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The regularized trace formula for a fourth order differential operator given in a finite interval

Serpil Karayel* and Yonca Sezer

*Correspondence:
serpil@yildiz.edu.tr
Faculty of Arts and Science, Yildiz Technical University, Davutpaşa, İstanbul, 34220, Turkey

Abstract

In this work, a regularized trace formula for a differential operator of fourth order with bounded operator coefficient is found.

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1 Introduction

Investigations into the regularized trace formulas of scalar differential operators started with the work [1] firstly. After that work, regularized trace formulas for several differential operators have been studied in some works as [2, 3] and [4]. In [5] a formula for the second regularized trace of the problem generated by a Sturm-Liouville operator equation with a spectral parameter dependent boundary condition is found. The list of the works on this subject is given in [6] and [7]. The trace formulas for differential operators with operator coefficient are investigated in the works [8–13] and [14]. The boundary conditions in our work are completely different from those in [9].

In this work, we find the following regularized trace formula for a self-adjoint differential operator L of fourth order with bounded operator coefficient:

$$\sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty}\left(\lambda_{mn}-\left(m+\frac{1}{2}\right)^4\right)-\frac{1}{\pi}\int_0^{\pi}\operatorname{tr}Q(x)\,dx\right]=\frac{1}{4}\left[\operatorname{tr}Q(\pi)-\operatorname{tr}Q(0)\right].$$

Here $\{\lambda_{mn}\}_{n=1}^{\infty}$ are the eigenvalues of the operator L which belong to the interval $[(m+\frac{1}{2})^4-\|Q\|,(m+\frac{1}{2})^4+\|Q\|]$.

1.1 Notation and preliminaries

Let H be a separable Hilbert space with infinite dimension. Let us consider the operators L_0 and L in the Hilbert space $H_1=L_2(0,\pi;H)$ which are formed by the following differential expressions:

$$\ell_0(y)=y^{\mathrm{IV}}(x),$$

$$\ell(y)=y^{\mathrm{IV}}(x)+Q(x)y(x)$$

with the same boundary conditions $y(0) = y''(0) = y'(\pi) = y'''(\pi) = 0$. Suppose that the operator function $Q(x)$ in the expression $\ell(y)$ satisfies the following conditions:

(Q1) $Q(x) : H \to H$ is a self-adjoint kernel operator for every $x \in [0,\pi]$. Moreover, $Q(x)$ has second order weak derivative in this interval and $Q^{(i)}(x) : H \to H$ $(i = 1,2)$ are self-adjoint kernel operators for every $x \in [0,\pi]$.

(Q2) $\|Q\| < \frac{5}{2}$.

(Q3) There is an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of the space H such that

$$\sum_{n=1}^{\infty} \|Q(x)\varphi_n\| < \infty.$$

(Q4) The functions $\|Q^{(i)}(x)\|_{\sigma_1(H)}$ are bounded and measurable in the interval $[0,\pi]$ $(i = 0,1,2)$.

Here $\sigma_1(H)$ is the space of kernel operators from H to H as in [15]. Moreover, we denote the norms by $\|\cdot\|_H$ and $\|\cdot\|$ and inner products by $(\cdot,\cdot)_H$ and $(\cdot,\cdot)$ in H and H_1, respectively, and we also denote the sum of eigenvalues of a kernel operator A by $\operatorname{tr} A = \operatorname{trace} A$.

Spectrum of the operator L_0 is the set

$$\sigma(L_0) = \left\{\left(\frac{1}{2}\right)^4, \left(\frac{3}{2}\right)^4, \ldots, \left(m + \frac{1}{2}\right)^4, \ldots\right\}.$$

Every point of this set is an eigenvalue of L_0 which has infinite multiplicity. The orthonormal eigenfunctions corresponding to eigenvalue $(m + \frac{1}{2})^4$ are in the form

$$\psi_{mn}(x) = \sqrt{\frac{2}{\pi}} \sin\left(m + \frac{1}{2}\right)x \cdot \varphi_n \quad (n = 1,2,\ldots). \tag{1.1}$$

2 Some relations between spectrums of operators L_0 and L

Let R_λ^0, R_λ be resolvents of the operators L_0 and L, respectively. If the operator $Q : H_1 \to H_1$ satisfies conditions (Q2) and (Q3), the following can be proved:

(a) $QR_\lambda^0 \in \sigma_1(H_1)$ for every $\lambda \notin \sigma(L_0)$.

