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A regularization algorithm for a common solution of generalized equilibrium problem, fixed point problem and the zero points of the sum of two operators

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Abstract

For finding a common solution of generalized equilibrium problem, fixed point problem and the zero points of the sum of two operators, a regularization algorithm is established in the framework of real Hilbert spaces. And the strong convergence theorem is obtained under certain assumptions. The main results presented in this paper are useful in nonlinear analysis and optimization. Moreover, the results and corollaries extend the corresponding conclusions proposed by many authors.

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1 Introduction

In this paper, assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, and let C be a nonempty, closed and convex subset of H. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. In the following, we recall some mappings which will often be used in this paper.

- $f: C \to C$ is said to be k-contractive iff there exists a constant $k \in (0,1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$ for all $x, y \in C$.
- $S: C \to C$ is said to be nonexpansive iff $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$.
- $T: H \to H$ is said to be firmly nonexpansive iff $\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$ for all $x, y \in H$.
- $P_C: H \to C$ is said to be metric projection iff $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$.
- $A: H \to H$ is said to be monotone iff $\langle x - y, Ax - Ay\rangle \geq 0$ for all $x, y \in H$.
- Given a number $\eta > 0$, $A: H \to H$ is said to be η-strongly monotone iff $\langle x - y, Ax - Ay\rangle \geq \eta\|x - y\|^2$ for all $x, y \in H$.
- Given a number $\alpha > 0$, $A: C \to H$ is said to be α-inverse-strongly monotone (α-ism) iff $\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2$ for all $x, y \in C$.

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- $Y : C \to H$ is a strict pseudo-contraction [1] if there exists $t \in \mathbb{R}$ with $0 \leq t < 1$ such that $\|Yx - Yy\|^2 \leq \|x - y\|^2 + t\|(I - Y)x - (I - Y)y\|^2$ for all $x, y \in C$.

First, we introduce the following generalized equilibrium problem.

Find $x \in C$ such that

$$F(x, y) + \langle Tx, y - x\rangle \geq 0, \quad \forall y \in C, \tag{1.1}$$

where $T : C \to H$ is a monotone mapping and $F : C \times C \to \mathbb{R}$ is a bifunction.

In this paper, we use $\mathrm{GEP}(F, T)$ to denote the set of such $x \in C$, *i.e.*, $\mathrm{GEP}(F, T) = \{x \in C : F(x, y) + \langle Tx, y - x\rangle \geq 0, \forall y \in C\}$.

In the case of $T \equiv 0$, problem (1.1) is reduced to the following equilibrium problem [2].

Find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

In this paper, we use $\mathrm{EP}(F)$ to denote the set of such $x \in C$, *i.e.*, $\mathrm{EP}(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}$.

In the case of $F \equiv 0$, problem (1.1) is reduced to the classical variational inequality.

To study equilibrium problems (1.1) and (1.2), we may assume that F satisfies the following conditions:

(A1) $F(x, x)=0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t\downarrow 0} F\bigl(tz + (1 - t)x, y\bigr) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and weakly lower semicontinuous.

Many problems can be transformed into finding solutions of equilibrium problems (1.1) and (1.2), for instance, image recovery, network allocation, inverse problems, transportation problems and optimization problems; see [2–7] and the references therein. Recently, many regularization methods have been extensively studied for solving solutions of equilibrium problems (1.1) and (1.2); see [7–13] and the references therein.

Second, we introduce the following fixed point problem for a family of infinitely nonexpansive mappings. Consider the fixed point problem

$$\mathrm{Fix}(S) := \{x \in C : x = Sx\},$$

where $S : C \to C$ is a mapping, we use $\mathrm{Fix}(S)$ to denote the fixed point set of S. If C is a bounded, closed and convex subset of H, then $\mathrm{Fix}(S)$ is not empty; see [1].

Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings and $\{\gamma_i\}$ be a nonnegative real sequence with $0 \leq \gamma_i < 1$, $\forall i \geq 1$. For $n \geq 1$, define a mapping $W_n : C \to C$ as follows:

$$U_{n,n+1} = I,$$

$$U_{n,n} = \gamma_n S_n U_{n,n+1} + (1 - \gamma_n)I,$$

$$U_{n,n-1} = \gamma_{n-1}S_{n-1}U_{n,n} + (1-\gamma_{n-1})I,$$
$$\ldots,$$
$$U_{n,k} = \gamma_k S_k U_{n,k+1} + (1-\gamma_k)I,$$
$$U_{n,k-1} = \gamma_{k-1}S_{k-1}U_{n,k} + (1-\gamma_{k-1})I,$$
$$\ldots,$$
$$U_{n,2} = \gamma_2 S_2 U_{n,3} + (1-\gamma_2)I,$$
$$W_n = U_{n,1} = \gamma_1 S_1 U_{n,2} + (1-\gamma_1)I.$$

Such a mapping W_n is nonexpansive from C to C and it is called a W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\gamma_n, \gamma_{n-1}, \ldots, \gamma_1$; see [14] and the references therein.

Third, we introduce the problem of zero points of a maximal monotone mapping

$$M^{-1}0 = \{x \in H : 0 \in Mx\}, \tag{1.3}$$

where M is a mapping of H into 2^H, the effective domain of M is denoted by $\operatorname{dom} M$ or $D(M)$, that is, $\operatorname{dom} M = \{x \in H : Mx \neq \emptyset\}$. A multi-valued mapping M is said to be a monotone mapping on H if $\langle x-y, u-v\rangle \geq 0$ for all $x, y \in \operatorname{dom} M$, $u \in Mx$, $v \in My$. A monotone mapping M on H is said to be maximal if its graph is not properly contained in the graph of any other monotone mapping on H. It is well known that a monotone mapping M is maximal if and only if, for any $(x,u) \in H \times H$, $\langle x-y, u-v\rangle \geq 0$ for all $(y,v) \in \operatorname{Graph}(M)$ implies $u \in Mx$.

For a maximal monotone mapping M on H and $r > 0$, we may define a single-valued mapping $J_r = (I + rM)^{-1} : H \to \operatorname{dom} M$, which is called the resolvent of M for r. It is easy to see that $M^{-1}0 = \operatorname{Fix}(J_r)$ for all $r > 0$, and the resolvent J_r is firmly nonexpansive, *i.e.*,

$$\|J_r x - J_r y\|^2 \leq \langle x - y, J_r x - J_r y\rangle, \quad \forall x, y \in H.$$

Finally, we introduce the subdifferential of a lower semicontinuous convex function and an indicator function.

Let h be a proper lower semicontinuous convex function on a Hilbert space H into $(-\infty, \infty]$. Then the subdifferential ∂h of h is defined as follows:

$$\partial h(x) = \bigl\{z \in H : h(x) + \langle z, y - x\rangle \leq h(y), \forall y \in H\bigr\} \tag{1.4}$$

for all $x \in H$. From Rockafellar [15] we know that ∂h is a maximal monotone operator. Let i_C be the indicator function of C (C is a nonempty closed convex subset of H), *i.e.*,

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases} \tag{1.5}$$

Then i_C is a proper lower semicontinuous convex function on H and the subdifferential ∂i_C of i_C is a maximal monotone mapping. So we can define the resolvent J_r of ∂i_C for $r > 0$, *i.e.*,

$$J_r x = (I + r\partial i_C)^{-1} x$$

for all $x \in H$. We have that for any $x \in H$ and $q \in C$,

$$\begin{aligned} q = J_r x \quad &\Longleftrightarrow \quad x \in q + r\partial i_C(q) \\ &\Longleftrightarrow \quad x \in q + rN_C(q) \\ &\Longleftrightarrow \quad x - q \in rN_C(q) \\ &\Longleftrightarrow \quad \frac{1}{r}\langle x - q, p - q\rangle \leq 0, \quad \forall p \in C \\ &\Longleftrightarrow \quad \langle x - q, p - q\rangle \leq 0, \quad \forall p \in C \\ &\Longleftrightarrow \quad q = P_C x, \end{aligned}$$

where $N_C(q)$ is the normal cone to C at q, *i.e.*,

$$N_C(q) = \{z \in H : \langle z, p - q\rangle \leq 0, \forall p \in C\}.$$

In the present paper, we study the equilibrium problems (1.1) and (1.2), the fixed point problem for a family of infinitely nonexpansive mappings, and the problem of zero points of maximal monotone mapping (1.3). Motivated and inspired by the research going on in this direction, we propose a new regularization algorithm, and it is proved that the sequence generated by this algorithm converges strongly to a common solution of the above three problems. The results presented in this paper improve and extend the corresponding results in Chang *et al.* [7], Takahashi and Takahashi [8], Hao [16] and Yuan and Zhang [17].

