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Sufficiency and duality in nondifferentiable multiobjective programming involving higher order strong invexity

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Abstract

In the present paper, we consider a nondifferentiable multiobjective programming problem with support functions and locally Lipschitz functions. Several sufficient optimality conditions are discussed for a strict minimizer of a nondifferentiable multiobjective programming problem under strong invexity and its generalizations of order σ. Weak and strong duality theorems are established for a Mond-Weir type dual.

Keywords: multiobjective programming; locally Lipschitz function; strict minimizer; optimality conditions; duality

1 Introduction

Optimality conditions and duality results in multiobjective programming problems have attracted many researchers in recent years. The concepts of weak efficient solution, efficient solution and properly efficient solution have played an important role in the analysis of these types of multiobjective optimization problems. Recently, much attention has been paid to other types of solution concepts, one of them is higher order strict minimizer [1]. This concept plays a role in stability results [2] and in the convergence analysis of iterative numerical methods [3]. In [4], Ward discussed the strict minimizer of order σ for a single objective programming problem. Jimenez [5] extended the notion of Ward [4] to introduce the notion of local efficient solution of a multiobjective programming problem and characterized it under tangent cone. Jimenez and Novo [6, 7] discussed optimality conditions for a multiobjective optimization problem. Gupta *et al.* [8] presented the equivalent definition of higher order strict local efficient solution for a multiobjective programming problem. The notion of Ward [4] was further extended for global strict minimizer in [9].

Agarwal *et al.* [10] presented the optimality and duality results for multiobjective optimization problems involving locally Lipschitz functions and type I invexity. In [11], Bae *et al.* formulated nondifferentiable multiobjective programming problem and discussed duality results under generalized convexity. Bae and Kim [12], and Kim and Bae [13] derived optimality conditions and duality theorems for a nondifferentiable multiobjective programming problem with support function. Recently, optimality conditions and duality for a strict minimizer of nonsmooth multiobjective optimization problems with normal cone were derived in [14].

In this paper, we consider the following nondifferentiable multiobjective problem:

$$\text{(MP)}\quad \text{Minimize } f(x) + s(x|D) = \bigl[f_1(x) + s(x|D_1), f_2(x) + s(x|D_2), \dots, f_k(x) + s(x|D_k)\bigr]$$
$$\text{subject to } x \in X = \bigl\{x \in S : g_j(x) \leqq 0, j = 1, 2, \dots, m\bigr\},$$

where $f : X \to R^k$ and $g : X \to R^m$ are locally Lipschitz functions and X is a convex set in R^n. D_i is a compact convex set of R^n.

The paper is organized as follows. In Section 2, we recall some known concepts in the literature and then introduce the concept of strong invexity of order σ for a locally Lipschitz function and its generalizations. Section 3 deals with several sufficient optimality conditions for higher order minimizers via introduced classes of functions. In Section 4, we establish the Mond-Weir type duality results, and conclusion is discussed in Section 5.

2 Notations and prerequisites

Throughout the paper, $\nabla g(x)$ will denote the $m \times n$ Jacobian matrix of g at x. For $\bar{x} \in X$, $I = \{j : g_j(\bar{x}) = 0\}$ and g_I will denote the vector of active constraints at $\bar{x}$. The index sets $K = \{1, 2, \dots, k\}$ and $M = \{1, 2, \dots, m\}$.

Definition 2.1 [15] Let D be a compact convex set in R^n. The support function $s(\cdot|D)$ is defined by

$$s(x|D) = \max\bigl\{x^T y : y \in D\bigr\}.$$

The support function $s(\cdot|D)$ has a subdifferential. The subdifferential of $s(\cdot|D)$ at x is given by

$$\partial s(x|D) = \bigl\{z \in D : z^T x = s(x|D)\bigr\}.$$

The support function $s(\cdot|D)$ is convex and everywhere finite, that is, there exists $z \in D$ such that

$$s(y|D) \geqq s(x|D) + z^T (y - x) \quad \text{for all } y \in D.$$

Equivalently,

$$z^T x = s(x|D).$$

A function $f : R^n \to R$ is said to be locally Lipschitz at $\bar{x} \in R^n$ if there exist scalars $\delta > 0$ and $\epsilon > 0$ such that

$$\bigl|f\bigl(x^1\bigr) - f\bigl(x^2\bigr)\bigr| \leqq \delta \bigl\|x^1 - x^2\bigr\| \quad \text{for all } x^1, x^2 \in \bar{x} + \epsilon B,$$

where $\bar{x} + \epsilon B$ is the open ball of radius ϵ about $\bar{x}$.