(b) Spectrum of the operator L is a subset of the union of pairwise disjoint intervals

$$F_m = \left[\left(m + \frac{1}{2}\right)^4 - \|Q\|, \left(m + \frac{1}{2}\right)^4 + \|Q\|\right] \quad (m = 0,1,2,\ldots), \sigma(L) \subset \bigcup_{m=0}^{\infty} F_m.$$

(c) Each point of the spectrum of L, different from $(m + \frac{1}{2})^4$ in F_m is an isolated eigenvalue which has finite multiplicity.

(d) The series $\sum_{n=1}^{\infty} [\lambda_{mn} - (m + \frac{1}{2})^4]$ $(m = 0,1,2,\ldots)$ are absolutely convergent where $\{\lambda_{mn}\}_{n=1}^{\infty}$ are eigenvalues of the operator L in the interval F_m.

Let $\rho(L)$ be resolvent set of the operator L; $\rho(L) = \mathbf{C}\backslash\sigma(L)$. Since $QR_\lambda^0 \in \sigma_1(H_1)$ for every $\lambda \in \rho(L)$, from the equation $R_\lambda = R_\lambda^0 - R_\lambda Q R_\lambda^0$ we obtain $R_\lambda - R_\lambda^0 \in \sigma_1(H_1)$.

On the other hand, if we consider the series

$$\sum_{n=1}^{\infty} \left[\lambda_{mn} - \left(m + \frac{1}{2}\right)^4\right] \quad (m = 0,1,2,\ldots)$$

are absolutely convergent then we have

$$\operatorname{tr}\bigl(R_\lambda - R_\lambda^0\bigr) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{\lambda_{mn}-\lambda} - \frac{1}{(m+\frac{1}{2})^4-\lambda}\right]$$

for every $\lambda \in \rho(L)$ [10]. If we multiply both sides of this equality with $\frac{\lambda}{2\pi i}$ and integrate this equality over the circle $|\lambda| = b_p = (p+1)^4$ $(p \in \mathbf{N}, p \geq 1)$, then we find

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{|\lambda|=b_p} \lambda \operatorname{tr}\bigl(R_\lambda - R_\lambda^0\bigr)\,d\lambda \\
&\quad = \frac{1}{2\pi i}\int_{|\lambda|=b_p} \lambda \sum_{m=0}^{p}\sum_{n=1}^{\infty}\left[\frac{1}{\lambda_{mn}-\lambda} - \frac{1}{(m+\frac{1}{2})^4-\lambda}\right]d\lambda \\
&\qquad + \frac{1}{2\pi i}\int_{|\lambda|=b_p} \lambda \sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{\lambda_{mn}-\lambda} - \frac{1}{(m+\frac{1}{2})^4-\lambda}\right]d\lambda.
\end{aligned} \tag{2.1}$$

If we consider the relations $|\lambda_{mn}| < b_p$ $(m = 0, 1, 2, \ldots, p)$ and $|\lambda_{mn}| > b_p$ $(m = p+1, p+2, \ldots)$ for $n = 1, 2, 3, \ldots$, then from (2.1) we get

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{|\lambda|=b_p} \lambda \operatorname{tr}\bigl(R_\lambda - R_\lambda^0\bigr)\,d\lambda \\
&\quad = \sum_{m=0}^{p}\sum_{n=1}^{\infty}\left[\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda\,d\lambda}{\lambda-(m+\frac{1}{2})^4} - \frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda\,d\lambda}{\lambda-\lambda_{mn}}\right] \\
&\qquad + \sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda\,d\lambda}{\lambda-(m+\frac{1}{2})^4} - \frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda\,d\lambda}{\lambda-\lambda_{mn}}\right] \\
&\quad = \sum_{m=0}^{p}\sum_{n=1}^{\infty}\left[\left(m+\frac{1}{2}\right)^4 - \lambda_{mn}\right].
\end{aligned} \tag{2.2}$$