The structure of this paper is set as follows. In Section 2, we introduce some lemmas which will be used in the proof of theorems. The main result, that is, the strong convergence of the regularization algorithm, is proved in Section 3. Corollaries to generalized equilibrium problem and the zero points of the sum of two operators are presented in Section 4. And the conclusion of this paper is given in the final section, *i.e.*, Section 5.

2 Preliminaries

In the following, we give some useful lemmas, which will often be used in the proof of the main results and their corollaries.

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. When $\{x_n\}$ is a sequence in H, we define that the strong convergence of $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. Let C be a nonempty closed convex subset of a Hilbert space H, and let $T : C \to H$ be a mapping. We denote by $\mathrm{Fix}(T)$ the set of fixed points for T. If $T : C \to H$ is a nonexpansive mapping, then $\mathrm{Fix}(T)$ is closed and convex; see [18].

Firstly, we recall the metric (nearest point) projection from H onto C is the mapping $P_C : H \to C$ which is defined as follows: given $x \in H$, $P_C x$ is the unique point in C with the property

$$\|x - P_C x\| = \inf_{y \in C}\|x - y\| =: d(x, C).$$

P_C is characterized as follows.

Lemma 2.1 *Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if the following inequality holds*:

$$\langle x - y, y - z \rangle \geq 0, \quad \forall z \in C.$$

Then we introduce the lemma below, which shows the uniqueness of solution of the variational inequality.

Lemma 2.2 [19] *Let H be a Hilbert space, C be a closed convex subset of H, and $f : C \to C$ be a contraction with coefficient $\alpha < 1$. Then*

$$\langle x - y, (I - f)x - (I - f)y \rangle \geq (1 - \alpha)\|x - y\|^2, \quad x, y \in C.$$

That is, $I - f$ is strongly monotone with coefficient $1 - \alpha$.

Lemma 2.3 [2] *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Define a mapping $T_r : H \to C$ as follows:

$$T_r x = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}, \quad x \in H,$$

then the following conclusions hold:

(1) *T_r is single-valued*;

(2) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

(3) $F(T_r) = \mathrm{EP}(F)$;

(4) $\mathrm{EP}(F)$ *is closed and convex.*

In the following, we introduce the property of W-mapping generated by a family of infinitely nonexpansive mappings.

Lemma 2.4 [14] *Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings with a nonempty common fixed point set, and let $\{\gamma_i\}$ be a real sequence such that $0 < \gamma_i \leq l < 1$, where l is some real number, $\forall i > 1$. Then*

(1) *W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^n F(S_i)$ for each $n \geq 1$*;

(2) *for each $x \in C$ and for each positive integer k, the limit $\lim_{n\to\infty} U_{n,k}$ exists*;

(3) *the mapping $W : C \to C$ defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x, \quad x \in C, \tag{2.1}$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^\infty F(S_i)$ and it is called the W-mapping generated by $S_1, S_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$.

Lemma 2.5 [7] *Let* $\{S_i : C \to C\}$ *be a family of infinitely nonexpansive mappings with a nonempty common fixed point set, and let* $\{\gamma_i\}$ *be a real sequence such that* $0 < \gamma_i \le l < 1$, $\forall i > 1$. *If* K *is any bounded subset of* C, *then*

$$\limsup_{n\to\infty, x\in K} \|Wx - W_n x\| = 0.$$

The next lemma which we introduce is about the resolvent of the maximal monotone operator.

Lemma 2.6 (see [20–22]) *Let* H *be a real Hilbert space, and let* B *be a maximal monotone operator on* H. *For* $r > 0$ *and* $x \in H$, *define the resolvent* $J_r x$. *Then the following holds*:

$$\frac{s-t}{s}\langle J_s x - J_t x, J_s x - x\rangle \ge \|J_s x - J_t x\|^2$$

for all $s, t > 0$ *and* $x \in H$. *In particular*,

$$\|J_s x - J_t x\| \le \bigl(|s-t|/s\bigr)\|x - J_s x\|$$

for all $s, t > 0$ *and* $x \in H$.

The following lemma will be used in the proof of the main results.

Lemma 2.7 [23] *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in* H, *and let* $\{\beta_n\}$ *be a sequence in* $(0,1)$ *with* $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. *Suppose that* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all* $n \ge 0$ *and*

$$\limsup_{n\to\infty}\bigl(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\bigr) \le 0.$$

Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Besides, the following two lemmas are extremely important in the proof of theorems. One is called a demiclosed principle for nonexpansive mappings, the other is called an important lemma.

Lemma 2.8 (Demiclosed principle [24]) *Let* $T : C \to C$ *be a nonexpansive mapping with* $F(T) \ne \emptyset$. *If* $\{x_n\}_{n=1}^{\infty}$ *is a sequence in* C *weekly converging to* x *and if* $\{(I-T)x_n\}_{n=1}^{\infty}$ *converges strongly to* y, *then* $(I-T)x = y$. *In particular, if* $y = 0$, *then* $x \in F(T)$.

Lemma 2.9 [25] *Assume that* $\{a_n\}_{n=0}^{\infty}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n + \beta_n, \quad n \ge 0,$$

where $\{\gamma_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *are sequences in* $(0,1)$ *and* $\{\delta_n\}_{n=0}^{\infty}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=0}^{\infty}\gamma_n = \infty$;
(ii) *either* $\limsup_{n\to\infty}\delta_n \le 0$ *or* $\sum_{n=0}^{\infty}\gamma_n|\delta_n| < \infty$;
(iii) $\sum_{n=0}^{\infty}\beta_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

Theorem 3.1 *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction which satisfies* (A1)-(A4), *and let $f : C \to C$ be a k-contraction with the constant $k \in (0,1)$. Let $A : C \to H$ be an α-inverse-strongly monotone (α-ism) mapping with $\alpha > 0$, let $B : C \to H$ be a β-inverse-strongly monotone (β-ism) mapping with $\beta > 0$, and let $T : C \to H$ be a τ-inverse-strongly monotone (τ-ism) mapping with $\tau > 0$. Let M be a maximal monotone operator on H such that the domain of M is included in C, and let $J_s = (I + sM)^{-1}$ be the resolvent of M for $s > 0$. Let N be a maximal monotone operator on H such that the domain of N is included in C, and let $J_r = (I + rN)^{-1}$ be the resolvent of N for $r > 0$. Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Omega = \mathrm{GEP}(F,T) \cap \bigcap_{i=1}^{\infty} F(S_i) \cap (A+N)^{-1}0 \cap (B+M)^{-1}0 \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{r_n\}$, $\{\lambda_n\}$ and $\{s_n\}$ be positive number sequences. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = J_{r_n}(u_n - r_n A u_n), \\ x_{n+1} = \alpha_n f(y_n) + \beta_n W_n J_{s_n}(y_n - s_n B y_n) + \gamma_n x_n, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{u_n\}$ is such that

$$F(u_n, y) + \langle T x_n, y - u_n \rangle + \frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$

and $\{W_n\}$ is the sequence generated in (2.1). *Assume that the following conditions hold*:

(i) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;
(ii) $0 < c \leq \lambda_n \leq d < 2\tau$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;
(iii) $0 < e \leq s_n \leq g < 2\beta$ *and* $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$;
(iv) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(v) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$,

where a, b, c, d, e, and g are real constants. Then $\{x_n\}$ converges strongly to a point $q \in \Omega$, which solves uniquely the following variational inequality:

$$\langle q - f(q), q - x \rangle \leq 0, \quad \forall x \in \Omega. \tag{3.2}$$

Equivalently, $q = P_\Omega f(q)$.

