Yonghui and Jiang *Journal of Inequalities and Applications* (2015) 2015:259
DOI 10.1186/s13660-015-0783-4

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

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The boundedness of Marcinkiewicz integrals commutators on non-homogeneous metric measure spaces

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Abstract

Let $(\mathcal{X},d,\mu)$ be a metric measure space satisfying the upper doubling condition and geometrically doubling condition in the sense of Hytönen. In this paper, the authors establish the boundedness of the commutator generated by the RBMO(μ) function and the Marcinkiewicz integral with kernel satisfying a Hörmander-type condition, respectively, from $L^p(\mu)$ with $1 < p < \infty$ to itself.

MSC: 42B20; 42B35; 47B47; 30L99

Keywords: non-homogeneous metric measure spaces; Marcinkiewicz integral; commutator; Lebesgue space; RBMO(μ)

1 Introduction

In 1938, Marcinkiewicz [1] introduced the integral on one-dimensional Euclidean space $\mathbb{R}$, which is today called the Marcinkiewicz integral, and conjectured that it is bounded on $L^p([0,2\pi])$, $1 < p < \infty$. Zygmund in [2] proved the Marcinkiewicz conjecture. In 1958, Stein [3] generalized the above Marcinkiewicz integral to the higher-dimensional case. Let Ω be homogeneous of degree zero in $\mathbb{R}^n$, $n \geq 2$, integrable and have mean value zero on the unit sphere $\mathcal{S}^{n-1}$. The higher-dimensional Marcinkiewicz integral is then defined by

$$\mathcal{M}_\Omega(f)(x) = \left\{\int_0^\infty \left|\int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\,dy\right|^2 \frac{dt}{t^3}\right\}^{\frac{1}{2}}, \quad x \in \mathbb{R}^n.$$

Stein [3] proved that if $\Omega \in \mathrm{Lip}_\alpha(\mathcal{S}^{n-1})$ for some $\alpha \in (0,1]$, then $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $p \in (1,2]$ and also bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. Since then, many papers focused on the boundedness of this operator on various function spaces. We refer the reader to [4–12] for its developments and applications.

The main purpose of this paper is to establish the bound of the commutator generated by the Marcinkiewicz integral and the RBMO(μ) function on the non-homogeneous metric measure spaces.

During the past 10 to 15 years, considerable attention has been paid to the study of the classical theory of harmonic analysis on Euclidean spaces with non-doubling measures only satisfying the polynomial growth condition (see [13–21]). To be precise, let μ be a positive Radon measure on $\mathbb{R}^d$ with satisfies the polynomial growth condition that, for all $x \in \mathbb{R}^d$ and $r > 0$,

$$\mu\big(B(x,r)\big) \le c_0 r^n, \tag{1.1}$$

where c_0 is a positive constants and $0 < n \le d$, and $B(x,r)$ is the open ball centered at x and having radius r. The analysis associated with such non-doubling measure μ has proved to play a striking role in solving the long-standing open Painlevé's problem and Vitushkin's conjecture by Tolsa [19]. The non-doubling measure μ may not satisfy the well-known doubling condition, which is a key assumption in harmonic analysis on spaces of homogeneous type in the sense of Coifman and Weiss [22, 23]. To unify both spaces of homogeneous type and the metric spaces endowed with measures satisfying the polynomial growth condition, Hytönen [24] introduced a new class of metric measure spaces satisfying both the so-called geometrically doubling and the upper doubling condition, which are called non-homogeneous metric measure spaces (see Definition 1.3 below). Many classical results have been proved still valid if the underlying spaces are non-homogeneous metric measure spaces (see [25–32]). From now on, we always assume that $(\mathcal{X}, d, \mu)$ is a non-homogeneous metric measure spaces in the sense of Hytönen [24]. In this setting, Hytönen [24] introduced the space RBMO(μ), and Hytönen and Martikainen [27] established a version of the Tb theorem. About Marcinkiewicz integral, Lin and Yang [31] have proved that the $L^p(\mu)$-boundedness with $p \in (1,\infty)$ is equivalent to either of its boundedness from $L^1(\mu)$ into $L^{1,\infty}(\mu)$ or from the atomic Hardy space $H^1(\mu)$ (see [28]) to $L^1(\mu)$. They also showed that if the Marcinkiewicz integral is bounded from $H^1(\mu)$ to $L^1(\mu)$, then it is bounded from $L^\infty(\mu)$ to RBLO(μ) (see [33]), which is a proper subset of RBMO(μ). These results essentially improve the existing results in [34].

Now we recall some necessary notions and notation.

The following notion of the geometrically doubling is well known in analysis on metric spaces, which was originally introduced by Coifman and Weiss in [22, 23] and is also known as metrically doubling.

Definition 1.1 A metric space $(\mathcal{X}, d)$ is said to be geometrically doubling if there exists some $N_0 \in \mathbb{N}$ such that, for all balls $B(x,r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, \frac{r}{2})\}_i$ of $B(x,r)$ such that the cardinality of this covering is at most N_0.