Moreover, from the formula $R_\lambda = R_\lambda^0 - R_\lambda Q R_\lambda^0$, we obtain the following equality:

$$R_\lambda - R_\lambda^0 = -R_\lambda^0 Q R_\lambda^0 + R_\lambda^0\bigl(QR_\lambda^0\bigr)^2 - R_\lambda\bigl(QR_\lambda^0\bigr)^3. \tag{2.3}$$

If we put this equality into (2.2), we have

$$\sum_{m=0}^{p}\sum_{n=1}^{\infty}\left[\lambda_{mn} - \left(m+\frac{1}{2}\right)^4\right] = M_{p1} + M_{p2} + M_p. \tag{2.4}$$

Here

$$M_{pj} = \frac{(-1)^j}{2\pi i}\int_{|\lambda|=b_p} \lambda \operatorname{tr}\bigl[R_\lambda^0\bigl(QR_\lambda^0\bigr)^j\bigr]\,d\lambda \quad (j = 1, 2), \tag{2.5}$$

$$M_p = \frac{-1}{2\pi i}\int_{|\lambda|=b_p} \lambda \operatorname{tr}\bigl[R_\lambda\bigl(QR_\lambda^0\bigr)^3\bigr]\,d\lambda. \tag{2.6}$$

Theorem 2.1 *If the operator function* $Q(x)$ *satisfies condition* (Q3), *then we have*

$$M_{pj} = \frac{(-1)^j}{2\pi ij} \int_{|\lambda|=b_p} \operatorname{tr}\big[\big(QR_\lambda^0\big)^j\big]\, d\lambda.$$

Proof It can be shown that the operator function QR_λ^0 is analytic with respect to the norm in the space $\sigma_1(H_1)$ in domain $\rho(L_0) = \mathbf{C} \setminus \sigma(L_0)$ and

$$\operatorname{tr}\big\{\big[\big(QR_\lambda^0\big)^j\big]'\big\} = j \operatorname{tr}\big[\big(QR_\lambda^0\big)'\big(QR_\lambda^0\big)^{j-1}\big]. \tag{2.7}$$

Considering $(QR_\lambda^0)' = (QR_\lambda^0)^2$, we can write the formula (2.7) as

$$\operatorname{tr}\big\{\big[\big(QR_\lambda^0\big)^j\big]'\big\} = j \operatorname{tr}\big[R_\lambda^0\big(QR_\lambda^0\big)^j\big]. \tag{2.8}$$

From (2.5) and (2.8), we obtain

$$M_{pj} = \frac{(-1)^{j+1}}{2\pi ij} \int_{|\lambda|=b_p} \lambda \operatorname{tr}\big\{\big[\big(QR_\lambda^0\big)^j\big]'\big\}\, d\lambda.$$

From here, we find

$$\begin{aligned} M_{pj} &= \frac{(-1)^{j+1}}{2\pi ij} \int_{|\lambda|=b_p} \operatorname{tr}\big\{\big[\lambda\big(QR_\lambda^0\big)^j\big]' - \big(QR_\lambda^0\big)^j\big\}\, d\lambda \\ &= \frac{(-1)^j}{2\pi ij} \int_{|\lambda|=b_p} \operatorname{tr}\big[\big(QR_\lambda^0\big)^j\big]\, d\lambda + \frac{(-1)^{j+1}}{2\pi ij} \int_{|\lambda|=b_p} \operatorname{tr}\big\{\big[\lambda\big(QR_\lambda^0\big)^j\big]'\big\}\, d\lambda. \end{aligned} \tag{2.9}$$

It can be easily shown that

$$\operatorname{tr}\big\{\big[\lambda\big(QR_\lambda^0\big)^j\big]'\big\} = \big\{\operatorname{tr}\big[\lambda\big(QR_\lambda^0\big)^j\big]\big\}'.$$

Therefore, we have

$$\int_{|\lambda|=b_p} \operatorname{tr}\big\{\big[\lambda\big(QR_\lambda^0\big)^j\big]'\big\}\, d\lambda = \int_{|\lambda|=b_p} \big\{\operatorname{tr}\big[\lambda\big(QR_\lambda^0\big)^j\big]\big\}'\, d\lambda. \tag{2.10}$$

