lambda_2)-j_0} b_n^q\right\}^{\frac{1}{q}}. \end{aligned}$$

Then by (18) and (19), we have (24).

For $0 < \varepsilon < \frac{p}{2}(\lambda_1+\gamma)$, $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}$, $\widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$, we set

$$\widetilde{a}_m = \|m\|_\alpha^{-i_0+\lambda_1-\frac{\varepsilon}{p}} = \|m\|_\alpha^{\widetilde{\lambda}_1-i_0}, \qquad \widetilde{b}_n = \|n\|_\beta^{\widetilde{\lambda}_2-j_0-\varepsilon} \quad \left(m \in \mathbf{N}^{i_0}, n \in \mathbf{N}^{j_0}\right).$$

Then by (8) and (21), we obtain

$$\begin{aligned} \|\widetilde{a}\|_{p,\Phi}\|\widetilde{b}\|_{q,\Psi} &= \left[\sum_m \|m\|_\alpha^{p(i_0-\lambda_1)-i_0}\widetilde{a}_m^p\right]^{\frac{1}{p}} \left[\sum_n \|n\|_\beta^{q(j_0-\lambda_2)-j_0}\widetilde{b}_n^q\right]^{\frac{1}{q}} \\ &= \left(\sum_m \|m\|_\alpha^{-i_0-\varepsilon}\right)^{\frac{1}{p}} \left(\sum_n \|n\|_\beta^{-j_0-\varepsilon}\right)^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon}\left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} + \varepsilon O(1)\right)^{\frac{1}{p}} \\ &\quad \times \left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon\widetilde{O}(1)\right)^{\frac{1}{q}}, \end{aligned} \tag{28}$$

$$\begin{aligned} \widetilde{I} &:= \sum_n \left[\sum_m \prod_{k=1}^s \frac{(\min\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\gamma}{s}}}{(\max\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\lambda+\gamma}{s}}} \widetilde{a}_m\right]\widetilde{b}_n \\ &= \sum_n w(\widetilde{\lambda}_1, n)\|n\|_\beta^{-j_0-\varepsilon} > \widetilde{K}_2^{(s)} \sum_n \left(1 - O\left(\frac{1}{\|n\|_\beta^{\gamma+\widetilde{\lambda}_1}}\right)\right)\|n\|_\beta^{-j_0-\varepsilon} \\ &= \widetilde{K}_2^{(s)}\left(\sum_n \|n\|_\beta^{-j_0-\varepsilon} - \sum_n O\left(\frac{1}{\|n\|_\beta^{\gamma+\lambda_1+j_0+\frac{\varepsilon}{q}}}\right)\right) \\ &= \widetilde{K}_2^{(s)}\left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\varepsilon j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \widetilde{O}(1) - O(1)\right). \end{aligned} \tag{29}$$

If there exists a constant $K \leq (K_1^{(s)})^{\frac{1}{p}}(K_2^{(s)})^{\frac{1}{q}}$, such that (24) is valid as we replace $(K_1^{(s)})^{\frac{1}{p}}(K_2^{(s)})^{\frac{1}{q}}$ by K, then using (28) and (29) we have

$$\begin{aligned}
&\bigl(K_2^{(s)} + o(1)\bigr)\left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon\widetilde{O}(1) - \varepsilon O(1)\right) < \varepsilon\widetilde{I} < \varepsilon K\|\widetilde{a}\|_{p,\varphi}\|\widetilde{b}\|_{q,\psi} \\
&\quad = K\left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} + \varepsilon O(1)\right)^{\frac{1}{p}}\left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon\widetilde{O}(1)\right)^{\frac{1}{q}}.
\end{aligned}$$

For $\varepsilon \to 0^+$, we find

$$\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}k_s(\lambda_1) \leq K\left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right)^{\frac{1}{p}}\left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right)^{\frac{1}{q}},$$

and then $(K_1^{(s)})^{\frac{1}{p}}(K_2^{(s)})^{\frac{1}{q}} \leq K$. Hence, $K = (K_1^{(s)})^{\frac{1}{p}}(K_2^{(s)})^{\frac{1}{q}}$ is the best possible constant factor of (24). □