Remark 1.2 Let $(\mathcal{X}, d)$ be a metric space. In [24], Hytönen showed that the following statements are mutually equivalent:

(1) $(\mathcal{X}, d)$ is geometrically doubling.
(2) For any $\varepsilon \in (0,1)$ and any ball $B(x,r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, \varepsilon r)\}_i$ of $B(x,r)$ such that the cardinality of this covering is at most $N_0\varepsilon^{-n}$, where $n = \log_2 N_0$.
(3) For any $\varepsilon \in (0,1)$ and any ball $B(x,r) \subset \mathcal{X}$ contains at most $N_0\varepsilon^{-n}$ centers of disjoint balls $\{B(x_i, \varepsilon r)\}_i$.
(4) There exists $M \in \mathbb{N}$ such that any ball $B(x,r) \subset \mathcal{X}$ contains at most M centers $\{x_i\}_i$ of disjoint balls $\{B(x_i, r/4)\}_{i=1}^M$.

Definition 1.3 A metric measure space $(\mathcal{X}, d, \mu)$ is said to be upper doubling if μ is a Borel measure on $\mathcal{X}$ and there exist a dominating function $\lambda : \mathcal{X} \times (0,\infty) \to (0,\infty)$ and a positive constant c_λ such that, for each $x \in \mathcal{X}$, $r \to \lambda(x,r)$ is non-decreasing and

$$\mu\big(B(x,r)\big) \leq \lambda(x,r) \leq c_\lambda \lambda(x,r/2), \quad \text{for all } x \in \mathcal{X}, r > 0. \tag{1.2}$$

It was proved in [28] that there exists a dominating function $\tilde{\lambda}$ related to λ satisfying the property that there exists a positive constant $c_{\tilde{\lambda}}$ such that $\tilde{\lambda} \leq \lambda$, $c_{\tilde{\lambda}} \leq c_\lambda$ and, for all $x, y \in \mathcal{X}$, $r > 0$ with $d(x,y) \leq r$, $\tilde{\lambda}(x,r) \leq c_{\tilde{\lambda}} \tilde{\lambda}(y,r)$. Based on this, in this paper, we always assume that the dominating function λ also satisfies it.

The following coefficients $\delta(B,S)$ for all ball B and S were introduced in [24] as analogs of Tolsa's number $K_{B,S}$ in [18].

Definition 1.4 For all balls $B \subset S$, let

$$\delta(B,S) = 1 + \int_{(2S-B)} \frac{d\mu(x)}{\lambda(c_B, d(x,c_B))}, \tag{1.3}$$

where above and in that follows, for a ball $B = B(c_B, r_B)$ and $\rho > 0$, $\rho B = B(c_B, \rho r_B)$.

Remark 1.5 The following discrete version $K_{B,S}$ of $\delta(B,S)$ was first introduced by Bui and Duong [25] in non-homogeneous metric measure spaces, which is more close to the quantity $K_{B,S}$ introduced by Tolsa [18] in the setting of non-doubling measures. For all balls $B \subset S$, let $K_{B,S}$ be defined by

$$K_{B,S} = 1 + \sum_{i=1}^{N_{B,S}} \frac{\mu(6^i B)}{\lambda(c_B, 6^i r_B)}, \tag{1.4}$$

where $N_{B,S}$ denote the smallest integer satisfying $6^{N_{B,S}} r_B \geq r_s$. Obviously $\delta(B,S) \lesssim K_{B,S}$. As was pointed out by Bui and Duong [25], it is not true that $\delta(B,S) \sim K_{B,S}$.

Definition 1.6 Let $\alpha, \beta \in (0,\infty)$. A ball $B \subset \mathcal{X}$ is called (α,β)-doubling if $\mu(\alpha B) \leq \beta \mu(B)$.

It was proved in [24] that if a metric measure space $(\mathcal{X}, d, \mu)$ is upper doubling and $\alpha, \beta \in (0,\infty)$ satisfying $\beta > c_\lambda^{\log_2 \alpha} = \alpha^\nu$, then, for any ball B, there exists some $j \in \mathbb{N} \cup \{0\}$ such that $\alpha^j B$ is (α,β)-doubling. Moreover, let $(\mathcal{X}, d, \mu)$ be geometrically doubling, $\beta > \alpha^n$ with $n = \log_2 N_0$ and μ a Borel measure on $\mathcal{X}$ which is finite on bounded sets. Hytönen [24] also showed that for μ-almost every $x \in \mathcal{X}$, there exist arbitrary small (α,β)-doubling balls centered at x. Furthermore, the radii of these balls may be chosen to be of the form $\alpha^{-j} B$ for $j \in \mathbb{N}$ and any preassigned number $r > 0$. Throughout this paper, for any $\alpha \in (1,\infty)$ and ball B, the smallest (α,β_α)-doubling ball of the form $\alpha^j B$ with $j \in \mathbb{N}$ is denoted by $\tilde{B}^\alpha$, where

$$\beta_\alpha = \max\{\alpha^{3n}, \alpha^{3\nu}\} + 30^n + 30^\nu.$$

In what follows, by a doubling ball we mean a $(6,\beta_6)$-doubling ball and $\tilde{B}^6$ is simply denoted by $\tilde{B}$.