The integral on the right-hand side of the last equality can be written as

$$\int_{|\lambda|=b_p} \big\{\operatorname{tr}\big[\lambda\big(QR_\lambda^0\big)^j\big]\big\}'\, d\lambda = \int_{\substack{|\lambda|=b_p \\ \operatorname{Im}\lambda\geq 0}} \big\{\operatorname{tr}\big[\lambda\big(QR_\lambda^0\big)^j\big]\big\}'\, d\lambda + \int_{\substack{|\lambda|=b_p \\ \operatorname{Im}\lambda\leq 0}} \big\{\operatorname{tr}\big[\lambda\big(QR_\lambda^0\big)^j\big]\big\}'\, d\lambda. \tag{2.11}$$

Let ε_0 be a constant satisfying the condition $0 < \varepsilon_0 < b_p - (p + \frac{1}{2})^4$. Consider the function $\operatorname{tr}[\lambda(QR_\lambda^0)^j]$ is analytic in simple connected domains

$$G_1 = \{\lambda \in \mathbf{C} : b_p - \varepsilon_0 < \lambda < b_p + \varepsilon_0, \operatorname{Im}\lambda > -\varepsilon_0\},$$

$$G_2 = \{\lambda \in \mathbf{C} : b_p - \varepsilon_0 < \lambda < b_p + \varepsilon_0, \operatorname{Im}\lambda < \varepsilon_0\}$$

and

$$\{\lambda \in \mathbf{C} : |\lambda| = b_p, \operatorname{Im}\lambda \geq 0\} \subset G_1, \qquad \{\lambda \in \mathbf{C} : |\lambda| = b_p, \operatorname{Im}\lambda \leq 0\} \subset G_2.$$

From (2.11) we obtain

$$\begin{aligned} \int_{|\lambda|=b_p} \{\operatorname{tr}[\lambda(QR_\lambda^0)^j]\}' \, d\lambda = {} & \operatorname{tr}[-b_p(QR_{-b_p}^0)^j] - \operatorname{tr}[b_p(QR_{b_p})^j] \\ & + \operatorname{tr}[b_p(QR_{b_p}^0)^j] - \operatorname{tr}[-b_p(QR_{-b_p}^0)^j] = 0. \end{aligned} \tag{2.12}$$

From (2.9), (2.10) and (2.12) we find

$$M_{pj} = \frac{(-1)^j}{2\pi ij} \int_{|\lambda|=b_p} \operatorname{tr}[(QR_\lambda^0)^j] \, d\lambda. \qquad \square$$

3 The formula of the regularized trace of the operator L

In this section, we find a formula for the regularized trace of the operator L. According to Theorem 2.1,

$$M_{p1} = -\frac{1}{2\pi i} \int_{|\lambda|=b_p} \operatorname{tr}[(QR_\lambda^0)] \, d\lambda. \tag{3.1}$$

Since $\{\psi_{mn}\}_{m=0,n=1}^{\infty\infty}$ is an orthonormal basis of the space H_1, from (3.1) we obtain

$$\begin{aligned} M_{p1} &= -\frac{1}{2\pi i} \int_{|\lambda|=b_p} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (QR_\lambda^0 \psi_{mn}, \psi_{mn}) \, d\lambda \\ &= -\frac{1}{2\pi i} \int_{|\lambda|=b_p} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{(Q\psi_{mn}, \psi_{mn})}{(m+\frac{1}{2})^4 - \lambda} \, d\lambda \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (Q\psi_{mn}, \psi_{mn}) \cdot \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{1}{\lambda - (m+\frac{1}{2})^4} \, d\lambda. \end{aligned} \tag{3.2}$$

Considering $(m+\frac{1}{2})^4 < b_p = (p+1)^4$ for $m \leq p$ and $(m+\frac{1}{2})^4 > b_p = (p+1)^4$ for $m > p$, from formula (3.2)

$$M_{p1} = \sum_{m=0}^{p} \sum_{n=1}^{\infty} (Q\psi_{mn}, \psi_{mn}) \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{1}{\lambda - (m+\frac{1}{2})^4} \, d\lambda = \sum_{m=0}^{p} \sum_{n=1}^{\infty} (Q\psi_{mn}, \psi_{mn}) \tag{3.3}$$

is obtained.