Proof We divide the proof into several steps.

Step 1. We prove that the sequence $\{x_n\}$ is bounded.

Since A is an α-ism mapping, we see from restriction (i) that $\forall x, y \in C$,

$$\begin{aligned} \|(I - r_n A)x - (I - r_n A)y\|^2 &= \|x - y\|^2 - 2r_n \langle x - y, Ax - Ay \rangle + r_n^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2r_n \alpha \|Ax - Ay\|^2 + r_n^2 \|Ax - Ay\|^2 \\ &= \|x - y\|^2 + r_n(r_n - 2\alpha) \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

This implies that $I - r_n A$ is nonexpansive.

In the same way, we find that $I - s_nB$ and $I - \lambda_n T$ are nonexpansive. Note that $u_n = T_{\lambda_n}(I - \lambda_n T)x_n$. Let $p \in \Omega$, it follows that

$$\|u_n - p\| \le \big\|(I - \lambda_n T)x_n - (I - \lambda_n T)p\big\| \le \|x_n - p\|.$$

Putting $z_n = J_{s_n}(y_n - s_nBy_n)$, we see that

$$\|z_n - p\| \le \|y_n - p\| \le \|u_n - p\| \le \|x_n - p\|. \tag{3.3}$$

From (3.3), we find that

$$\begin{aligned}\|x_{n+1} - p\| &\le \alpha_n\big\|f(y_n) - p\big\| + \beta_n\|W_nz_n - p\| + \gamma_n\|x_n - p\| \\ &\le \alpha_n k\|y_n - p\| + \alpha_n\big\|f(p) - p\big\| + \beta_n\|z_n - p\| + \gamma_n\|x_n - p\| \\ &\le \big(1 - \alpha_n(1-k)\big)\|x_n - p\| + \alpha_n(1-k)\frac{\|f(p) - p\|}{1-k}.\end{aligned}$$

By induction, we derive that

$$\|x_n - p\| \le \max\left\{\|x_1 - p\|, \frac{\|f(p) - p\|}{1-k}\right\}, \quad \forall n \ge 1.$$

Therefore, it turns out that $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Without loss of generality, we can assume that there exists a bounded set $K \subset C$ such that $x_n, y_n, z_n, u_n \in K$. Since $u_n = T_{\lambda_n}(I - \lambda_n T)x_n$, we find that

$$F(u_{n+1}, y) + \frac{1}{\lambda_{n+1}}\big\langle y - u_{n+1}, u_{n+1} - (I - \lambda_{n+1}T)x_{n+1}\big\rangle \ge 0, \quad \forall y \in C \tag{3.4}$$

and

$$F(u_n, y) + \frac{1}{\lambda_n}\big\langle y - u_n, u_n - (I - \lambda_n T)x_n\big\rangle \ge 0, \quad \forall y \in C. \tag{3.5}$$

Let $y = u_n$ in (3.4) and $y = u_{n+1}$ in (3.5). Then we add up (3.4) and (3.5) to derive that

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - (I - \lambda_n T)x_n - \frac{\lambda_n}{\lambda_{n+1}}\big(u_{n+1} - (I - \lambda_{n+1}T)x_{n+1}\big)\right\rangle \ge 0.$$

This implies that

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\le \bigg\langle u_{n+1} - u_n, (I - \lambda_{n+1}T)x_{n+1} - (I - \lambda_n T)x_n \\ &\qquad + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)\cdot\big(u_{n+1} - (I - \lambda_{n+1}T)x_{n+1}\big)\bigg\rangle \\ &\le \|u_{n+1} - u_n\|\cdot\bigg(\big\|(I - \lambda_{n+1}T)x_{n+1} - (I - \lambda_n T)x_n\big\| \\ &\qquad + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right|\cdot\big\|u_{n+1} - (I - \lambda_{n+1}T)x_{n+1}\big\|\bigg).\end{aligned}$$

From $I - \lambda_n T$ is nonexpansive and condition (ii), we obtain that

$$\begin{aligned}\|u_{n+1} - u_n\| &\leq \big\|(I - \lambda_{n+1}T)x_{n+1} - (I - \lambda_n T)x_n\big\| \\ &\quad + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}}\big\|u_{n+1} - (I - \lambda_{n+1}T)x_{n+1}\big\| \\ &\leq \big\|(I - \lambda_{n+1}T)x_{n+1} - (I - \lambda_{n+1}T)x_n\big\| + |\lambda_{n+1} - \lambda_n| \cdot \|Tx_n\| \\ &\quad + \frac{|\lambda_{n+1} - \lambda_n|}{c}\big\|u_{n+1} - (I - \lambda_{n+1}T)x_{n+1}\big\| \\ &\leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|L_1, \end{aligned} \tag{3.6}$$

where L_1 is an appropriate constant such that

$$L_1 = \sup_{n\geq 1}\left\{\|Tx_n\| + \frac{1}{c}\big\|u_{n+1} - (I - \lambda_{n+1}T)x_{n+1}\big\|\right\}.$$

Since both J_{r_n} and $I - r_n A$ are nonexpansive, it follows from Lemma 2.6, condition (i) and (3.6) that

$$\begin{aligned}\|y_{n+1} - y_n\| &= \big\|J_{r_{n+1}}(I - r_{n+1}A)u_{n+1} - J_{r_n}(I - r_n A)u_n\big\| \\ &\leq \big\|J_{r_{n+1}}(I - r_{n+1}A)u_{n+1} - J_{r_{n+1}}(I - r_{n+1}A)u_n\big\| \\ &\quad + \big\|J_{r_{n+1}}(I - r_{n+1}A)u_n - J_{r_{n+1}}(I - r_n A)u_n\big\| \\ &\quad + \big\|J_{r_{n+1}}(I - r_n A)u_n - J_{r_n}(I - r_n A)u_n\big\| \\ &\leq \|u_{n+1} - u_n\| + \big\|(I - r_{n+1}A)u_n - (I - r_n A)u_n\big\| \\ &\quad + \frac{|r_{n+1} - r_n|}{r_{n+1}}\big\|J_{r_{n+1}}(I - r_n A)u_n - (I - r_n A)u_n\big\| \\ &\leq \|u_{n+1} - u_n\| + |r_{n+1} - r_n| \cdot \|Au_n\| \\ &\quad + \frac{|r_{n+1} - r_n|}{a}\big\|J_{r_{n+1}}(I - r_n A)u_n - (I - r_n A)u_n\big\| \\ &\leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|L_1 + |r_{n+1} - r_n|L_2, \end{aligned} \tag{3.7}$$

where L_2 is an appropriate constant such that

$$L_2 = \sup_{n\geq 1}\left\{\|Au_n\| + \frac{1}{a}\big\|J_{r_{n+1}}(I - r_n A)u_n - (I - r_n A)u_n\big\|\right\}.$$

Thus, from both J_{s_n} and $I - s_n B$ are nonexpansive, we have from Lemma 2.6, condition (iii) and (3.7) that

$$\begin{aligned}\|z_{n+1} - z_n\| &= \big\|J_{s_{n+1}}(I - s_{n+1}B)y_{n+1} - J_{s_n}(I - s_n B)y_n\big\| \\ &\leq \big\|J_{s_{n+1}}(I - s_{n+1}B)y_{n+1} - J_{s_{n+1}}(I - s_{n+1}B)y_n\big\| \\ &\quad + \big\|J_{s_{n+1}}(I - s_{n+1}B)y_n - J_{s_{n+1}}(I - s_n B)y_n\big\| \\ &\quad + \big\|J_{s_{n+1}}(I - s_n B)y_n - J_{s_n}(I - s_n B)y_n\big\| \\ &\leq \|y_{n+1} - y_n\| + \big\|(I - s_{n+1}B)y_n - (I - s_n B)y_n\big\| \end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{|s_{n+1} - s_n|}{s_{n+1}} \bigl\| J_{s_{n+1}}(I - s_n B)y_n - (I - s_n B)y_n \bigr\| \\
&\le \|y_{n+1} - y_n\| + |s_{n+1} - s_n| \cdot \|By_n\| \\
&\quad + \frac{|s_{n+1} - s_n|}{e} \bigl\| J_{s_{n+1}}(I - s_n B)y_n - (I - s_n B)y_n \bigr\| \\
&\le \|x_{n+1} - x_n\| \\
&\quad + L_3 \bigl(|\lambda_{n+1} - \lambda_n| + |r_{n+1} - r_n| + |s_{n+1} - s_n| \bigr),
\end{aligned} \tag{3.8}
$$

where $L_3 = \max\{L_1, L_2, \sup_{n\geq 1}\{\|By_n\| + \frac{1}{e}\|J_{s_{n+1}}(I - s_n B)y_n - (I - s_n B)y_n\|\}\}$.