Now we recall the definition of $\mathrm{RBMO}(\mu)$ from [24].

Definition 1.7 Let $\rho \in (1,\infty)$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the space RBMO(μ) if there exist a positive constant c and, for any ball $B \subset \mathcal{X}$, a number f_B such that

$$\frac{1}{\mu(\rho B)}\int_B \left|f(x) - f_B\right| d\mu(x) \le c$$

and, for any two balls $B \subset S$,

$$|f_B - f_S| \le c\delta_{B,S}.$$

The infimum of the positive constant c is defined to be the RBMO(μ) norm of f and denote by $\|f\|_{\mathrm{RBM}(\mu)}$.

In [24], it follows that the definition of RBMO(μ) is independent of the choice of $\rho \in (1,\infty)$.

The following equivalent characterization of RBMO(μ) was established in [28].

Lemma 1.8 *Let $\rho \in (1,\infty)$ and $f \in L^1_{\mathrm{loc}}(\mu)$. Then the following statements are equivalent*:

(1) $f \in \mathrm{RBMO}(\mu)$;

(2) *there exist a positive constant c and, for any ball $B \subset \mathcal{X}$, such that*

$$\frac{1}{\mu(\rho B)}\int_B \left|f(x) - m_{\tilde{B}}f\right| d\mu(x) \le c$$

and, for any doubling balls $B \subset S$,

$$|m_B - m_S| \le c\delta_{B,S}.$$

Moreover, let $\|f\|_*$ be the infimum of the positive constant c in (2). *Then there exists a constant $\tilde{c}$ such that $\frac{\|f\|_*}{\tilde{c}} \le \|f\|_{\mathrm{RBMO}(\mu)} \le \tilde{c}\|f\|_*$.*

Now we give the definition of Marcinkiewicz integral (see [31]).

Definition 1.9 Let K be a locally integrable function on $(\mathcal{X}\times\mathcal{X})\setminus\{(x,x) : x \in \mathcal{X}\}$. Assume that there exists a positive constant c such that, for all $x, y \in \mathcal{X}$ with $x \neq y$,

$$\left|K(x,y)\right| \le c\frac{d(x,y)}{\lambda(x,d(x,y))} \tag{1.5}$$

and, for all $y, y' \in \mathcal{X}$,

$$\int_{d(x,y)\ge 2d(y,y')} \left[\left|K(x,y) - K\left(x,y'\right)\right| + \left|K(y,x) - K\left(y',x\right)\right|\right]\frac{1}{d(x,y)}\,d\mu(x) \le c. \tag{1.6}$$

The Marcinkiewicz integral $\mathcal{M}f$ associated to the above kernel K is defined by setting, for all $x \in \mathcal{X}$,

$$\mathcal{M}(f)(x) = \left[\int_0^\infty \left|\int_{d(x,y)<t} K(x,y)f(y)\,d\mu(y)\right|^2 \frac{dt}{t^3}\right]^{\frac{1}{2}}. \tag{1.7}$$

We show that the commutator $\mathcal{M}_b$, associating with $b \in \mathrm{RBMO}(\mu)$ and $\mathcal{M}$, which is defined by

$$\mathcal{M}_b(f)(x) = [b, \mathcal{M}](f)(x) = b(x)\mathcal{M}(f)(x) - \mathcal{M}(bf)(x), \quad x \in \mathcal{X}. \tag{1.8}$$

In [31], the authors have proven that the Marcinkiewicz integral $\mathcal{M}$ is bounded from $L^p(\mu)$ to $L^p(\mu)$, $1 < p < \infty$. Our main result is as follows.

Theorem 1.10 *Let K satisfy* (1.5) *and the following Hörmander-type condition*:

$$\begin{aligned}\sup_{r>0, d(y,y')\le r} \sum_{i=1}^{\infty} i \int_{6^i r < d(x,y) \le 6^{i+1} r} \big[\big|K(x,y) - K(x,y')\big| \\ + \big|K(y,x) - K(y',x)\big|\big] \frac{1}{d(x,y)}\, d\mu(x) \le c.\end{aligned} \tag{1.9}$$

If $\mathcal{M}$ is bounded on $L^2(\mu)$, then, for any $b \in \mathrm{RBMO}(\mu)$, the commutator $\mathcal{M}_b$ is bounded on $L^p(\mu)$ with the bound no more than $c_p\|b\|_{\mathrm{RBMO}(\mu)}$, where $1 < p < \infty$.

Remark 1.11 The Hörmander-type condition (1.9) is slightly stronger than (1.6).

The organization of this paper is as follows. In Section 2, we introduce the sharp maximal operator $M^{\#}$, associated with $K_{B,S}$ and prove Lemma 2.6. This technical lemma is of independent interest. Section 3 is devoted to the proof of Theorem 1.10.

Throughout this paper, we denote c a positive constant which is independent of the main parameters involved, but may vary from line to line. For any ball $B \subset \mathcal{X}$, we denote its center and radius by c_B and r_B. $m_B f$ means that $\frac{1}{\mu(B)} \int_B f(y)\, d\mu(y)$.