The generalized directional derivative [16] of a locally Lipschitz function f at x in the direction v, denoted by $f^\circ(x;v)$, is as follows:

$$f^\circ(x;v) = \lim_{y\to x}\sup_{t\downarrow 0}\frac{f(y+tv)-f(y)}{t}.$$

The generalized gradient [17] of f at x is denoted by

$$\partial f(x) = \{\xi \in R^n : f^\circ(x;v) \geqq \xi^t v \text{ for all } v \in R^n\}.$$

We now consider the following multiobjective problem:

$$\text{(P)}\quad \text{Minimize } f(x) = [f_1(x), f_2(x), \dots, f_k(x)]$$
$$\text{subject to } x \in X.$$

Since the objectives in such problems generally conflict with one another, an optimal solution is chosen from the set of strict minimizer solutions in the following sense.

Definition 2.2 [5] A point $\bar{x} \in X$ is a strict minimizer for (P) if there exists $\epsilon > 0$ such that

$$f(x) \nless f(\bar{x}) \quad \text{for all } x \in B(\bar{x},\epsilon)\cap X,$$

that is, there exists no $x \in B(\bar{x},\epsilon)\cap X$ such that

$$f(x) < f(\bar{x}).$$

Let $\sigma \geq 1$ be an integer throughout the paper.

Definition 2.3 [9] A point $\bar{x} \in X$ is a local strict minimizer of order σ for (P) if there exist $\epsilon > 0$ and a constant $c \in \operatorname{int} R^k_+$ such that

$$f(x) \nless f(\bar{x}) + c\|x-\bar{x}\|^\sigma \quad \text{for all } x \in B(\bar{x},\epsilon)\cap X.$$

The notion of a local strict minimizer reduces to the global sense if the ball $B(\bar{x},\epsilon)$ is replaced by the whole space R^n.

Bhatia and Sahay [17] introduced the following notion of a strict minimizer of order σ with respect to a nonlinear function for the multiobjective programming problem.

Definition 2.4 A point $\bar{x} \in X$ is a local strict minimizer of order σ for (P) with respect to a nonlinear function $\psi : X \times X \to R^n$ if there exists a constant $c \in \operatorname{int} R^k_+$ such that

$$f(x) \nless f(\bar{x}) + c\|\psi(x,\bar{x})\|^\sigma \quad \text{for all } x \in B(\bar{x},\epsilon)\cap X.$$

Definition 2.5 A point $\bar{x} \in X$ is a strict minimizer of order σ for (P) with respect to a nonlinear function $\psi : X \times X \to R^n$ if there exists a constant $c \in \operatorname{int} R^k_+$ such that

$$f(x) \nless f(\bar{x}) + c\|\psi(x,\bar{x})\|^\sigma \quad \text{for all } x \in X.$$

We now introduce the higher order strong invexity and its generalizations for nonsmooth locally Lipschitz functions.

Let $f : S \to R$ be a locally Lipschitz function on S.

Definition 2.6 f is said to be strongly invex of order σ with respect to η, ψ on S if there exists a constant $c > 0$ such that for all $x, \bar{x} \in S$,

$$f(x) - f(\bar{x}) \geqq \xi^T \eta(x,\bar{x}) + c\|\psi(x,\bar{x})\|^\sigma \text{ for all } \xi \in \partial f(\bar{x}).$$

Definition 2.7 f is said to be strongly pseudo-invex type I of order σ with respect to η, ψ on S if there exists a constant $c > 0$ such that for all $x, \bar{x} \in S$,

$$\xi^T \eta(x,\bar{x}) \geqq 0 \quad \text{for some } \xi \in \partial f(\bar{x}) \quad \text{implies} \quad f(x) \geqq f(\bar{x}) + c\|\psi(x,\bar{x})\|^\sigma.$$

Or equivalently

$$f(x) < f(\bar{x}) + c\|\psi(x,\bar{x})\|^\sigma \quad \text{implies} \quad \xi^t \eta(x,\bar{x}) < 0.$$

Definition 2.8 f is said to be strongly pseudo-invex type II of order σ with respect to η, ψ on S if there exists a constant $c > 0$ such that for all $x, \bar{x} \in S$,

$$\xi^T \eta(x,\bar{x}) + c\|\psi(x,\bar{x})\|^\sigma \geqq 0 \quad \text{for some } \xi \in \partial f(\bar{x}) \quad \text{implies} \quad f(x) \geqq f(\bar{x}).$$