Theorem 2 *As regards the assumptions of Theorem* 1, *for* $0 < \|a\|_{p,\Phi} < \infty$, *we have the following inequality with the best constant factor* $(K_1^{(s)})^{\frac{1}{p}}(K_2^{(s)})^{\frac{1}{q}}$:

$$\begin{aligned}
J &:= \left\{\sum_n \|n\|_\beta^{p\lambda_2-j_0}\left[\sum_m \prod_{k=1}^{s} \frac{(\min\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\gamma}{s}}}{(\max\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\lambda+\gamma}{s}}}a_m\right]^p\right\}^{\frac{1}{p}} \\
&< \bigl(K_1^{(s)}\bigr)^{\frac{1}{p}}\bigl(K_2^{(s)}\bigr)^{\frac{1}{q}}\|a\|_{p,\Phi},
\end{aligned} \tag{30}$$

which is equivalent to (24). *In particular, for* $s = 1$ (*or* $c_s = \cdots = c_1$), *we have the following inequality equivalent to* (26):

$$\left\{\sum_n \|n\|_\beta^{p\lambda_2-j_0}\left[\sum_m \frac{(\min\{\|m\|_\alpha, c_1\|n\|_\beta\})^{\gamma}}{(\max\{\|m\|_\alpha, c_1\|n\|_\beta\})^{\lambda+\gamma}}a_m\right]^p\right\}^{\frac{1}{p}} < \bigl(K_1^{(1)}\bigr)^{\frac{1}{p}}\bigl(K_2^{(1)}\bigr)^{\frac{1}{q}}\|a\|_{p,\Phi}. \tag{31}$$

Proof We set b_n as follows:

$$b_n := \|n\|_\beta^{p\lambda_2-j_0}\left(\sum_m \prod_{k=1}^{s} \frac{(\min\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\gamma}{s}}}{(\max\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\lambda+\gamma}{s}}}a_m\right)^{p-1}, \quad n \in \mathbf{N}^{j_0}.$$

Then it follows that $J^p = \|b\|_{q,\Psi}^q$. If $J = 0$, then (30) is trivially valid for $0 < \|a\|_{p,\Phi} < \infty$; if $J = \infty$, then it is impossible since the right hand side of (30) is finite. Suppose that $0 < J < \infty$. Then by (24), we find

$$\|b\|_{q,\Psi}^q = J^p = I < \bigl(K_1^{(s)}\bigr)^{\frac{1}{p}}\bigl(K_2^{(s)}\bigr)^{\frac{1}{q}}\|a\|_{p,\Phi}\|b\|_{q,\Psi},$$

namely,

$$\|b\|_{q,\Psi}^{q-1} = J < \bigl(K_1^{(s)}\bigr)^{\frac{1}{p}}\bigl(K_2^{(s)}\bigr)^{\frac{1}{q}}\|a\|_{p,\Phi},$$

and then (30) follows.

On the other hand, assuming that (30) is valid, by Hölder's inequality, we have

$$I = \sum_n \bigl(\Psi(n)\bigr)^{\frac{-1}{q}} \left[\sum_m \prod_{k=1}^{s} \frac{(\min\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\gamma}{s}} a_m}{(\max\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\lambda+\gamma}{s}}} \right] \bigl[\bigl(\Psi(n)\bigr)^{\frac{1}{q}} b_n\bigr]$$

$$\leq J\|b\|_{q,\Psi}. \tag{32}$$

Then by (30), we have (24). Hence (30) and (24) are equivalent.