2 The sharp maximal function

For a locally integrable function f, let $M^{\#}f$ be the sharp maximal function of f, namely, for $x \in \mathcal{X}$,

$$M^{\#}f(x) = \sup_{x\in B} \frac{1}{\mu(6B)} \int_B \big|f(y) - m_{\tilde{B}} f\big|\, d\mu(y) + \sup_{(B,S)\in\Delta_x} \frac{|m_{\tilde{B}} f - m_{\tilde{S}} f|}{K_{B,S}}, \tag{2.1}$$

where $\Delta_x = \{(B,S) : x \in B \subset S \text{ and } B, S \text{ are doubling balls}\}$.

For $0 < r < \infty$, let $M_r^{\#}f(x) = [M^{\#}(|f|^r)(x)]^{\frac{1}{r}}$ for $x \in \mathcal{X}$. A simple computation proves that if $0 < r < 1$,

$$M_r^{\#}f(x) \le c_r M^{\#}f(x), \tag{2.2}$$

where $c_r > 0$ is independent of f and x.

We recall some results in [32].

Lemma 2.1

(1) *Let $p \in (1, \infty)$, $r \in (1, p)$ and $\rho \in [5, \infty)$. The following maximal operators defined, respectively, by setting, for all $f \in L^1_{\mathrm{loc}}(\mu)$ and $x \in \mathcal{X}$:*

$$M_{r,\rho} f(x) = \sup_{x\in B} \left\{ \frac{1}{\mu(\rho B)} \int_B \big|f(y)\big|^r\, d\mu(y) \right\}^{\frac{1}{r}},$$

$$Nf(x) = \sup_{x\in B:\text{doubling}} \frac{1}{\mu(B)} \int_B |f(y)|\,d\mu(y),$$

and

$$M_\rho f(x) = \sup_{x\in B} \frac{1}{\mu(\rho B)} \int_B |f(y)|\,d\mu(y)$$

are bounded on $L^p(\mu)$ and also bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$.

(2) *For all $f \in L^1_{\text{loc}}(\mu)$ it holds true that $|f(x)| \le Nf(x)$ for μ-almost every $x \in \mathcal{X}$.*

In Lemma 2.1, if $0 < r < 1$, using the Hölder inequality, we have $M_{r,\rho}f(x) < M_\rho f(x)$. So Lemma 2.1 is right when $0 < r < 1$.

We also need the following Calderón-Zygmund decomposition theorem obtained by Bui and Duong [25]. Let γ be a fixed positive constant satisfying that $\gamma > \max\{c_\lambda^{3\log_2 6}, 6^{3n}\}$ where c_λ is as in Definition 1.3 and n as in Remark 1.2.

Lemma 2.2 *Let $p \in [1,\infty)$, $f \in L^p(\mu)$ and $t \in (0,\infty)$ ($t > \frac{\gamma^{\frac{1}{p}}\|f\|_{L^p(\mu)}}{[\mu(\mathcal{X})]^{1/p}}$ when $\mu(\mathcal{X}) < \infty$). Then*

(1) *there exists a family of finite overlapping balls $\{B_j\}_j$, pairwise disjoint,*

$$\frac{1}{\mu(6^2B_j)} \int_{B_j} |f(x)|^p\,d\mu(x) > \frac{t^p}{\gamma} \quad \text{for all } j,$$

$$\frac{1}{\mu(6^2\eta B_j)} \int_{\eta B_j} |f(x)|^p\,d\mu(x) \le \frac{t^p}{\gamma} \quad \text{for all } j \text{ and all } \eta \in (2,\infty),$$

and

$$|f(x)| \le t \quad \text{for } \mu\text{-almost every } x \in \mathcal{X} \setminus \Bigl(\bigcup_j 6B_j\Bigr);$$

(2) *for each j, let S_j be a $(3\times 6^2, c_\lambda^{\log_2 3\times 6^2+1})$-doubling ball of the family $\{(3\times 6^2)^k B_j\}_{k\in\mathbb{N}}$, and $\omega_j = \frac{\chi_{6B_j}}{\sum_k \chi_{6B_k}}$. Then there exists a family $\{\varphi_j\}_j$ of functions such that, for each j, $supp(\varphi_j) \subset S_j$, φ_j has a constant sign on S_j,*

$$\int_{\mathcal{X}} \varphi_j(x)\,d\mu(x) = \int_{6B_j} f(x)\omega_j(x)\,d\mu(x),$$

$$\sum_j |\varphi_j(x)| \le \gamma_0 t \quad \text{for } \mu\text{-almost every } x \in \mathcal{X},$$

where γ_0 is some positive constant depending only on $(\mathcal{X},\mu)$, and there exists a positive constant c, independent of f, t and j such that, when $p = 1$, it holds true that

$$\|\varphi_j\|_{L^\infty(\mu)}\mu(S_j) \le c\int_{\mathcal{X}} |f(x)\omega_j(x)|\,d\mu(x)$$

and, if $p \in (1,\infty)$, it holds true that

$$\biggl\{\int_{S_j} |\varphi_j(x)|^p\,d\mu(x)\biggr\}^{1/p} \bigl[\mu(S_j)\bigr]^{1/p'} \le \frac{c}{t^{p-1}} \int_{\mathcal{X}} |f(x)\omega_j(x)|^p\,d\mu(x).$$

The following John-Nirenberg inequality was established by Hytönen in [24].