Definition 2.9 f is said to be strongly quasi-invex type I of order σ with respect to η, ψ on S if there exists a constant $c > 0$ such that for all $x, \bar{x} \in S$,

$$f(x) \leqq f(\bar{x}) \quad \text{implies} \quad \xi^T \eta(x,\bar{x}) + c\|\psi(x,\bar{x})\|^\sigma \leqq 0 \quad \text{for all } \xi \in \partial f(\bar{x}).$$

Definition 2.10 f is said to be strongly quasi-invex type II of order σ with respect to η, ψ on S if there exists a constant $c > 0$ such that for all $x, \bar{x} \in S$,

$$f(x) \leqq f(\bar{x}) + c\|\psi(x,\bar{x})\|^\sigma \quad \text{implies} \quad \xi^T \eta(x,\bar{x}) \leqq 0 \quad \text{for all } \xi \in \partial f(\bar{x}).$$

3 Karush-Kuhn-Tucker type sufficiency

In this section, we discuss various Karush-Kuhn-Tucker type sufficient optimality conditions for a feasible solution to be a strict minimizer of order σ of (MP).

Theorem 3.1 *Let $f_i(\cdot) + (\cdot)^T w_i$, $i \in K$ be strongly invex of order σ and g_j, $j \in I$ be strongly quasi-invex type* I *of order σ with respect to the same η and ψ. If there exist $\bar{\lambda}_i \geqq 0$, $i = 1, 2, \ldots, k$, $\bar{\mu}_j \geqq 0$, $j = 1, 2, \ldots, m$ and $\bar{w}_i \in D_i$, $i \in K$ satisfying*

$$0 \in \sum_{i=1}^{k} \bar{\lambda}_i \big(\partial f_i(\bar{x}) + \bar{w}_i\big) + \sum_{j=1}^{m} \bar{\mu}_j \partial g_j(\bar{x}), \tag{1}$$

$$\bar{x}^T \bar{w}_i = s(\bar{x}|D_i), \quad i \in K, \tag{2}$$

$$\bar{\mu}_j g_j(\bar{x}) = 0, \quad j \in M, \tag{3}$$

$$\bar{\lambda}^T e = 1, \quad e = (1, 1, \dots, 1) \in R^k, \tag{4}$$

then $\bar{x}$ *is a strict minimizer of order of* σ *with respect to* ψ *of* (MP).

Proof Let J = $\{j : g_j(\bar{x}) < 0\}$. Therefore $I \cup J = M$. Also $\bar{\mu} \geqq 0$, $g(\bar{x}) \leqq 0$ and $\bar{\mu}_j g_j(\bar{x}) = 0$, $j \in M$ implies $\bar{\mu}_j = 0$.

Condition (1) implies that there exist $\bar{\xi}_i \in \partial f_i(\bar{x})$ and $\bar{\zeta}_i \in \partial g_j(\bar{x})$ satisfying

$$0 = \sum_{i=1}^{k} \bar{\lambda}_i(\bar{\xi} + w_i) + \sum_{j \in I} \bar{\mu}_j \bar{\zeta}_j. \tag{5}$$

Now suppose that $\bar{x}$ is not a strict minimizer of order σ with respect to ψ for (MP). Then, for $c_i > 0$, $i = 1, 2, \dots, k$, there exists some $x \in X$ such that

$$f_i(x) + s(x|D_i) < f_i(\bar{x}) + s(\bar{x}|D_i) + c_i \|\psi(x, \bar{x})\|^{\sigma}, \quad i \in K.$$

Since $x^T w_i \leqq s(x|D_i)$ and $(\bar{x})^T w_i = s(x|D_i)$,

$$\begin{aligned} f_i(x) + x^T w_i &\leqq f_i(x) + s(x|D_i) \\ &< f_i(\bar{x}) + s(\bar{x}|D_i) + c_i \|\psi(x, \bar{x})\|^{\sigma}, \quad i \in K \\ &= f_i(\bar{x}) + \bar{x}^T w_i + c_i \|\psi(x, \bar{x})\|^{\sigma}, \quad i \in K. \end{aligned}$$

Using $\bar{\lambda}_i \geqq 0$ and $\bar{\lambda}^T e = 1$, we get

$$\sum_{i=1}^{k} \bar{\lambda}_i \bigl(f_i(x) + x^T w_i\bigr) < \sum_{i=1}^{k} \bar{\lambda}_i \bigl(f_i(\bar{x}) + \bar{x}^T w_i\bigr) + \sum_{i=1}^{k} \bar{\lambda}_i c_i \|\psi(x, \bar{x})\|^{\sigma}. \tag{6}$$