From (1.1) and (3.3), we have

$$\begin{aligned} M_{p1} &= \sum_{m=0}^{p} \sum_{n=1}^{\infty} \int_0^\pi \left(Q(x) \sqrt{\frac{2}{\pi}} \sin\left(m+\frac{1}{2}\right)x \cdot \varphi_n, \sqrt{\frac{2}{\pi}} \sin\left(m+\frac{1}{2}\right)x \cdot \varphi_n \right)_H dx \\ &= \frac{2}{\pi} \sum_{m=0}^{p} \sum_{n=1}^{\infty} \int_0^\pi (Q(x)\varphi_n, \varphi_n)_H \sin^2\left(m+\frac{1}{2}\right)x \, dx \\ &= \frac{1}{\pi} \sum_{m=0}^{p} \int_0^\pi \left[\sum_{n=1}^{\infty} (Q(x)\varphi_n, \varphi_n)_H \right] (1 - \cos(2m+1)x) \, dx. \end{aligned} \tag{3.4}$$

If we consider the formula $\sum_{n=1}^{\infty}(Q(x)\varphi_n,\varphi_n)_H = \operatorname{tr} Q(x)$, then we get

$$M_{p1} = \frac{p+1}{\pi}\int_0^{\pi} \operatorname{tr} Q(x)\,dx - \frac{1}{\pi}\sum_{m=0}^{p}\int_0^{\pi} \operatorname{tr} Q(x)\cos(2m+1)x\,dx. \tag{3.5}$$

Lemma 3.1 *If the operator function* $Q(x)$ *satisfies conditions* (Q2) *and* (Q3), *then we have*

$$\|R_\lambda\| < p^{-3}$$

over the circle $|\lambda| = b_p$.

Proof Since the operator function $Q(x)$ satisfies conditions (Q2) and (Q3), we have

$$\{\lambda_{mn}\}_{n=1}^{\infty} \subset \left(\left(m+\frac{1}{2}\right)^4 - \|Q\|, \left(m+\frac{1}{2}\right)^4 + \|Q\|\right) \quad (m = 0,1,2,\ldots),$$

$$\left|\lambda_{mn} - \left(m+\frac{1}{2}\right)^4\right| < \|Q\| < \frac{5}{2} \quad (m = 0,1,2,\ldots; n = 1,2,\ldots).$$

If we consider this relation, we get

$$\begin{aligned}
|\lambda_{mn} - \lambda| &= \left|\lambda - \left(m+\frac{1}{2}\right)^4 - \left(\lambda_{mn} - \left(m+\frac{1}{2}\right)^4\right)\right| \\
&\geq \left|\lambda - \left(m+\frac{1}{2}\right)^4\right| - \left|\lambda_{mn} - \left(m+\frac{1}{2}\right)^4\right| \\
&> |\lambda| - \left(m+\frac{1}{2}\right)^4 - \frac{5}{2} = (p+1)^4 - \left(m+\frac{1}{2}\right)^4 - \frac{5}{2} \\
&\geq (p+1)^4 - \left(p+\frac{1}{2}\right)^4 - \frac{5}{2} > 2\left(p+\frac{1}{2}\right)^3 - \frac{5}{2} > p^3
\end{aligned} \tag{3.6}$$

for $m \leq p$ and

$$\begin{aligned}
|\lambda_{mn} - \lambda| &= \left|\left(m+\frac{1}{2}\right)^4 - \lambda - \left(\left(m+\frac{1}{2}\right)^4 - \lambda_{mn}\right)\right| \\
&\geq \left|\left(m+\frac{1}{2}\right)^4 - \lambda\right| - \left|\left(m+\frac{1}{2}\right)^4 - \lambda_{mn}\right| \geq \left(m+\frac{1}{2}\right)^4 - |\lambda| - \frac{5}{2} \\
&\geq \left(p+\frac{3}{2}\right)^4 - (p+1)^4 - \frac{5}{2} > 2(p+1)^3 - \frac{5}{2} > p^3
\end{aligned} \tag{3.7}$$

for $m \geq p+1$.