Lemma 2.3 *let* $(\mathcal{X}, d, \mu)$ *be geometrically doubling and upper doubling. For every* $\rho > 1$, *there is a constant c so that, for every* $f \in \mathrm{RBMO}(\mu)$ *and every ball B,*

$$\mu\big(x \in B : \big|f(x) - f_B\big| > t\big) \leq 2\mu(\rho B)\exp\big(-ct/\|f\|_{\mathrm{RBMO}(\mu)}\big),$$

where f_B *can be seen in definition of* $\mathrm{RBMO}(\mu)$.

From Lemma 2.3, it is easy to prove that there are two positive c_1, c_2 such that, for any ball B and $b \in \mathrm{RBMO}(\mu)$,

$$\frac{1}{\mu(\rho B)} \int_B \exp\left(\frac{|b(x) - m_{\widetilde{B}}(b)|}{c_1\|b\|_{\mathrm{RBMO}(\mu)}}\right) d\mu(x) \leq c_2. \tag{2.3}$$

Lemma 2.4 *There is a constant c such that, for any* $a > 0$ *and* $t_1, t_2 > 0$,

$$t_1 t_2 \leq c\big[t_1 \log(2 + at_1) + a^{-1}\exp t_2\big].$$

This lemma had been established in [34].

We also need some useful properties of $K_{B,S}$, which were proved in [25, 32].

Lemma 2.5

(1) *For all balls* $B \subset R \subset S$, $K_{B,R} \leq 2K_{B,S}$.
(2) *For any* $\rho \in [1, \infty)$, *there exists a positive constant* c_ρ, *depending only on* ρ, *such that, for all balls* $B \subset S$ *with* $r_S \leq \rho r_B$, $K_{B,S} \leq c_\rho$.
(3) *There exists a positive constant c, such that, for all balls B,* $K_{B,\tilde{B}} \leq c$.
(4) *There exists a positive constant c, depending on* c_λ, *such that, for all balls* $B \subset R \subset S$, $K_{B,S} \leq K_{B,R} + cK_{R,S}$.
(5) *There exists a positive constant c, depending on* c_λ, *such that, for all balls* $B \subset R \subset S$, $K_{R,S} \leq cK_{B,S}$.

Now we give and prove the main result about the sharp maximal function $M^\#$.

Lemma 2.6 *Let K satisfy* (1.5) *and the Hörmander-type condition* (1.9). *We have* $s \in (1, \infty)$, $p_0 \in (1, \infty)$ *and* $b \in L^\infty(\mu)$. *If* $\mathcal{M}$ *is bounded on* $L^2(\mu)$, *then there is a positive constant c such that, for all* $f \in L^\infty(\mu) \cap L^{p_0}(\mu)$ *and for all* $x \in \mathcal{X}$,

$$M^\#\big[\mathcal{M}_b(f)\big](x) \leq c\big[\|b\|_{\mathrm{RBMO}(\mu)} M_{s,6}\big[\mathcal{M}(f)\big](x) + \|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{L^\infty(\mu)}\big].$$

Proof Without loss of generality, we may assume $\|b\|_{\mathrm{RBMO}(\mu)} = 1$. To prove Lemma 2.6, it suffices to prove that

$$\frac{1}{\mu(6B)} \int_B \big|\mathcal{M}_b(f)(y) - h_B\big|\, d\mu(y) \leq cM_{s,6}\big[\mathcal{M}(f)\big](x) + \|f\|_{L^\infty(\mu)} \tag{2.4}$$

for all $x \in B$ and

$$|h_B - h_S| \leq c(K_{B,S})^2\big[M_{s,6}\big[\mathcal{M}(f)\big](x) + \|f\|_{L^\infty(\mu)}\big] \tag{2.5}$$

for all balls $B \subset S$ with $x \in B$, where B is an arbitrary ball and S is a doubling ball,

$$h_B = m_B\big[\mathcal{M}\big(\big(b - m_{\tilde{B}}(b)\big)f\chi_{\mathcal{X}\setminus\frac{6}{5}B}\big)\big]$$

and

$$h_S = m_S\big[\mathcal{M}\big(\big(b - m_S(b)\big)f\chi_{\mathcal{X}\setminus\frac{6}{5}S}\big)\big].$$

To prove (2.4), for a fixed ball B, $x \in B$ and $f \in L^\infty(\mu)$, we write

$$f(y) = f(y)\chi_{\frac{6}{5}B}(y) + f(y)\chi_{\mathcal{X}\setminus\frac{6}{5}B}(y) = f_1(y) + f_2(y)$$

and

$$\mathcal{M}_b(f)(y) = \big(b(y) - m_{\tilde{B}}(b)\big)\mathcal{M}(f)(y) - \mathcal{M}\big(\big(b(y) - m_{\tilde{B}}(b)\big)f_1\big)(y) - \mathcal{M}\big(\big(b(y) - m_{\tilde{B}}(b)\big)f_2\big)(y).$$