The strong invexity of $f_i(\cdot) + (\cdot)^T w_i$, $i \in K$ of order σ with respect to η and ψ,

$$\bigl(f_i(x) + x^T w_i\bigr) - \bigl(f_i(\bar{x}) + \bar{x}^T w_i\bigr) \geqq \bigl\langle \xi_i + w_i, \eta(x, \bar{x})\bigr\rangle + c_i \|\psi(x, \bar{x})\|^{\sigma}, \quad i = 1, 2, \dots, k.$$

For $\bar{\lambda}_i \geqq 0$, we obtain

$$\begin{aligned} &\sum_{i=1}^{k} \bar{\lambda}_i \bigl(f_i(x) + x^T w_i\bigr) - \sum_{i=1}^{k} \bar{\lambda}_i \bigl(f_i(\bar{x}) + \bar{x}^T w_i\bigr) \\ &\quad \geqq \sum_{i=1}^{k} \bar{\lambda}_i \bigl\langle \xi_i + w_i, \eta(x, \bar{x})\bigr\rangle + \sum_{i=1}^{k} \bar{\lambda}_i c_i \|\psi(x, \bar{x})\|^{\sigma}. \end{aligned} \tag{7}$$

As $x \in X$, we have

$$g_j(x) \leqq g_j(\bar{x}), \quad j \in I.$$

The strongly quasi-invex type I of g_j, $j \in I$ of order σ with respect to η and ψ gives

$$\bigl\langle \zeta_j, \eta(x, \bar{x})\bigr\rangle + \beta_j \|\psi(x, \bar{x})\|^{\sigma} \leqq 0, \quad \beta_j > 0, j \in I.$$

The above inequality along with $\bar{\mu}_j \geqq 0$, $j \in I$ yields

$$\sum_{j\in I}\langle \bar{\mu}_j\zeta_j, \eta(x,\bar{x})\rangle + \sum_{j\in I}\bar{\mu}_j\beta_j\|\psi(x,\bar{x})\|^{\sigma} \leqq 0.$$

As $\bar{\mu}_j = 0$ for $j \in J$, we have

$$\sum_{j=1}^{m}\langle \bar{\mu}_j\zeta_j, \eta(x,\bar{x})\rangle + \sum_{j\in I}\bar{\mu}_j\beta_j\|\psi(x,\bar{x})\|^{\sigma} \leqq 0. \tag{8}$$

Adding (7), (8) and using (5), we get

$$\sum_{i=1}^{k}\bar{\lambda}_i\bigl(f_i(x) + x^T w_i\bigr) - \sum_{i=1}^{k}\bar{\lambda}_i\bigl(f_i(\bar{x}) + \bar{x}^T w_i\bigr) \geqq \alpha\|\psi(x,\bar{x})\|^{\sigma},$$

where $\alpha = \sum_{i=1}^{k}\bar{\lambda}_i c_i + \sum_{j\in I}\bar{\mu}_j\beta_j$. This implies that

$$\sum_{i=1}^{k}\bar{\lambda}_i\bigl[\bigl(f_i(x) + x_i^T w_i\bigr) - \bigl(f_i(\bar{x}) + \bar{x}_i^T w_i\bigr) - a_i\|\psi(x,\bar{x})\|^{\sigma}\bigr],$$

where $a = \alpha e$, since $\bar{\lambda}^T e = 1$, which contradicts (5). Hence $\bar{x}$ is a strict minimizer of order σ with respect to ψ for (MP). □

Remark 3.1 If g_j, $j \in I$ are strongly invex of order σ with respect to ψ on S, then the above Theorem 3.1 holds.

Theorem 3.2 *Let $f_i(\cdot) + (\cdot)^T w_i$, $i = 1, 2, \ldots, k$, be strongly pseudo-invex type* I *of order σ and g_j, $j \in I$ be strongly quasi-invex type* I *of order σ with respect to the same η and ψ. If conditions* (1)-(4) *are satisfied, then $\bar{x}$ is a strict minimizer of order σ of* (MP).