By the equivalency, the constant factor $(K_1^{(s)})^{\frac{1}{p}}(K_2^{(s)})^{\frac{1}{q}}$ in (30) is the best possible. Otherwise, we would reach a contradiction by (32) that the constant factor $(K_1^{(s)})^{\frac{1}{p}}(K_2^{(s)})^{\frac{1}{q}}$ in (24) is not the best possible. □

4 Operator expressions and some particular cases

For $p > 1$, we define two real weight normal discrete spaces $l_{p,\varphi}$ and $l_{q,\psi}$ as follows:

$$l_{p,\varphi} := \left\{ a = \{a_m\}; \|a\|_{p,\Phi} = \left(\sum_m \Phi(m) a_m^p \right)^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\psi} := \left\{ b = \{b_n\}; \|b\|_{q,\Psi} = \left(\sum_n \Psi(n) b_n^q \right)^{\frac{1}{q}} < \infty \right\}.$$

As regards the assumptions of Theorem 1, in view of $J < (K_1^{(s)})^{\frac{1}{p}}(K_2^{(s)})^{\frac{1}{q}}\|a\|_{p,\Phi}$, we give the following definition.

Definition 2 Define a multidimensional Hilbert-type operator $T : l_{p,\Phi} \to l_{p,\Psi^{1-p}}$ as follows: For $a \in l_{p,\Phi}$, there exists an unique representation $Ta \in l_{p,\Psi^{1-p}}$, satisfying

$$Ta(n) := \sum_m \prod_{k=1}^{s} \frac{(\min\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\gamma}{s}}}{(\max\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\lambda+\gamma}{s}}} a_m \quad \bigl(n \in \mathbf{N}^{j_0}\bigr). \tag{33}$$

For $b \in l_{q,\Psi}$, we define the following formal inner product of Ta and b as follows:

$$(Ta, b) := \sum_n \sum_m \prod_{k=1}^{s} \frac{(\min\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\gamma}{s}}}{(\max\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\lambda+\gamma}{s}}} a_m b_n. \tag{34}$$

Then by Theorem 1 and Theorem 2, for $0 < \|a\|_{p,\varphi}, \|b\|_{q,\psi} < \infty$, we have the following equivalent inequalities:

$$(Ta, b) < \bigl(K_1^{(s)}\bigr)^{\frac{1}{p}} \bigl(K_2^{(s)}\bigr)^{\frac{1}{q}} \|a\|_{p,\Phi} \|b\|_{q,\Psi}, \tag{35}$$

$$\|Ta\|_{p,\Psi^{1-p}} < \bigl(K_1^{(s)}\bigr)^{\frac{1}{p}} \bigl(K_2^{(s)}\bigr)^{\frac{1}{q}} \|a\|_{p,\Phi}. \tag{36}$$

It follows that T is bounded with

$$\|T\| := \sup_{a(\neq\theta)\in l_{p,\Phi}} \frac{\|Ta\|_{p,\Psi^{1-p}}}{\|a\|_{p,\Phi}} \leq \bigl(K_1^{(s)}\bigr)^{\frac{1}{p}} \bigl(K_2^{(s)}\bigr)^{\frac{1}{q}}. \tag{37}$$

Since the constant factor $(K_1^{(s)})^{\frac{1}{p}}(K_2^{(s)})^{\frac{1}{q}}$ in (36) is the best possible, we have the following.

Corollary 1 *As regards the assumptions of Theorem* 2, T *is defined by Definition* 2, *it follows that*

$$\begin{aligned}\|T\| &= \left(K_1^{(s)}\right)^{\frac{1}{p}}\left(K_2^{(s)}\right)^{\frac{1}{q}} \\ &= \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} k_s(\lambda_1). \qquad (38)\end{aligned}$$

Remark 1 (i) For $i_0 = j_0 = 1$ in (24), we have the inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{m, c_k n\})^{\frac{\gamma}{s}}}{(\max\{m, c_k n\})^{\frac{\lambda+\gamma}{s}}} a_m b_n < k_s(\lambda_1)\|a\|_{p,\phi}\|b\|_{q,\psi}. \qquad (39)$$

Hence, (24) is an extension of (4) for

$$k_\lambda(m,n) = \prod_{k=1}^{s}\frac{(\min\{m, c_k n\})^{\frac{\gamma}{s}}}{(\max\{m, c_k n\})^{\frac{\lambda+\gamma}{s}}}.$$