It yields from (3.8) that

$$
\begin{aligned}
\|W_{n+1}z_{n+1} - W_n z_n\| &\le \|W_{n+1}z_{n+1} - Wz_{n+1}\| + \|Wz_{n+1} - Wz_n\| \\
&\quad + \|Wz_n - W_n z_n\| \\
&\le \sup_{x\in K} \bigl\{ \|W_{n+1}x - Wx\| + \|Wx - W_n x\| \bigr\} + \|x_{n+1} - x_n\| \\
&\quad + L_3 \bigl(|\lambda_{n+1} - \lambda_n| + |r_{n+1} - r_n| + |s_{n+1} - s_n| \bigr),
\end{aligned} \tag{3.9}
$$

where K is the bounded subset of C defined above. Let $x_{n+1} = (1 - \gamma_n)t_n + \gamma_n x_n$.

It follows that

$$
\begin{aligned}
t_{n+1} - t_n &= \frac{\alpha_{n+1} f(y_{n+1}) + \beta_{n+1} W_{n+1} z_{n+1}}{1 - \gamma_{n+1}} - \frac{\alpha_n f(y_n) + \beta_n W_n z_n}{1 - \gamma_n} \\
&= \frac{\alpha_{n+1}}{1 - \gamma_{n+1}} f(y_{n+1}) + \frac{1 - \alpha_{n+1} - \gamma_{n+1}}{1 - \gamma_{n+1}} W_{n+1} z_{n+1} \\
&\quad - \left(\frac{\alpha_n}{1 - \gamma_n} f(y_n) + \frac{1 - \alpha_n - \gamma_n}{1 - \gamma_n} W_n z_n \right) \\
&= \frac{\alpha_{n+1}}{1 - \gamma_{n+1}} \bigl(f(y_{n+1}) - W_{n+1} z_{n+1} \bigr) - \frac{\alpha_n}{1 - \gamma_n} \bigl(f(y_n) - W_n z_n \bigr) \\
&\quad + W_{n+1} z_{n+1} - W_n z_n.
\end{aligned}
$$

From (3.9), we derive that

$$
\begin{aligned}
\|t_{n+1} - t_n\| &\le \frac{\alpha_{n+1}}{1 - \gamma_{n+1}} \bigl\| f(y_{n+1}) - W_{n+1} z_{n+1} \bigr\| + \frac{\alpha_n}{1 - \gamma_n} \bigl\| f(y_n) - W_n z_n \bigr\| \\
&\quad + \|W_{n+1} z_{n+1} - W_n z_n\| \\
&\le \frac{\alpha_{n+1}}{1 - \gamma_{n+1}} \bigl\| f(y_{n+1}) - W_{n+1} z_{n+1} \bigr\| + \frac{\alpha_n}{1 - \gamma_n} \bigl\| f(y_n) - W_n z_n \bigr\| \\
&\quad + \sup_{x\in K} \bigl\{ \|W_{n+1}x - Wx\| + \|Wx - W_n x\| \bigr\} + \|x_{n+1} - x_n\| \\
&\quad + L_3 \bigl(|\lambda_{n+1} - \lambda_n| + |r_{n+1} - r_n| + |s_{n+1} - s_n| \bigr),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|t_{n+1} - t_n\| - \|x_{n+1} - x_n\| &\le \frac{\alpha_{n+1}}{1 - \gamma_{n+1}} \bigl\| f(y_{n+1}) - W_{n+1} z_{n+1} \bigr\| \\
&\quad + \frac{\alpha_n}{1 - \gamma_n} \bigl\| f(y_n) - W_n z_n \bigr\|
\end{aligned}
$$

$$+ \sup_{x\in K}\{\|W_{n+1}x - Wx\| + \|Wx - W_n x\|\}$$
$$+ L_3(|\lambda_{n+1} - \lambda_n| + |r_{n+1} - r_n| + |s_{n+1} - s_n|).$$

It follows from conditions (i)-(v) that

$$\lim_{n\to\infty} \sup(\|t_{n+1} - t_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

From Lemma 2.7, we derive that $\lim_{n\to\infty} \|t_n - x_n\| = 0$.

Since $t_n - x_n = \frac{1}{1-\gamma_n}(x_{n+1} - x_n)$, it follows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.10}$$

Step 3. We prove that $\lim_{n\to\infty} \|x_n - u_n\| = 0$.

Since T is τ-ism, we find from (3.3) that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \alpha_n\|f(y_n) - p\|^2 + \beta_n\|u_n - p\|^2 + \gamma_n\|x_n - p\|^2\\ &\leq \alpha_n\|f(y_n) - p\|^2 + \beta_n\|x_n - p - \lambda_n(Tx_n - Tp)\|^2 + \gamma_n\|x_n - p\|^2\\ &\leq \alpha_n\|f(y_n) - p\|^2 + \|x_n - p\|^2 - \lambda_n\beta_n(2\tau - \lambda_n)\|Tx_n - Tp\|^2,\end{aligned}$$

it turns out that

$$\begin{aligned}&\lambda_n\beta_n(2\tau - \lambda_n)\|Tx_n - Tp\|^2\\ &\quad\leq \alpha_n\|f(y_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\|.\end{aligned}$$

By virtue of conditions (ii), (iv), (v), we derive from (3.10) that

$$\lim_{n\to\infty} \|Tx_n - Tp\| = 0. \tag{3.11}$$

Since T_{λ_n} is firmly nonexpansive, we find from Lemma 2.3 that

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{\lambda_n}(I - \lambda_n T)x_n - T_{\lambda_n}(I - \lambda_n T)p\|^2\\ &\leq \langle (I - \lambda_n T)x_n - (I - \lambda_n T)p, u_n - p\rangle\\ &\leq \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2\\ &\quad + 2\lambda_n\|Tx_n - Tp\| \cdot \|x_n - u_n\|),\end{aligned}$$

it implies that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\lambda_n\|Tx_n - Tp\| \cdot \|x_n - u_n\|.$$

Therefore, we derive that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \alpha_n\|f(y_n) - p\|^2 + \beta_n\|u_n - p\|^2 + \gamma_n\|x_n - p\|^2\\ &\leq \alpha_n\|f(y_n) - p\|^2 - \beta_n\|x_n - u_n\|^2\\ &\quad + 2\lambda_n\beta_n\|Tx_n - Tp\| \cdot \|x_n - u_n\| + \|x_n - p\|^2,\end{aligned}$$

which yields that

$$\begin{aligned}\beta_n\|x_n-u_n\|^2 &\le \alpha_n\big\|f(y_n)-p\big\|^2+2\lambda_n\beta_n\|Tx_n-Tp\|\cdot\|x_n-u_n\| \\ &\quad+\big(\|x_n-p\|+\|x_{n+1}-p\|\big)\|x_{n+1}-x_n\|.\end{aligned}$$

By use of conditions (ii), (iv), (v), we obtain from (3.10) and (3.11) that

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.12}$$

Step 4. We prove that $\lim_{n\to\infty}\|u_n-y_n\|=0$.