Proof Condition (1) implies that there exist $\bar{\xi}_i \in \partial f_i(\bar{x})$ and $\bar{\zeta}_i \in \partial g_j(\bar{x})$ satisfying

$$0 = \sum_{i=1}^{k}\bar{\lambda}_i(\bar{\xi} + w_i) + \sum_{j\in I}\bar{\mu}_j\bar{\zeta}_j. \tag{9}$$

Now suppose that $\bar{x}$ is not a strict minimizer of order σ with respect to ψ for (MP).Then, for $c_i > 0$, $i = 1, 2, \ldots, k$, there exists some $x \in X$ such that

$$f_i(x) + s(x|D_i) < f_i(\bar{x}) + s(\bar{x}|D_i) + c_i\|\psi(x,\bar{x})\|^{\sigma}, \quad i \in K.$$

Since $x_i^T w_i \leqq s(x|D_i)$ and $\bar{x}_i^T w_i = s(x|D_i)$,

$$\begin{aligned} f_i(x) + x^T w_i &\leqq f_i(x) + s(x|D_i) \\ &< f_i(\bar{x}) + s(\bar{x}|D_i) + c_i\|\psi(x,\bar{x})\|^{\sigma}, \quad i \in K \\ &= f_i(\bar{x}) + \bar{x}^T w_i + c_i\|\psi(x,\bar{x})\|^{\sigma}, \quad i \in K. \end{aligned}$$

As $f_i(\cdot) + (\cdot)^T w_i$, $i = 1, 2, \dots, k$, are strongly pseudo-invex type I of order σ with respect to η and ψ,

$$\langle \xi_i + w_i, \eta(x, \bar{x}) \rangle < 0, \quad i \in K.$$

For $\bar{\lambda}_i \geqq 0$ and $\lambda^T e = 1$, we obtain

$$\sum_{i=1}^{k} \bar{\lambda}_i \langle \xi_i + w_i, \eta(x, \bar{x}) \rangle < 0. \tag{10}$$

As $x \in X$, we have

$$g_j(x) \leqq g_j(\bar{x}), \quad j \in I.$$

The strongly quasi-invex type I of g_j, $j \in I$ of order σ with respect to η and ψ yields

$$\langle \zeta_j, \eta(x, \bar{x}) \rangle + \beta_j \| \psi(x, \bar{x}) \|^{\sigma} \leqq 0, \quad \beta_j > 0, j \in I.$$

The above inequality along with $\bar{\mu}_j \geqq 0$, $j \in I$ yields

$$\sum_{j \in I} \langle \bar{\mu}_j \zeta_j, \eta(x, \bar{x}) \rangle + \sum_{j \in I} \bar{\mu}_j \beta_j \| \psi(x, \bar{x}) \|^{\sigma} \leqq 0.$$

As $\bar{\mu}_j = 0$ for $j \in J$, we have

$$\sum_{j=1}^{m} \langle \bar{\mu}_j \zeta_j, \eta(x, \bar{x}) \rangle + \sum_{j \in I} \bar{\mu}_j \beta_j \| \psi(x, \bar{x}) \|^{\sigma} \leqq 0. \tag{11}$$

On adding (10) and (11), we obtain

$$\eta^t(x, \bar{x}) \left[\sum_{i=1}^{k} \bar{\lambda}_i \xi_i + \sum_{i=1}^{k} \bar{\mu}_j \zeta_j \right] + \sum_{j=1}^{m} \bar{\mu}_j \beta_j \| \psi(x, \bar{x}) \|^{\sigma} < 0.$$

The above inequality along with (9) gives $\sum_{j=1}^{m} \bar{\mu}_j \beta_j \| \psi(x, \bar{x}) \|^{\sigma} < 0$, which is not possible. Hence the result. □

Theorem 3.3 *Let conditions* (1)-(4) *be satisfied. Suppose that* $f_i(\cdot) + (\cdot)^T w_i$, $i = 1, 2, \dots, k$, *are strongly pseudo-invex type* I *of order* σ *and that* g_j, $j \in I$ *are strongly quasi-invex type* II *of order* σ *with respect to* η *and* ψ. *Then* $\bar{x}$ *is a strict minimizer of order* σ *of with respect to* ψ *of* (MP).

Proof Condition (1) implies that there exist $\bar{\xi}_i \in \partial f_i(\bar{x})$ and $\bar{\zeta}_i \in \partial g_j(\bar{x})$ satisfying (9).

Now suppose that $\bar{x}$ is not a strict minimizer of order σ with respect to ψ for (MP). Then, for $c_i > 0$, $i = 1, 2, \dots, k$, there exists some $x \in X$ such that

$$f_i(x) + s(x|D_i) < f_i(\bar{x}) + s(\bar{x}|D_i) + c_i \| \psi(x, \bar{x}) \|^{\sigma}, \quad i \in K.$$

Since $x^T w_i \leqq s(x|D_i)$ and $\bar{x}^T w_i = s(x|D_i)$,

$$\begin{aligned} f_i(x) + x^T w_i &\leqq f_i(x) + s(x|D_i) \\ &< f_i(\bar{x}) + s(\bar{x}|D_i) + c_i \|\psi(x,\bar{x})\|^\sigma, \quad i \in K \\ &= f_i(\bar{x}) + \bar{x}^T w_i + c_i \|\psi(x,\bar{x})\|^\sigma, \quad i \in K. \end{aligned}$$