So we can write

$$\begin{aligned}
&\frac{1}{\mu(6B)}\int_B \big|\mathcal{M}_b(f)(y) - h_B\big|\,d\mu(y)\\
&\quad\le \frac{1}{\mu(6B)}\int_B \big|b(y) - m_{\tilde{B}}(b)\big|\mathcal{M}(f)(y)\,d\mu(y)\\
&\qquad + \frac{1}{\mu(6B)}\int_B \mathcal{M}\big(\big(b(y) - m_{\tilde{B}}(b)\big)f_1\big)(y)\,d\mu(y)\\
&\qquad + \frac{1}{\mu(6B)}\int_B \big|\mathcal{M}\big(\big(b(y) - m_{\tilde{B}}(b)\big)f_2\big)(y) - h_B\big|\,d\mu(y)\\
&\quad= A_1 + A_2 + A_3.
\end{aligned}$$

By the Hölder inequality and Corollary 2.3 in [32], we see that

$$\begin{aligned}
A_1 &\le \frac{1}{\mu(6B)^{\frac{1}{s}+\frac{1}{s'}}}\left[\int_B \big|b(y) - m_{\tilde{B}}(b)\big|^{s'}\,d\mu(y)\right]^{1/s'}\left[\int_B \big(\mathcal{M}(f)\big)^s(y)\right]^{1/s}\\
&\le M_{s,6}\big[\mathcal{M}(f)\big](x).
\end{aligned}$$

To estimate A_2, from the Hölder inequality, the $L^2(\mu)$-boundedness of $\mathcal{M}$ and Corollary 2.3 in [32], it follows that

$$\begin{aligned}
A_2 &\le \frac{\mu(B)^{1/2}}{\mu(6B)}\left[\int_B \big|\mathcal{M}\big[\big(b(y) - m_{\tilde{B}}(b)\big)f_1\big](y)\big|^2\,d\mu(y)\right]^{1/2}\\
&\le \left[\frac{1}{\mu(6B)}\int_B \big|\big(b(y) - m_{\tilde{B}}(b)\big)f_1(y)\big|^2\,d\mu(y)\right]^{1/2}\\
&\le \mu(6B)^{-1/2}\big\|\big(b(y) - m_{\widetilde{\frac{6}{5}B}}(b)\big)f_1(y)\big\|_{L^2(\mu)} + \mu(6B)^{-1/2}\big\|\big(m_{\widetilde{\frac{6}{5}B}}(b) - m_{\tilde{B}}(b)\big)f_1(y)\big\|_{L^2(\mu)}\\
&\le \|f\|_{L^\infty(\mu)}\left[\frac{1}{\mu(6B)}\int_{\frac{6}{5}B}\big|b(y) - m_{\widetilde{\frac{6}{5}B}}(b)\big|^2\,d\mu(y)\right]^{1/2}
\end{aligned}$$

$$+ c\|f\|_{L^\infty(\mu)}\left[\frac{\mu(\frac{6}{5}B)}{\mu(6B)}\right]^{1/2}$$
$$\le c\|f\|_{L^\infty(\mu)},$$

where we use the fact that $|m_{\widetilde{\frac{6}{5}B}}(b) - m_{\tilde{B}}(b)| \le c(K_{B,\tilde{B}} + K_{\frac{6}{5}B,\widetilde{\frac{6}{5}B}} + K_{B,\frac{6}{5}B}) \le c$.

To obtain (2.4), we still need to estimate A_3. Set

$$M_1(x,y) = \left(\int_0^\infty\left[\int_{d(y,z)\le t\le d(x,z)}\big|K(y,z)\big(b(z) - m_{\tilde{B}}(b)\big)f_2(z)\big|\,d\mu(z)\right]^2\frac{dt}{t^3}\right)^{1/2},$$
$$M_2(x,y) = \left(\int_0^\infty\left[\int_{d(x,z)\le t\le d(y,z)}\big|K(y,z)\big(b(z) - m_{\tilde{B}}(b)\big)f_2(z)\big|\,d\mu(z)\right]^2\frac{dt}{t^3}\right)^{1/2},$$

and

$$M_3(x,y) = \left(\int_0^\infty\left[\int_{\max\{d(y,z),d(x,z)\}}\big|\big(K(y,z) - K(x,z)\big)\big(b(z) - m_{\tilde{B}}(b)\big)f_2(z)\big|\,d\mu(z)\right]^2\frac{dt}{t^3}\right)^{1/2}.$$

For any $x, y \in \mathcal{X}$, we have (see also [31], p.134)

$$\big|\mathcal{M}\big[\big(b - m_{\tilde{B}}(b)\big)f_2\big](y) - \mathcal{M}\big[\big(b - m_{\tilde{B}}(b)\big)f_2\big](x)\big| \le \sum_{i=1}^{3} M_i(x,y).$$