Since J_{r_n} is nonexpansive, A is α-ism, we find from (3.3) that

$$\begin{aligned}\|y_n-p\|^2 &= \big\|J_{r_n}(I-r_nA)u_n-J_{r_n}(I-r_nA)p\big\|^2 \\ &\le \big\|(I-r_nA)u_n-(I-r_nA)p\big\|^2 \\ &\le \|u_n-p\|^2-2r_n\alpha\|Au_n-Ap\|^2+r_n^2\|Au_n-Ap\|^2 \\ &\le \|x_n-p\|^2+r_n(r_n-2\alpha)\|Au_n-Ap\|^2.\end{aligned}$$

Therefore, we derive that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\big\|f(y_n)-p\big\|^2+\beta_n\|W_nz_n-p\|^2+\gamma_n\|x_n-p\|^2 \\ &\le \alpha_n\big\|f(y_n)-p\big\|^2+\beta_n\|y_n-p\|^2+\gamma_n\|x_n-p\|^2 \\ &\le \alpha_n\big\|f(y_n)-p\big\|^2+r_n(r_n-2\alpha)\beta_n\|Au_n-Ap\|^2+\|x_n-p\|^2,\end{aligned}$$

it turns out that

$$\begin{aligned}&r_n(2\alpha-r_n)\beta_n\|Au_n-Ap\|^2 \\ &\quad\le \alpha_n\big\|f(y_n)-p\big\|^2+\big(\|x_n-p\|+\|x_{n+1}-p\|\big)\|x_{n+1}-x_n\|.\end{aligned}$$

In view of conditions (i), (iv), (v), we obtain from (3.10) that

$$\lim_{n\to\infty}\|Au_n-Ap\|=0. \tag{3.13}$$

Since J_{r_n} is firmly nonexpansive, $I-r_nA$ is nonexpansive, we find from (3.3) that

$$\begin{aligned}\|y_n-p\|^2 &= \big\|J_{r_n}(I-r_nA)u_n-J_{r_n}(I-r_nA)p\big\|^2 \\ &\le \big\langle (I-r_nA)u_n-(I-r_nA)p, y_n-p\big\rangle \\ &= \frac{1}{2}\Big\{\big\|(I-r_nA)u_n-(I-r_nA)p\big\|^2+\|y_n-p\|^2 \\ &\quad-\big\|(I-r_nA)u_n-(I-r_nA)p-(y_n-p)\big\|^2\Big\} \\ &\le \frac{1}{2}\Big\{\|u_n-p\|^2+\|y_n-p\|^2-\big\|u_n-y_n-r_n(Au_n-Ap)\big\|^2\Big\} \\ &\le \frac{1}{2}\Big\{\|x_n-p\|^2+\|y_n-p\|^2-\|u_n-y_n\|^2 \\ &\quad+2r_n\|u_n-y_n\|\cdot\|Au_n-Ap\|\Big\},\end{aligned}$$

which implies that

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|u_n - y_n\|^2 + 2r_n\|u_n - y_n\| \cdot \|Au_n - Ap\|.$$

From (3.3), this further implies that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n \|f(y_n) - p\|^2 + \beta_n \|W_n z_n - p\|^2 + \gamma_n \|x_n - p\|^2 \\
&\le \alpha_n \|f(y_n) - p\|^2 + \beta_n \|y_n - p\|^2 + \gamma_n \|x_n - p\|^2 \\
&\le \alpha_n \|f(y_n) - p\|^2 - \beta_n \|u_n - y_n\|^2 \\
&\quad + 2r_n\beta_n \|u_n - y_n\| \cdot \|Au_n - Ap\| + \|x_n - p\|^2,
\end{aligned}$$

which yields that

$$\begin{aligned}
\beta_n \|u_n - y_n\|^2 &\le \alpha_n \|f(y_n) - p\|^2 + 2r_n\beta_n \|u_n - y_n\| \cdot \|Au_n - Ap\| \\
&\quad + \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_{n+1} - x_n\|.
\end{aligned}$$

By use of conditions (i), (iv), (v), we derive from (3.10) and (3.13) that

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.14}$$

Step 5. We prove that $\lim_{n\to\infty} \|y_n - z_n\| = 0$.

Since J_{s_n} is nonexpansive, B is β-ism, we find from (3.3) that

$$\begin{aligned}
\|z_n - p\|^2 &= \|J_{s_n}(I - s_nB)y_n - J_{s_n}(I - s_nB)p\|^2 \\
&\le \|(I - s_nB)y_n - (I - s_nB)p\|^2 \\
&\le \|y_n - p\|^2 - 2s_n\beta \|By_n - Bp\|^2 + s_n^2 \|By_n - Bp\|^2 \\
&\le \|x_n - p\|^2 + s_n(s_n - 2\beta)\|By_n - Bp\|^2.
\end{aligned}$$

Hence, we derive that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \big\|\alpha_n\big(f(y_n) - p\big) + \beta_n(W_nz_n - p) + \gamma_n(x_n - p)\big\|^2 \\
&\le \alpha_n \|f(y_n) - p\|^2 + \beta_n \|z_n - p\|^2 + \gamma_n \|x_n - p\|^2 \\
&\le \alpha_n \|f(y_n) - p\|^2 + s_n(s_n - 2\beta)\beta_n \|By_n - Bp\|^2 + \|x_n - p\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&s_n(2\beta - s_n)\beta_n \|By_n - Bp\|^2 \\
&\quad \le \alpha_n \|f(y_n) - p\|^2 + \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_{n+1} - x_n\|.
\end{aligned}$$

By virtue of conditions (iii), (iv), (v), we derive from (3.10) that

$$\lim_{n\to\infty} \|By_n - Bp\| = 0. \tag{3.15}$$

Since J_{s_n} is firmly nonexpansive, $I - s_nB$ is nonexpansive, we find from (3.3) that

$$\begin{aligned}
\|z_n - p\|^2 &= \big\|J_{s_n}(I - s_nB)y_n - J_{s_n}(I - s_nB)p\big\|^2 \\
&\leq \big\langle (I - s_nB)y_n - (I - s_nB)p, z_n - p\big\rangle \\
&= \frac{1}{2}\big\{\big\|(I - s_nB)y_n - (I - s_nB)p\big\|^2 + \|z_n - p\|^2 \\
&\quad - \big\|(I - s_nB)y_n - (I - s_nB)p - (z_n - p)\big\|^2\big\} \\
&\leq \frac{1}{2}\big\{\|y_n - p\|^2 + \|z_n - p\|^2 - \big\|y_n - z_n - s_n(By_n - Bp)\big\|^2\big\} \\
&\leq \frac{1}{2}\big\{\|x_n - p\|^2 + \|z_n - p\|^2 - \|y_n - z_n\|^2 \\
&\quad + 2s_n\|y_n - z_n\| \cdot \|By_n - Bp\|\big\}.
\end{aligned}$$

It turns out that

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - \|y_n - z_n\|^2 + 2s_n\|y_n - z_n\| \cdot \|By_n - Bp\|.$$

Hence, we obtain that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \alpha_n\big\|f(y_n) - p\big\|^2 + \beta_n\|W_nz_n - p\|^2 + \gamma_n\|x_n - p\|^2 \\
&\leq \alpha_n\big\|f(y_n) - p\big\|^2 + \beta_n\|z_n - p\|^2 + \gamma_n\|x_n - p\|^2 \\
&\leq \alpha_n\big\|f(y_n) - p\big\|^2 - \beta_n\|y_n - z_n\|^2 \\
&\quad + 2s_n\beta_n\|y_n - z_n\| \cdot \|By_n - Bp\| + \|x_n - p\|^2,
\end{aligned}$$

it follows that

$$\begin{aligned}
\beta_n\|y_n - z_n\|^2 &\leq \alpha_n\big\|f(y_n) - p\big\|^2 + 2s_n\beta_n\|y_n - z_n\| \cdot \|By_n - Bp\| \\
&\quad + \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_{n+1} - x_n\|.
\end{aligned}$$

By use of conditions (iii), (iv), (v), we obtain from (3.10) and (3.15) that

$$\lim_{n\to\infty}\|y_n - z_n\| = 0. \tag{3.16}$$

Step 6. We show that

$$\lim_{n\to\infty}\sup\big\langle f(q) - q, x_n - q\big\rangle \leq 0, \quad \forall x_n \in \Omega,$$

where $q = P_\Omega f(q)$. It is equivalent to show that $q \in \Omega = \mathrm{GEP}(F, T) \cap \bigcap_{i=1}^{\infty} F(S_i) \cap (A + N)^{-1}0 \cap (B + M)^{-1}0$.