As $f_i(\cdot) + (\cdot)^T w_i$, $i = 1, 2, \dots, k$, are strongly pseudo-invex type I of order σ with respect to η and ψ,

$$\langle \xi_i + w_i, \eta(x,\bar{x}) \rangle < 0, \quad i \in K.$$

For $\bar{\lambda}_i \geqq 0$ and $\lambda^T e = 1$, we obtain

$$\sum_{i=1}^{k} \bar{\lambda}_i \langle \xi_i + w_i, \eta(x,\bar{x}) \rangle < 0. \tag{12}$$

As $x \in X$, we have

$$g_j(x) \leqq g_j(\bar{x}), \quad j \in I$$

or

$$g_j(x) \leqq g_j(\bar{x}) + \beta_j \|\psi(x,\bar{x})\|^\sigma \quad \text{for } \beta_j > 0, j \in M.$$

Since g_j, $j \in I$ is strongly quasi-invex type II of order σ with respect to η and ψ, therefore

$$\langle \zeta_j, \eta(x,\bar{x}) \rangle \leqq 0.$$

The above inequality along with $\bar{\mu}_j \geqq 0$, $j \in I$ yields

$$\sum_{j \in I} \langle \bar{\mu}_j \zeta_j, \eta(x,\bar{x}) \rangle \leqq 0.$$

As $\bar{\mu}_j = 0$ for $j \in J$, we have

$$\sum_{j=1}^{m} \langle \bar{\mu}_j \zeta_j, \eta(x,\bar{x}) \rangle \leqq 0. \tag{13}$$

On adding (12) and (13) we get

$$\sum_{i=1}^{k} \left\langle \bar{\lambda}_i (\xi_i + w_i) + \sum_{j=1}^{m} \bar{\mu}_j \zeta_j, \eta(x,\bar{x}) \right\rangle < 0.$$

This contradicts (9). □

4 Mond-Weir type duality

For the primal problem (MP), we formulate the following Mond-Weir type dual problem:

$$\text{(MD)}\quad \text{Maximize } f(u) = \bigl[f_1(u) + u^T w_1, f_2(u) + u^T w_2, \ldots, f_k(u) + u^T w_k\bigr]$$

$$\text{subject to } 0 \in \sum_{i=1}^{k} \lambda_i\bigl(\partial f_i(u) + w_i\bigr) + \sum_{j=1}^{m} \mu_j \partial g_j(u), \tag{14}$$

$$\mu^T g(u) \geqq 0, \tag{15}$$

$$\mu \geqq 0, \qquad w_i \in D_i, \quad i \in K, \tag{16}$$

$$\lambda \geqq 0, \qquad \lambda^T e = 1, \quad e = (1,1,\ldots,1) \in R^k. \tag{17}$$

Theorem 4.1 (Weak duality) *Let x and $(u,\lambda,\mu,w_1,w_2,\ldots,w_k)$ be feasible solutions for* (MP) *and* (MD) *respectively. Suppose $\sum_{i=1}^{k} \lambda_i(f_i(\cdot) + (\cdot)^T w_i)$, $i \in K$ is strongly pseudo-invex type* I *and $\sum_{j=1}^{m} \mu_j g_j$ is strongly quasi-invex type* I *of order σ with respect to η and ψ, then there exists $c \in \operatorname{int} R_+^k$ such that*

$$f_i(x) + s(x|D_i) \nless f_i(u) + u^T w_i + c_i \bigl\| \psi(x,u) \bigr\|^\sigma, \quad i \in K.$$

Proof Since $(u,\lambda,\mu,w_1,w_2,\ldots,w_k)$ is a feasible solution for (MD), there exist $\xi_i \in \partial f_i(u)$ and $\zeta_j \in \partial g_j(u)$ such that

$$0 = \sum_{i=1}^{k} \lambda_i(\xi_i + w_i) + \sum_{j=1}^{m} \mu_j \zeta_j. \tag{18}$$