On the other hand, we can write

$$\|R_\lambda\| = \max_{\substack{m=0,1,\ldots \\ n=1,2,\ldots}} \left\{|\lambda_{mn} - \lambda|^{-1}\right\}. \tag{3.8}$$

From (3.6), (3.7) and (3.8) we get

$$\|R_\lambda\| < p^{-3} \quad \left(|\lambda| = b_p\right). \qquad \square$$

Lemma 3.2 *If the operator function* $Q(x)$ *satisfies condition* (Q3), *then we have*

$$\left\| QR_\lambda^0 \right\|_{\sigma_1(H_1)} < 5p^{-2} \sum_{n=1}^{\infty} \left\| Q(x)\varphi_n \right\|$$

over the circle $|\lambda| = b_p$.

Proof Let us show that the series $\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \| QR_\lambda^0 \psi_{mn} \|$ is convergent.

For $\lambda \notin \sigma(L_0)$, we get

$$\begin{aligned}
&\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left\| QR_\lambda^0 \psi_{mn} \right\| \\
&\quad = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left| \left(m + \frac{1}{2} \right)^4 - \lambda \right|^{-1} \| Q\psi_{mn} \| \\
&\quad = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left| \left(m + \frac{1}{2} \right)^4 - \lambda \right|^{-1} \left[\int_0^\pi \left\| Q(x) \sqrt{\frac{2}{\pi}} \sin\left(m + \frac{1}{2} \right) x \cdot \varphi_n \right\|_H^2 dx \right]^{\frac{1}{2}} \\
&\quad = \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left| \left(m + \frac{1}{2} \right)^4 - \lambda \right|^{-1} \int_0^\pi \left\| Q(x)\varphi_n \right\|_H^2 \sin^2\left(m + \frac{1}{2} \right) x \, dx \\
&\quad < \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left| \left(m + \frac{1}{2} \right)^4 - \lambda \right|^{-1} \left[\int_0^\pi \left\| Q(x)\varphi_n \right\|_H^2 dx \right]^{\frac{1}{2}} \\
&\quad = \sum_{m=0}^{\infty} \left| \left(m + \frac{1}{2} \right)^4 - \lambda \right|^{-1} \sum_{n=1}^{\infty} \left\| Q(x)\varphi_n \right\|.
\end{aligned} \tag{3.9}$$

From this relation we obtain

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left\| QR_\lambda^0 \psi_{mn} \right\| < \infty \quad \bigl(\lambda \notin \sigma(L_0) \bigr).$$

On the other hand, since the sequence $\{\psi_{mn}\}_{m=0,n=1}^{\infty\infty}$ is an orthonormal basis of the space H_1, we get

$$\left\| QR_\lambda^0 \right\|_{\sigma_1(H_1)} \leq \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left\| QR_\lambda^0 \psi_{mn} \right\| \tag{3.10}$$

[15]. From (3.9) and (3.10) we obtain

$$\left\| QR_\lambda^0 \right\|_{\sigma_1(H_1)} \leq \sum_{n=1}^{\infty} \left\| Q(x)\varphi_n \right\| \sum_{m=0}^{\infty} \left| \left(m + \frac{1}{2} \right)^4 - \lambda \right|^{-1}. \tag{3.11}$$

Furthermore, over the circle $|\lambda| = b_p$ we get

$$\begin{aligned}
&\sum_{m=0}^{\infty} \left| \left(m + \frac{1}{2} \right)^4 - \lambda \right|^{-1} \\
&\quad = \sum_{m=0}^{p} \left| \left(m + \frac{1}{2} \right)^4 - \lambda \right|^{-1} + \sum_{m=p+1}^{\infty} \left| \left(m + \frac{1}{2} \right)^4 - \lambda \right|^{-1}
\end{aligned}$$