Applying the Minkowski inequality and (1.9) we conclude that, for all $x, y \in B$,

$$\begin{aligned}
M_1(x,y) &\le \int_{d(y,z)<d(x,z)}\big|K(y,z)f_2(z)\big(b(z) - m_{\tilde{B}}(b)\big)\big|\left[\int_{d(y,z)\le t<d(x,z)}\frac{dt}{t^3}\right]^{1/2}d\mu(z)\\
&\le c\int_{\mathcal{X}\setminus 5B}\frac{r_B^{1/2}}{d(z,c_B)^{1/2}}\frac{|b(z) - m_{\tilde{B}}(b)|}{\lambda(c_B,d(z,c_B))}f(z)\,d\mu(z)\\
&\le c\sum_{i=1}^{\infty}\int_{6^i5B\setminus 6^{i-1}5B}\frac{r_B^{1/2}}{d(z,c_B)^{1/2}}\frac{|m_{\widetilde{6^i5B}}(b) - m_{\tilde{B}}(b)|}{\lambda(c_B,d(z,c_B))}f(z)\,d\mu(z)\\
&\quad + \sum_{i=1}^{\infty}\int_{6^i5B\setminus 6^{i-1}5B}\frac{r_B^{1/2}}{d(z,c_B)^{1/2}}\frac{|m_{\widetilde{6^i5B}}(b) - b(z)|}{\lambda(c_B,d(z,c_B))}f(z)\,d\mu(z)\\
&\le c\sum_{i=1}^{\infty}i6^{(-i/2)}\frac{1}{\lambda(c_B,6^{i-1}5r_B)}\int_{6^i5B}\big|f(z)\big|d\mu(z)\\
&\quad + \sum_{i=1}^{\infty}6^{(-i/2)}\frac{1}{\lambda(c_B,6^{i-1}5r_B)}\int_{6^i5B}\big|m_{\widetilde{6^i5B}}(b) - b(z)\big|\big|f(z)\big|\,d\mu(z)\\
&\le c\|f\|_{L^\infty(\mu)}\sum_{i=1}^{\infty}i6^{(-i/2)}\frac{\mu(6^{i+1}5B)}{\lambda(c_B,6^{i-1}5r_B)}\\
&\le c\|f\|_{L^\infty(\mu)},
\end{aligned}$$

where we use the doubling condition of λ, $|m_{\widetilde{6^i5B}}(b) - m_{\tilde{B}}(b)| \le ci$ and $\lambda(c_B,d(x,c_B)) \backsim \lambda(x,d(x,c_B)) \backsim \lambda(x,d(x,y))$ for $y \in B$ and $x \in \mathcal{X}\setminus kB$ $(k > 1)$.

Similarly, $M_2(x,y) \le c\|f\|_{L^\infty(\mu)}$. Now for all $x, y \in B$, by the Minkowski inequality we have

$$\begin{aligned}
&M_3(x,y)\\
&\quad\le \int_{\mathcal{X}} \bigl|\bigl(K(y,z)-K(x,z)\bigr)f_2(z)\bigl(b(z)-m_{\widetilde{B}}(b)\bigr)\bigr| \biggl[\int_{\max\{d(y,z),d(x,z)\}\le t} \frac{dt}{t^3}\biggr]^{1/2} d\mu(z)\\
&\quad\le c\int_{\mathcal{X}} \bigl|\bigl(K(y,z)-K(x,z)\bigr)f_2(z)\bigl(b(z)-m_{\widetilde{B}}(b)\bigr)\bigr| \frac{1}{d(y,z)}\,d\mu(z)\\
&\quad\le c\|f\|_{L^\infty(\mu)} \sum_{i=1}^{\infty} \int_{6^i5B\setminus 6^{i-1}5B} \bigl|\bigl(K(y,z)-K(x,z)\bigr)\bigl(b(z)-m_{\widetilde{6^i5B}}(b)\bigr)\bigr| \frac{1}{d(y,z)}\,d\mu(z)\\
&\qquad + c\|f\|_{L^\infty(\mu)} \sum_{i=1}^{\infty} \int_{6^i5B\setminus 6^{i-1}5B} \bigl|\bigl(K(y,z)-K(x,z)\bigr)\bigl(m_{\widetilde{6^i5B}}(b)\bigr)-m_{\widetilde{B}}(b)\bigr| \frac{1}{d(y,z)}\,d\mu(z)\\
&\quad= M_{31}+M_{32}.
\end{aligned}$$