Since x is feasible for (MP) and $(u,\lambda,\mu,w_1,w_2,\ldots,w_k)$ is feasible for (MD), we have

$$\sum_{j=1}^{m} \mu_j g_j(x) \leqq \sum_{j=1}^{m} \mu_j g_j(u).$$

The strong quasi-invexity type I of $\sum_{j=1}^{m} \mu_j g_j(\cdot)$ of order σ with respect to η and ψ at u implies that there exists a constant $\beta > 0$ such that

$$\left\langle \sum_{j=1}^{m} \mu_j \zeta_j, \eta(x,u) \right\rangle + \beta \bigl\| \psi(x,u) \bigr\|^\sigma \leqq 0, \quad \forall \zeta_j \in \partial g_j(u), j \in M.$$

Using (18), we have

$$\left\langle \sum_{i=1}^{k} \lambda_i(\xi_i + w_i), \eta(x,u) \right\rangle - \beta \bigl\| \psi(x,u) \bigr\|^\sigma \geqq 0$$

or

$$\left\langle \sum_{i=1}^{k} \lambda_i(\xi_i + w_i), \eta(x,u) \right\rangle \geqq 0.$$

Now strong pseudo-invexity of type I of order σ of $\sum_{i=1}^{k}\lambda_i(f_i(\cdot)+(\cdot)^T w_i)$ with respect to η and ψ at u implies that there exists a constant $\gamma > 0$ such that

$$\sum_{i=1}^{k}\lambda_i\bigl(f_i(x)+x^T w_i\bigr) \geqq \sum_{i=1}^{k}\lambda_i\bigl(f_i(u)+u^T w_i\bigr)+\gamma\bigl\|\psi(x,u)\bigr\|^{\sigma}$$

or

$$\sum_{i=1}^{k}\lambda_i\bigl(f_i(x)+x^T w_i\bigr) \geqq \sum_{i=1}^{k}\lambda_i\bigl(f_i(u)+u^T w_i+c_i\bigl\|\psi(x,u)\bigr\|^{\sigma}\bigr), \tag{19}$$

where $c=\gamma e$ and $\lambda^T e=1$.

Suppose to the contrary that

$$f_i(x)+s(x|D_i) < f_i(u)+u^T w_i+c_i\bigl\|\psi(x,u)\bigr\|^{\sigma}, \quad i\in K.$$

Since $x^T w_i \leqq s(x|D_i)$, $i\in K$, we have

$$f_i(x)+x^T w < f_i(u)+u^T w_i+c_i\bigl\|\psi(x,u)\bigr\|^{\sigma}, \quad i\in K.$$

Using $\lambda \geqq 0$ and $\lambda^T e=1$, we get

$$\sum_{i=1}^{k}\lambda_i\bigl(f_i(x)+x^T w_i\bigr) < \sum_{i=1}^{k}\lambda_i\bigl(f_i(u)+u^T w_i+c_i\bigl\|\psi(x,u)\bigr\|^{\sigma}\bigr).$$

This contradicts (19). Hence the result. □

The following definition is needed in the proof of the strong duality theorem.

Definition 4.1 [14] A point $\bar{x}\in X$ is a strict maximizer of order σ for (MP) with respect to a nonlinear function $\psi : X\times X\to R^n$ if there exists a constant $c\in \operatorname{int} R_+^k$ such that

$$f(\bar{x})+\bar{x}^T w+c\bigl\|\psi(x,\bar{x})\bigr\|^{\sigma} \nless f(x) \quad \text{for all } x\in X.$$

Theorem 4.2 (Strong duality) *Let $\bar{x}$ be a strict minimizer of order σ with respect to ψ of* (MP), *and let the basic regularity hold at $\bar{x}$. Then there exist $\bar{\lambda}_i \geqq 0$, $\bar{w}_i\in D_i$, $i\in K$ and $\bar{\mu}_j \geqq 0$, $j\in M$ such that $(\bar{x},\bar{\lambda},\bar{\mu},\bar{w}_1,\bar{w}_2,\ldots,\bar{w}_k)$ is a feasible solution of* (MD) *and $\bar{x}^T w_i = s(\bar{x}|D_i)$, $i\in K$. Moreover, if the hypothesis of Theorem* 4.1 *is satisfied, then $(\bar{x},\bar{\lambda},\bar{\mu},\bar{w}_1,\bar{w}_2,\ldots,\bar{w}_k)$ is a strict minimizer of order m with respect to ψ of* (MD).