$$
\begin{aligned}
&< \sum_{m=0}^{p}\left(|\lambda| - \left(m+\frac{1}{2}\right)^4\right)^{-1} + \sum_{m=p+1}^{\infty}\left(\left(m+\frac{1}{2}\right)^4 - |\lambda|\right)^{-1} \\
&= \sum_{m=0}^{p}\left((p+1)^4 - \left(m+\frac{1}{2}\right)^4\right)^{-1} + \sum_{m=p+1}^{\infty}\left(\left(m+\frac{1}{2}\right)^4 - (p+1)^4\right)^{-1} \\
&< \sum_{m=0}^{p}\left((p+1)^4 - \left(p+\frac{1}{2}\right)^4\right)^{-1} \\
&\quad + \sum_{m=p+1}^{\infty}\left(\left(m+\frac{1}{2}\right)^2 - (p+1)^2\right)^{-1}\left(\left(m+\frac{1}{2}\right)^2 + (p+1)^2\right)^{-1} \\
&< \frac{1}{2}(p+1)\left(p+\frac{1}{2}\right)^{-3} + \frac{1}{2}p^{-2}\sum_{m=p+1}^{\infty}\left(\left(m+\frac{1}{2}\right)^2 - (p+1)^2\right)^{-1}.
\end{aligned}
\tag{3.12}
$$

It can be easily shown that

$$
\left(m+\frac{1}{2}\right)^2 - (p+1)^2 > \frac{1}{4}\left(m^2 - p^2\right) \quad (m \geq p+1)
\tag{3.13}
$$

and

$$
\sum_{m=p+1}^{\infty}\left(m^2 - p^2\right)^{-1} < 2p^{-\frac{1}{2}}.
\tag{3.14}
$$

From (3.12), (3.13) and (3.14) we get

$$
\sum_{m=o}^{\infty}\left|\left(m+\frac{1}{2}\right)^4 - \lambda\right|^{-1} < p^{-2} + 4p^{-\frac{5}{2}} < 5p^{-2}.
\tag{3.15}
$$

From (3.11) and (3.15) we obtain

$$
\left\|QR_\lambda^0\right\|_{\sigma_1(H_1)} < 5p^{-2}\sum_{n=1}^{\infty}\left\|Q(x)\varphi_n\right\| \quad \left(|\lambda| = b_p\right).
$$

□

Theorem 3.3 *If the operator function* $Q(x)$ *satisfies conditions* (Q1), (Q2), (Q3), *and* (Q4), *then we have the formula*

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\left\{\sum_{n=1}^{\infty}\left[\lambda_{mn} - \left(m+\frac{1}{2}\right)^4\right] - \frac{1}{\pi}\int_0^{\pi}\operatorname{tr}Q(x)\,dx\right\} \\
&\quad = \frac{1}{4}\left[\operatorname{tr}Q(\pi) - \operatorname{tr}Q(0)\right].
\end{aligned}
$$

Proof By using Theorem 2.1, Lemma 3.1 and Lemma 3.2, we find

$$
\begin{aligned}
|M_{p2}| &= \frac{1}{4\pi}\left|\int_{|\lambda|=b_p}\operatorname{tr}\left[\left(QR_\lambda^0\right)^2\right]d\lambda\right| \\
&< \frac{1}{4\pi}\int_{|\lambda|=b_p}\left|\operatorname{tr}\left[\left(QR_\lambda^0\right)^2\right]\right||d\lambda|
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{1}{4\pi}\int_{|\lambda|=b_p}\left\|\left(QR_\lambda^0\right)^2\right\|_{\sigma_1(H_1)}|d\lambda| \\
&\le \frac{1}{4\pi}\int_{|\lambda|=b_p}\left\|QR_\lambda^0\right\|\left\|QR_\lambda^0\right\|_{\sigma_1(H_1)}|d\lambda| \\
&\le \frac{\|Q\|}{4\pi}\int_{|\lambda|=b_p}\left\|R_\lambda^0\right\|\left\|QR_\lambda^0\right\|_{\sigma_1(H_1)}|d\lambda| \\
&< c_1\int_{|\lambda|=b_p} p^{-5}\,d\lambda \\
&= 2\pi\cdot b_p\cdot c_1\cdot p^{-5} = 2\pi c_1(p+1)^4p^{-5} < c_2p^{-1}.
\end{aligned} \tag{3.16}$$

Here c is a positive constant.