First, we show that $q \in \mathrm{GEP}(F, T)$. From Lemma 2.3, we get $u_n = T_{\lambda_n}(I - \lambda_nT)x_n$, for any $y \in C$, we find from (A2) that

$$\langle Tx_{n_i}, y - u_{n_i}\rangle + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle \geq F(y, u_{n_i}), \quad \forall y \in C. \tag{3.17}$$

Putting $y_h = hy + (1-h)q$ for any $h \in (0,1]$ and $y \in C$, we see that $y_h \in C$. From (3.17), we derive that

$$\begin{aligned}\langle y_h - u_{n_i}, Ty_h\rangle &\geq \langle y_h - u_{n_i}, Ty_h\rangle - \langle Tx_{n_i}, y_h - u_{n_i}\rangle \\ &\quad - \left\langle y_h - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle + F(y_h, u_{n_i}) \\ &= \langle y_h - u_{n_i}, Ty_h - Tu_{n_i}\rangle + \langle y_h - u_{n_i}, Tu_{n_i} - Tx_{n_i}\rangle \\ &\quad - \left\langle y_h - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle + F(y_h, u_{n_i}).\end{aligned}$$

By virtue of the monotonicity of T and condition (ii), we obtain from (A4) that

$$\langle y_h - q, Ty_h\rangle \geq F(y_h, q). \tag{3.18}$$

From (A1) and (A4), we see that

$$\begin{aligned}0 &= F(y_h, y_h) \leq hF(y_h, y) + (1-h)F(y_h, q) \\ &\leq hF(y_h, y) + (1-h)\langle y_h - q, Ty_h\rangle \\ &= hF(y_h, y) + (1-h)h\langle y - q, Ty_h\rangle.\end{aligned}$$

It turns out from (A3) that $q \in \mathrm{GEP}(F, T)$.

Then we show that $q \in \bigcap_{i=1}^{\infty} F(S_i)$. Indeed, choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty} \sup\bigl\langle (f - I)q, x_n - q\bigr\rangle = \lim_{i\to\infty}\bigl\langle (f - I)q, x_{n_i} - q\bigr\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $\{x_{n_{i_j}}\}$ converges weakly to q. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup q$. In view of (3.12), (3.14) and (3.16), we know $x_{n_i} \rightharpoonup q$ is equivalent to $u_{n_i} \rightharpoonup q$, $y_{n_i} \rightharpoonup q$ and $z_{n_i} \rightharpoonup q$.

Since $x_{n+1} = \alpha_n f(y_n) + \beta_n W_n z_n + \gamma_n x_n$, it implies that

$$\beta_n \| W_n z_n - x_n\| \leq \|x_{n+1} - x_n\| + \alpha_n \bigl\|f(y_n) - x_n\bigr\|.$$

By virtue of conditions (iv), (v), we derive from (3.10) that

$$\lim_{n\to\infty} \| W_n z_n - x_n\| = 0. \tag{3.19}$$

Observing that

$$\| W_n z_n - z_n\| \leq \| W_n z_n - x_n\| + \|x_n - u_n\| + \|u_n - y_n\| + \|y_n - z_n\|,$$

and from (3.12), (3.14), (3.16) and (3.19), we derive that

$$\lim_{n\to\infty} \| W_n z_n - z_n\| = 0. \tag{3.20}$$

It is not hard to find that

$$\| W z_n - z_n\| \leq \| W z_n - W_n z_n\| + \| W_n z_n - z_n\| \leq \sup_{x\in K} \| Wx - W_n x\| + \| W_n z_n - z_n\|.$$

From Lemma 2.5 and (3.20), we obtain that

$$\lim_{n\to\infty} \|Wz_n - z_n\| = 0.$$

Since $z_{n_i} \rightharpoonup q$, $W : C \to C$ is a nonexpansive mapping, we get by Lemma 2.8 that $q \in F(W)$. Then by Lemma 2.4, we know that $q \in F(W)$ is equivalent to $q \in \bigcap_{i=1}^{\infty} F(S_i)$.

In the following, we show that $q \in (A + N)^{-1}0$.

As in [26], we have that for any $r > 0$,

$$\begin{aligned} q \in (A+N)^{-1}0 \quad &\Longleftrightarrow \quad 0 \in Aq + Nq \\ &\Longleftrightarrow \quad 0 \in rAq + rNq \\ &\Longleftrightarrow \quad q - rAq \in q + rNq \\ &\Longleftrightarrow \quad q = J_r(I - rA)q \\ &\Longleftrightarrow \quad q \in \operatorname{Fix}\bigl(J_r(I - rA)\bigr). \end{aligned} \tag{3.21}$$

In view of condition (i), we can take $r_0 \in [a, b]$. From Lemma 2.6 and J_{r_0} is nonexpansive, we derive that

$$\begin{aligned} \bigl\|J_{r_0}(I - r_0A)u_n - y_n\bigr\| &\le \bigl\|J_{r_0}(I - r_0A)u_n - J_{r_0}(I - r_nA)u_n\bigr\| \\ &\quad + \bigl\|J_{r_0}(I - r_nA)u_n - y_n\bigr\| \\ &\le \bigl\|(I - r_0A)u_n - (I - r_nA)u_n\bigr\| \\ &\quad + \bigl\|J_{r_0}(I - r_nA)u_n - J_{r_n}(I - r_nA)u_n\bigr\| \\ &\le |r_n - r_0| \cdot \bigl\|A(u_n)\bigr\| \\ &\quad + \frac{|r_n - r_0|}{r_0}\bigl\|J_{r_0}(I - r_nA)u_n - (I - r_nA)u_n\bigr\| \to 0. \end{aligned} \tag{3.22}$$

Observing that

$$\bigl\|J_{r_0}(I - r_0A)u_n - u_n\bigr\| \le \bigl\|J_{r_0}(I - r_0A)u_n - y_n\bigr\| + \|y_n - u_n\|,$$

and from (3.14) and (3.22), we have that

$$\bigl\|J_{r_0}(I - r_0A)u_n - u_n\bigr\| \to 0. \tag{3.23}$$

From the boundedness of $\{x_n\}$, we may assume that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q$, $q \in C$. By (3.12), we also have that $u_{n_i} \rightharpoonup q$, $q \in C$. On the other hand, from $r_n \to r_0 \in [a, b]$, we have that $r_{n_i} \to r_0 \in [a, b]$.

By use of (3.23), we have that

$$\bigl\|J_{r_0}(I - r_0A)u_{n_i} - u_{n_i}\bigr\| \to 0.$$

Since $J_{r_0}(I - r_0A)$ is nonexpansive, we have from Lemma 2.8 that $q = J_{r_0}(I - r_0A)q$. In virtue of (3.21), this means that $q \in (A + N)^{-1}0$.

By use of the similar proof method, we can also derive that

$$\left\|J_{s_0}(I-s_0B)y_n - z_n\right\| \to 0, \tag{3.24}$$

and we also have from (3.16) and (3.24) that

$$\left\|J_{s_0}(I-s_0B)y_n - y_n\right\| \le \left\|J_{s_0}(I-s_0B)y_n - z_n\right\| + \|y_n - z_n\| \to 0.$$

From the above proof processing, $x_{n_i} \rightharpoonup q$ and $\|x_n - y_n\| \to 0$, we have that $y_{n_i} \rightharpoonup q$, where $q \in C$. On the other hand, from $s_n \to s_0 \in [e,g]$, we have that $s_{n_i} \to s_0 \in [e,g]$.