(ii) For $\gamma = 0$ in (24) and (30), we have $0 < \lambda_1 \leq i_0$, $0 < \lambda_2 \leq j_0$ and the following equivalent inequalities:

$$\sum_n\sum_m \frac{a_m b_n}{\prod_{k=1}^{s}(\max\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\lambda}{s}}} < \widetilde{K}_s(\lambda_1)\|a\|_{p,\Phi}\|b\|_{q,\Psi}, \qquad (40)$$

$$\left\{\sum_n \|n\|_\beta^{p\lambda_2-j_0}\left[\sum_m \frac{a_m}{\prod_{k=1}^{s}(\max\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\lambda}{s}}}\right]^p\right\}^{\frac{1}{p}} < \widetilde{K}_s(\lambda_1)\|a\|_{p,\Phi}, \qquad (41)$$

where the best possible constant factor is defined by

$$\widetilde{K}_s(\lambda_1) := \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\beta^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}}\widetilde{k}_s(\lambda_1), \qquad (42)$$

and $\widetilde{k}_s(\lambda_1)$ is indicated by (13).

(iii) For $\gamma = -\lambda$ in (24) and (30), we have $\lambda < \lambda_1 \leq i_0 + \lambda$, $\lambda < \lambda_2 \leq j_0 + \lambda$, $\lambda_1, \lambda_2 < 0$ and the following equivalent inequalities:

$$\sum_n\sum_m \frac{a_m b_n}{\prod_{k=1}^{s}(\min\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\lambda}{s}}} < \widehat{K}_s(\lambda_1)\|a\|_{p,\Phi}\|b\|_{q,\Psi}, \qquad (43)$$

$$\left\{\sum_n \|n\|_\beta^{p\lambda_2-j_0}\left[\sum_m \frac{a_m}{\prod_{k=1}^{s}(\min\{\|m\|_\alpha, c_k\|n\|_\beta\})^{\frac{\lambda}{s}}}\right]^p\right\}^{\frac{1}{p}} < \widetilde{K}_s(\lambda_1)\|a\|_{p,\Phi}, \qquad (44)$$

where the best possible constant factor is defined by

$$\widehat{K}_s(\lambda_1) := \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\beta^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}}\widehat{k}_s(\lambda_1), \qquad (45)$$

and $\widehat{k}_s(\lambda_1)$ is indicated by (14).

(iv) For $\lambda = 0$ in (24) and (30), we have $\lambda_2 = -\lambda_1$, $0 < \gamma + \lambda_1 \le i_0$, $0 < \gamma - \lambda_1 \le j_0$ ($\gamma > 0$), and the following equivalent inequalities:

$$\sum_n \sum_m \prod_{k=1}^{s} \left(\frac{\min\{\|m\|_\alpha, c_k\|n\|_\beta\}}{\max\{\|m\|_\alpha, c_k\|n\|_\beta\}} \right)^{\frac{\gamma}{s}} a_m b_n < K_s^{(0)}(\lambda_1)\|a\|_{p,\Phi}\|b\|_{q,\Psi}, \tag{46}$$

$$\left\{ \sum_n \frac{1}{\|n\|_\beta^{p\lambda_1+j_0}} \left[\sum_m \prod_{k=1}^{s} \left(\frac{\min\{\|m\|_\alpha, c_k\|n\|_\beta\}}{\max\{\|m\|_\alpha, c_k\|n\|_\beta\}} \right)^{\frac{\gamma}{s}} a_m \right]^p \right\}^{\frac{1}{p}} < K_s^{(0)}(\lambda_1)\|a\|_{p,\Phi}, \tag{47}$$

where the best possible constant factor is defined by

$$K_s^{(0)}(\lambda_1) := \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\beta^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} k_s^{(0)}(\lambda_1), \tag{48}$$

and $k_s^{(0)}(\lambda_1)$ is indicated by (15).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. YS participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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