By using formula (2.6), Lemma 3.1 and Lemma 3.2, we find

$$\begin{aligned}
|M_p| &= \frac{1}{2\pi}\left|\int_{|\lambda|=b_p}\lambda\operatorname{tr}\left[R_\lambda\left(QR_\lambda^0\right)^3\right]d\lambda\right| \\
&\le \frac{b_p}{2\pi}\int_{|\lambda|=b_p}\left\|R_\lambda\left(QR_\lambda^0\right)^3\right\||d\lambda| \\
&\le \frac{b_p}{2\pi}\int_{|\lambda|=b_p}\|R_\lambda\|\left\|\left(QR_\lambda^0\right)^3\right\|_{\sigma_1(H_1)}|d\lambda| \\
&\le \frac{b_p}{2\pi}\int_{|\lambda|=b_p}\|R_\lambda\|\left\|\left(QR_\lambda^0\right)^2\right\|\left\|\left(QR_\lambda^0\right)\right\|_{\sigma_1(H_1)}|d\lambda| \\
&\le \frac{b_p}{2\pi}\int_{|\lambda|=b_p}\|R_\lambda\|\|Q\|^2\left\|R_\lambda^0\right\|^2\left\|QR_\lambda^0\right\|_{\sigma_1(H_1)}|d\lambda| \\
&< c_3\cdot b_p^2p^{-11} \\
&= c_3(p+1)^8p^{-11} < c_4p^{-3}.
\end{aligned} \tag{3.17}$$

From (3.16) and (3.17) we get

$$\lim_{p\to\infty} M_{p2} = \lim_{p\to\infty} M_p = 0. \tag{3.18}$$

From (2.4) and (3.5) we obtain

$$\begin{aligned}
&\sum_{m=0}^{p}\left\{\sum_{n=1}^{\infty}\left[\lambda_{mn}-\left(m+\frac{1}{2}\right)^4\right]-\frac{1}{\pi}\int_0^\pi \operatorname{tr} Q(x)\,dx\right\} \\
&\quad = -\frac{1}{\pi}\sum_{m=0}^{p}\int_0^\pi \operatorname{tr} Q(x)\cos(2m+1)x\,dx + M_{p2} + M_p.
\end{aligned} \tag{3.19}$$

From (3.18) and (3.19) we find

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left\{\sum_{n=1}^{\infty}\left[\lambda_{mn}-\left(m+\frac{1}{2}\right)^4\right]-\frac{1}{\pi}\int_0^\pi \operatorname{tr} Q(x)\,dx\right\} \\
&\quad = -\frac{1}{\pi}\sum_{m=0}^{\infty}\int_0^\pi \operatorname{tr} Q(x)\cos(2m+1)x\,dx.
\end{aligned} \tag{3.20}$$

Moreover, using conditions (Q1) and (Q4), we get

$$\begin{aligned}
&-\frac{1}{\pi}\sum_{m=0}^{\infty}\int_0^{\pi}\operatorname{tr}Q(x)\cos(2m+1)x\,dx \\
&= -\frac{1}{2\pi}\sum_{m=1}^{\infty}\left[\int_0^{\pi}\operatorname{tr}Q(x)\cos mx\,dx-(-1)^m\int_0^{\pi}\operatorname{tr}Q(x)\cos mx\,dx\right] \\
&= -\frac{1}{4}\left\{\sum_{m=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi}\operatorname{tr}Q(x)\cos mx\,dx\right]\cos m0+\left[\frac{1}{\pi}\int_0^{\pi}\operatorname{tr}Q(x)\,dx\right]\cos 0\right\} \\
&\quad+\frac{1}{4}\left\{\sum_{m=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi}\operatorname{tr}Q(x)\cos mx\,dx\right]\cos m\pi+\left[\frac{1}{\pi}\int_0^{\pi}\operatorname{tr}Q(x)\,dx\right]\cos 0\pi\right\} \\
&= \frac{1}{4}\left[\operatorname{tr}Q(\pi)-\operatorname{tr}Q(0)\right].
\end{aligned}\tag{3.21}$$

From (3.20) and (3.21) we find

$$\sum_{m=0}^{\infty}\left\{\sum_{n=1}^{\infty}\left[\lambda_{mn}-\left(m+\frac{1}{2}\right)^4\right]-\frac{1}{\pi}\int_0^{\pi}\operatorname{tr}Q(x)\,dx\right\}=\frac{1}{4}\left[\operatorname{tr}Q(\pi)-\operatorname{tr}Q(0)\right].$$

The theorem is proved. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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