In view of the above inequality, we have that

$$\left\|J_{s_0}(I-s_0B)y_{n_i} - y_{n_i}\right\| \to 0.$$

Since $J_{s_0}(I-s_0B)$ is nonexpansive, we also have from Lemma 2.8 that $q = J_{s_0}(I-s_0B)q$. By (3.21), we obtain that $q \in (B+M)^{-1}0$.

Step 7. We finally prove that $x_n \to q$ in norm.

Indeed, we derive from (3.1) and (3.3) that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \alpha_n\big\langle f(y_n)-q, x_{n+1}-q\big\rangle + \beta_n\langle W_n z_n - q, x_{n+1}-q\rangle \\
&\quad + \gamma_n\langle x_n - q, x_{n+1}-q\rangle \\
&\le \alpha_n k\|y_n - q\| \cdot \|x_{n+1}-q\| + \alpha_n\big\langle f(q)-q, x_{n+1}-q\big\rangle \\
&\quad + \beta_n\|z_n - q\| \cdot \|x_{n+1}-q\| + \gamma_n\|x_n - q\| \cdot \|x_{n+1}-q\| \\
&\le \big(1-\alpha_n(1-k)\big)\|x_n - q\| \cdot \|x_{n+1}-q\| + \alpha_n\big\langle f(q)-q, x_{n+1}-q\big\rangle \\
&\le \frac{1}{2}\big(1-\alpha_n(1-k)\big)\big(\|x_n-q\|^2 + \|x_{n+1}-q\|^2\big) \\
&\quad + \alpha_n\big\langle f(q)-q, x_{n+1}-q\big\rangle,
\end{aligned}$$

it turns out that

$$\|x_{n+1}-q\|^2 \le \big(1-\alpha_n(1-k)\big)\|x_n-q\|^2 + 2\alpha_n\big\langle f(q)-q, x_{n+1}-q\big\rangle.$$

By use of condition (iv), we obtain from Lemma 2.9 that

$$\lim_{n\to\infty}\|x_n - q\|^2 = 0,$$

i.e., $x_n \to q$ as $n \to \infty$.

It is easy to see that the variational inequality (3.2) can be rewritten as

$$\big\langle f(q)-q, q-x\big\rangle \ge 0, \quad \forall x \in \Omega.$$

From Lemma 2.1, it is equivalent to the following fixed point equation:

$$P_\Omega f(q) = q.$$

This completes the proof. □

4 Theorems

Theorem 4.1 *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction which satisfies* (A1)-(A4), *and let $f : C \to C$ be a k-contraction with the constant $k \in (0,1)$. Let $T : C \to H$ be a τ-ism mapping with $\tau > 0$. Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Omega = \mathrm{GEP}(F,T) \cap \bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{\lambda_n\}$ be a positive number sequence. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(y_n) + \beta_n W_n u_n + \gamma_n x_n, \quad \forall n \in \mathbb{N}, \tag{4.1}$$

where $\{u_n\}$ is such that

$$F(u_n, y) + \langle Tx_n, y - u_n\rangle + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$

and $\{W_n\}$ is the sequence generated in (2.1). *Assume that the following conditions hold*:

(1) $0 < c \leq \lambda_n \leq d < 2\tau$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;

(2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(3) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$,

where c and d are real constants. Then $\{x_n\}$ converges strongly to a point $q \in \Omega$, which solves uniquely the variational inequality (3.2).

Theorem 4.2 *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction which satisfies* (A1)-(A4), *and let $f : C \to C$ be a k-contraction with the constant $k \in (0,1)$. Let $A : C \to H$ be an α-ism mapping with $\alpha > 0$. Let $T : C \to H$ be a τ-ism mapping with $\tau > 0$. Let N be a maximal monotone operator on H such that the domain of N is included in C. Let $J_r = (I + rN)^{-1}$ be the resolvent of N for $r > 0$. Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Omega = \mathrm{GEP}(F,T) \cap \bigcap_{i=1}^{\infty} F(S_i) \cap (A+N)^{-1}0 \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{r_n\}$ and $\{\lambda_n\}$ be positive number sequences. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = J_{r_n}(u_n - r_n A u_n), \\ x_{n+1} = \alpha_n f(y_n) + \beta_n W_n y_n + \gamma_n x_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{4.2}$$

where $\{u_n\}$ is such that

$$F(u_n, y) + \langle Tx_n, y - u_n\rangle + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$

and $\{W_n\}$ is the sequence generated in (2.1). *Assume that the following conditions hold*:

(1) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

(2) $0 < c \leq \lambda_n \leq d < 2\tau$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;

(3) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(4) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$,

where a, b, c, and d are real constants. Then $\{x_n\}$ converges strongly to a point $q \in \Omega$, which solves uniquely the variational inequality (3.2).

Theorem 4.3 *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction which satisfies* (A1)-(A4), *and let $f : C \to C$ be a k-contraction with the constant $k \in (0,1)$. Let $A : C \to H$ be an α-ism mapping with $\alpha > 0$, let $B : C \to H$ be a β-ism mapping with $\beta > 0$, and let $T : C \to H$ be a τ-ism mapping with $\tau > 0$. Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Omega = \mathrm{GEP}(F,T) \cap \bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{VI}(C,A) \cap \mathrm{VI}(C,B) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{r_n\}$, $\{\lambda_n\}$ and $\{s_n\}$ be positive number sequences. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = P_C(u_n - r_n A u_n), \\ x_{n+1} = \alpha_n f(y_n) + \beta_n W_n P_C(y_n - s_n B y_n) + \gamma_n x_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{4.3}$$

where $\{u_n\}$ is such that

$$F(u_n, y) + \langle T x_n, y - u_n \rangle + \frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$

and $\{W_n\}$ is the sequence generated in (2.1). *Assume that the following conditions hold*:

(1) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

(2) $0 < c \leq \lambda_n \leq d < 2\tau$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;

(3) $0 < e \leq s_n \leq g < 2\beta$ *and* $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$;

(4) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(5) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$,

where a, b, c, d, e, and g are real constants. Then $\{x_n\}$ converges strongly to a point $q \in \Omega$, which solves uniquely the variational inequality (3.2).

Proof Put $N = \partial i_C$ and $M = \partial i_C$ in Theorem 3.1. Then, for $r_n > 0$ and $s_n > 0$, we have that $J_{r_n} = P_C$ and $J_{s_n} = P_C$. Furthermore, we have $(A + \partial i_C)^{-1}0 = \mathrm{VI}(C,A)$ and $(B + \partial i_C)^{-1}0 = \mathrm{VI}(C,B)$. Indeed, for $q \in C$, we have

$$\begin{aligned} q \in (A + \partial i_C)^{-1}0 \quad &\Longleftrightarrow \quad 0 \in Aq + \partial i_C(q) \\ &\Longleftrightarrow \quad 0 \in Aq + N_C q \\ &\Longleftrightarrow \quad -Aq \in N_C q \\ &\Longleftrightarrow \quad \langle -Aq, p - q \rangle \leq 0, \quad \forall p \in C \\ &\Longleftrightarrow \quad \langle Aq, p - q \rangle \geq 0, \quad \forall p \in C \\ &\Longleftrightarrow \quad q \in \mathrm{VI}(C,A). \end{aligned}$$

Similarly, for $q \in C$, we also have

$$q \in (B + \partial i_C)^{-1}0 \quad \Longleftrightarrow \quad q \in \mathrm{VI}(C,B).$$

Thus, we obtain the desired result by Theorem 3.1. □

Theorem 4.4 *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction which satisfies* (A1)-(A4), *and let $f : C \to C$ be a*

k-contraction with the constant $k \in (0,1)$. Let $A : C \to H$ be an α-ism mapping with $\alpha > 0$, and let $B : C \to H$ be a β-ism mapping with $\beta > 0$. Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Omega = \mathrm{EP}(F) \cap \bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{VI}(C,A) \cap \mathrm{VI}(C,B) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{r_n\}$, $\{\lambda_n\}$ and $\{s_n\}$ be positive number sequences. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = P_C(u_n - r_n A u_n), \\ x_{n+1} = \alpha_n f(y_n) + \beta_n W_n P_C(y_n - s_n B y_n) + \gamma_n x_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{4.4}$$

where $\{u_n\}$ is such that

$$F(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$

and $\{W_n\}$ is the sequence generated in (2.1). Assume that the following conditions hold:

(1) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;
(2) $0 < c \leq \lambda_n \leq d < 2\tau$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;
(3) $0 < e \leq s_n \leq g < 2\beta$ *and* $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$;
(4) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(5) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$,

where a, b, c, d, e, and g are real constants. Then $\{x_n\}$ converges strongly to a point $q \in \Omega$, which solves uniquely the variational inequality (3.2).