Proof Since $\bar{x}$ is a strict minimizer of order σ with respect to ψ for (NMP), by Theorem 3.1 there exist $\bar{\lambda}_i \geqq 0$, $i\in K$, $\bar{\mu}_j \geqq 0$, $j\in M$ and $\bar{w}_i\in D_i$, $i\in K$,

$$0\in \sum_{i=1}^{k}\bar{\lambda}_i\bigl(\partial f_i(\bar{x})+\bar{w}_i\bigr)+\sum_{j=1}^{m}\bar{\mu}_j\partial g_j(\bar{x}),$$

$$\bar{x}^T\bar{w}_i = s(\bar{x}|D_i), \quad i\in K,$$

$$\bar{\mu}_j g_j(\bar{x}) = 0, \quad j \in M,$$
$$\bar{\lambda}^T e = 1, \quad e = (1, 1, \ldots, 1) \in R^k.$$

Therefore $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k)$ is feasible for (MD). Now a strict minimizer of order σ with respect to ψ at $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k)$ for (MD) follows from the weak duality theorem. □

5 Conclusion

In this paper, we have presented several Khun-Tucker type sufficient optimality conditions and Mond-Weir type duality results for a nondifferentiable multiobjective problem involving a support function of a compact convex set. The present results can be further generalized for the following fractional analogue of (MP):

$$\text{(FP)} \quad \text{Minimize} \left(\frac{f_1(x) + s(x|D_1)}{h_1(x) - s(x|E_1)}, \frac{f_2(x) + s(x|D_2)}{h_2(x) - s(x|E_2)}, \ldots, \frac{f_k(x) + s(x|D_k)}{h_k(x) - s(x|E_k)} \right)$$
$$\text{subject to } -g(x) \in C^*, \quad x \in C,$$

where $f_i : X \to R$, $h_i : X \to R$, $i \in K$, $g : X \to R^m$; D_i and E_i, $i \in K$ are compact sets in R^n. C is a closed cone with nonempty interior in R^m.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors carried out the proof. Both authors conceived of the study and participated in its design and coordination. Both authors read and approved the final manuscript.

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References

1. Auslender, A: Stability in mathematical programming with non-differentiable data. SIAM J. Control Optim. **22**, 239-254 (1984)
2. Cramme, L: Strong uniqueness for reaching criterion for the convergence of iterative procedures. Numer. Math. **29**, 179-193 (1978)
3. Studniarski, M: Necessary and sufficient conditions for isolated local minima of nonsmooth functions. SIAM J. Control Optim. **24**, 1044-1049 (1986)
4. Ward, DE: Characterizations of strict local minima and necessary conditions for weak sharp minima. J. Optim. Theory Appl. **80**, 551-571 (1994)
5. Jimenez, B: Strict efficiency in vector optimization. J. Math. Anal. Appl. **265**, 264-284 (2002)
6. Jimenez, B, Novo, V: First and second order sufficient conditions for strict minimality in multiobjective programming. Numer. Funct. Anal. Optim. **23**, 303-322 (2002)
7. Jimenez, B, Novo, V: First and second order sufficient conditions for strict minimality in nonsmooth vector optimization. J. Math. Anal. Appl. **284**, 496-510 (2003)
8. Gupta, A, Bhatia, D, Mehra, A: Higher order efficiency, saddle point optimality and duality for vector optimization problems. Numer. Funct. Anal. Optim. **28**, 339-352 (2007)
9. Bhatia, G: Optimality and mixed saddle point criteria in multiobjective optimization. J. Math. Anal. Appl. **342**, 135-145 (2008)
10. Agarwal, RP, Ahmad, I, Husain, Z, Jayswal, A: Optimality and duality in nonsmooth multiobjective optimization involving V-type I invex functions. J. Inequal. Appl. **2010**, Article ID 898624 (2010)
11. Bae, KD, Kang, YM, Kim, DS: Efficiency and generalized convex duality for nondifferentiable multiobjective programs. J. Inequal. Appl. **2010**, Article ID 930457 (2010)
12. Bae, KD, Kim, DS: Optimality and duality theorems in nonsmooth multiobjective optimization. Fixed Point Theory Appl. **2011**, Article ID 42 (2011)
13. Kim, DS, Bae, KD: Optimality conditions and duality for a class of nondifferentiable multiobjective programming problems. Taiwan. J. Math. **13**(2B), 789-804 (2009)

14. Bae, KD, Kim, DS: Optimality and duality for nonsmooth multiobjective optimization problems. J. Inequal. Appl. **2013**, Article ID 554 (2013)
15. Mond, B, Schechter, M: Nondifferentiable symmetric duality. Bull. Aust. Math. Soc. **53**, 177-188 (1996)
16. Clarke, FH: Optimization and Nonsmooth Analysis. Wiley-Interscience, New York (1983)
17. Bhatia, G, Sahay, RR: Strict global minimizers and higher-order generalized strong invexity in multiobjective optimization. J. Inequal. Appl. **2013**, Article ID 31 (2013)