Proof In Theorem 4.3, put $T = 0$. Then, for all $\tau \in (0,\infty)$, we have

$$\langle x - y, Tx - Ty\rangle \geq \tau \|Tx - Ty\|^2, \quad \forall x, y \in C.$$

Taking $c, d \in (0,\infty)$ with $0 < c < d < \infty$ and choosing a sequence $\{\lambda_n\}$ of real numbers with $c \leq \lambda_n \leq d$, we obtain the desired result by Theorem 4.3. □

Theorem 4.5 *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $P_C : H \to C$ be the metric projection. Let $f : C \to C$ be a k-contraction with the constant $k \in (0,1)$. Let $A : C \to H$ be an α-ism mapping with $\alpha > 0$, let $B : C \to H$ be a β-ism mapping with $\beta > 0$, and let $T : C \to H$ be a τ-ism mapping with $\tau > 0$. Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Omega = \mathrm{VI}(C,T) \cap \bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{VI}(C,A) \cap \mathrm{VI}(C,B) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{r_n\}$, $\{\lambda_n\}$ and $\{s_n\}$ be positive number sequences. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n = P_C(x_n - \lambda_n T x_n), \\ y_n = P_C(u_n - r_n A u_n), \\ x_{n+1} = \alpha_n f(y_n) + \beta_n W_n P_C(y_n - s_n B y_n) + \gamma_n x_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{4.5}$$

where $\{W_n\}$ is the sequence generated in (2.1). $\{\lambda_n\}$, $\{s_n\}$, $\{r_n\}$, $\{\alpha_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(1) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;
(2) $0 < c \leq \lambda_n \leq d < 2\tau$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;

(3) $0 < e \le s_n \le g < 2\beta$ *and* $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$;
(4) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(5) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$,

where a, b, c, d, e, and g are real constants. Then $\{x_n\}$ *converges strongly to a point* $q \in \Omega$, *which solves uniquely the variational inequality* (3.2).

Proof In Theorem 4.3, put $F = 0$. Then we find that

$$\langle Tx_n, y - u_n\rangle + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C$$

is equivalent to

$$\langle y - u_n, x_n - \lambda_n Tx_n - u_n\rangle \le 0, \quad \forall y \in C,$$

i.e., by Lemma 2.1, $u_n = P_C(x_n - \lambda_n Tx_n)$. This completes the proof. □

Theorem 4.6 *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let* $P_C : H \to C$ *be the metric projection. Let* $f : C \to C$ *be a k-contraction with the constant* $k \in (0,1)$. *Let* $A : C \to H$ *be an* α*-ism mapping with* $\alpha > 0$, *let* $B : C \to H$ *be a* β*-ism mapping with* $\beta > 0$, *and let* $T : C \to H$ *be a* τ*-ism mapping with* $\tau > 0$. *Let* $\{S_i : C \to C\}$ *be a family of infinitely nonexpansive mappings. Let* $Y : C \to H$ *be a widely r-strict pseudo-contraction with* $r < 1$ $(r \in \mathbb{R})$. $Z : C \to H$ *be a widely s-strict pseudo-contraction with* $s < 1$ $(s \in \mathbb{R})$. *Assume that* $\Omega = \mathrm{VI}(C,T) \cap \bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{Fix}(Y) \cap \mathrm{Fix}(Z) \neq \emptyset$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$. *Let* $\{r_n\}$, $\{\lambda_n\}$ *and* $\{s_n\}$ *be positive number sequences. Let* $x_1 \in C$ *and let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} u_n = P_C(x_n - \lambda_n Tx_n), \\ y_n = P_C(u_n - r_n Au_n), \\ x_{n+1} = \alpha_n f(y_n) + \beta_n W_n P_C(y_n - s_n By_n) + \gamma_n x_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{4.6}$$

where $\{W_n\}$ *is the sequence generated in* (2.1). $\{\lambda_n\}$, $\{s_n\}$, $\{r_n\}$, $\{\alpha_n\}$ *and* $\{\gamma_n\}$ *satisfy conditions* (i)-(v) *respectively, which appear in Theorem* 3.1. $\{t_n\}$ *satisfies*

(1) $\{t_n\} \subset (-\infty, 1)$;
(2) $r \le t_n \le l < 1$;
(3) $\sum_{n=1}^{\infty} |t_n - t_{n+1}| < \infty$.

Then $\{x_n\}$ *converges strongly to a point* $q_0 \in \Omega$.

Proof Put $N = \partial i_C$ and $A = I - Y$ in Theorem 3.1. Furthermore, put $p = 1 - l$, $r_n = 1 - t_n$ and $2\alpha = 1 - r$ in Theorem 3.1. From $\{t_n\} \subset (-\infty, 1)$ and $r \le t_n \le l < 1$, we get $\{r_n\} \subset (0, \infty)$ and $0 < p \le r_n \le 2\alpha$. We also get

$$\sum_{n=1}^{\infty} |r_{n+1} - r_n| = \sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$$

and

$$I - r_n A = I - (1 - t_n)(I - Y) = (1 - t_n)Y + t_n I.$$

Furthermore, we have $(A + \partial i_C)^{-1}0 = \mathrm{Fix}(Y)$. Indeed, for $q \in C$, we have

$$\begin{aligned} q \in (A + \partial i_C)^{-1}0 \quad &\Longleftrightarrow \quad 0 \in Aq + \partial i_C(q) \\ &\Longleftrightarrow \quad 0 \in q - Yq + N_C(q) \\ &\Longleftrightarrow \quad Yq - q \in N_C(q) \\ &\Longleftrightarrow \quad \langle Yq - q, p - q\rangle \leq 0, \quad \forall p \in C \\ &\Longleftrightarrow \quad P_C Y(q) = q. \end{aligned}$$

Since $\mathrm{Fix}(Y) \neq \emptyset$, we get from [27] that $\mathrm{Fix}(P_C Y) = \mathrm{Fix}(Y)$. Thus we obtain the desired result by Theorem 3.1.

Similarly, put $M = \partial i_C$ and $B = I - Z$ in Theorem 3.1. Furthermore, put $p = 1 - l$, $s_n = 1 - t_n$ and $2\beta = 1 - s$ in Theorem 3.1. From $\{t_n\} \subset (-\infty, 1)$ and $s \leq t_n \leq l < 1$, we get $\{s_n\} \subset (0, \infty)$ and $0 < p \leq s_n \leq 2\beta$. We also get

$$\sum_{n=1}^{\infty} |s_{n+1} - s_n| = \sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$$

and

$$I - s_n B = I - (1 - t_n)(I - Z) = (1 - t_n)Z + t_n I.$$

Furthermore, we also obtain $(B + \partial i_C)^{-1}0 = \mathrm{Fix}(Z)$. □

Due to Section 3 and Section 4, we will give our conclusion in the next section.

5 Conclusion

Methods for solving a generalized equilibrium problem, a fixed point problem and the zero points of the sum of two operators have been studied by many authors respectively. However, in this paper, for finding a common solution of the above three problems, we proposed a new regularization algorithm, and it is proved that the sequence generated by this algorithm has the strong convergence. And then some corollaries to this strong convergence theorem are presented, which play important roles in nonlinear analysis and optimization problem.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors read and approved the final manuscript.

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