In Lemma 2.4, we write $a = 6^i\mu(6^{i+1}5B)$, $t_1 = \frac{|K(y,z)-K(x,z)|}{d(y,z)}$, and $t_2 = \frac{|b(z)-m_{\widetilde{6^i5B}}|}{c_2}$. From this we have

$$\begin{aligned}
M_{31}(x,y) &\le c\|f\|_{L^\infty(\mu)} \sum_{i=1}^{\infty} \int_{6^i5B\setminus 6^{i-1}5B} \frac{|K(y,z)-K(x,z)|}{d(y,z)} \bigl|b(z)-m_{\widetilde{6^i5B}}(b)\bigr|\,d\mu(z)\\
&\le c\|f\|_{L^\infty(\mu)} \sum_{i=1}^{\infty} \int_{6^i5B\setminus 6^{i-1}5B} \biggl[\frac{|K(y,z)-K(x,z)|}{d(y,z)} \log\biggl[2+6^i\mu\bigl(6^{i+1}5B\bigr)\\
&\qquad\times \frac{|K(y,z)-K(x,z)|}{d(y,z)}\biggr] + \frac{1}{6^i\mu(6^{i+1}5B)} \exp\biggl(\frac{|b(z)-m_{\widetilde{6^i5B}}|}{c_2}\biggr)\biggr]\,d\mu(z)\\
&\le c\|f\|_{L^\infty(\mu)} \sum_{i=1}^{\infty} i \int_{6^i5B\setminus 6^{i-1}5B} \frac{|K(y,z)-K(x,z)|}{d(y,z)} \log\biggl(2+\frac{\mu(6^{i+1}5B)}{\lambda(c_B,d(y,z))}\biggr)\,d\mu(z)\\
&\qquad + c\|f\|_{L^\infty(\mu)} \sum_{i=1}^{\infty} \int_{6^i5B\setminus 6^{i-1}5B} \frac{1}{6^i\mu(6^{i+1}5B)} \exp\biggl(\frac{|b(z)-m_{\widetilde{6^i5B}}|}{c_2}\biggr)\,d\mu(z)\\
&\le c\|f\|_{L^\infty(\mu)} \sum_{i=1}^{\infty} i \int_{6^i5B\setminus 6^{i-1}5B} \frac{|K(y,z)-K(x,z)|}{d(y,z)}\,d\mu(z)\\
&\qquad + c\|f\|_{L^\infty(\mu)} \sum_{i=1}^{\infty} \frac{1}{6^i} \frac{1}{\mu(6^{i+1}5B)} \int_{6^i5B} \exp\biggl(\frac{|b(z)-m_{\widetilde{6^i5B}}|}{c_2}\biggr)\,d\mu(z)\\
&\le c\|f\|_{L^\infty(\mu)},
\end{aligned}$$

where we use (2.3). For M_{32} we estimate

$$\begin{aligned}
M_{32} &\le c\|f\|_{L^\infty(\mu)} \sum_{i=1}^{\infty} \bigl|m_{\widetilde{6^i5B}}(b))-m_{\widetilde{Q}}(b)\bigr| \int_{6^i5B\setminus 6^{i-1}5B} \bigl|\bigl(K(y,z)-K(x,z)\bigr)\bigr| \frac{1}{d(y,z)}\,d\mu(z)\\
&\le c\|f\|_{L^\infty(\mu)} \sum_{i=1}^{\infty} i \int_{6^i5B\setminus 6^{i-1}5B} \bigl|\bigl(K(y,z)-K(x,z)\bigr)\bigr| \frac{1}{d(y,z)}\,d\mu(z)\\
&\le c\|f\|_{L^\infty(\mu)}.
\end{aligned}$$

Combining these estimates above, we get

$$\begin{aligned}
A_3 &\le \frac{1}{\mu(6B)}\int_B \bigl|\mathcal{M}\bigl(\bigl(b(y)-m_{\widetilde{B}}(b)\bigr)f_2\bigr)(y)-h_B\bigr|\,d\mu(y)\\
&\le c\frac{1}{\mu(6B)}\frac{1}{\mu(B)}\int_B\int_B\sum_{i=1}^{3}M_i(x,y)\,d\mu(x)\,d\mu(y)\\
&\le c\|f\|_{L^\infty(\mu)}.
\end{aligned}$$

So the estimate (2.4) is proved.

Now we prove (2.5). Consider two balls $B\subset S$ with $x\in B$ and let $N=N_{B,S}+1$, where S is a doubling ball. Write $|h_B-h_S|$ as

$$\begin{aligned}
&|h_B-h_S|\\
&\quad\le \bigl|m_S\bigl[\mathcal{M}\bigl(\bigl(b-m_{\widetilde{B}}(b)\bigr)f\chi_{\mathcal{X}\setminus 6^N B}\bigr)\bigr]-m_B\bigl[\mathcal{M}\bigl(\bigl(b-m_{\widetilde{B}}(b)\bigr)f\chi_{\mathcal{X}\setminus 6^N B}\bigr)\bigr]\bigr|\\
&\qquad+\bigl|m_S\bigl[\mathcal{M}\bigl(\bigl(b-m_S(b)\bigr)f\chi_{\mathcal{X}\setminus 6^N B}\bigr)\bigr]-m_S\bigl[\mathcal{M}\bigl(\bigl(b-m_{\widetilde{B}}(b)\bigr)f\chi_{\mathcal{X}\setminus 6^N B}\bigr)\bigr]\bigr|\\
&\qquad+\bigl|m_B\bigl[\mathcal{M}\bigl(\bigl(b-m_{\widetilde{B}}(b)\bigr)f\chi_{6^N B\setminus\frac{6}{5}B}\bigr)\bigr]\bigr|+\bigl|m_S\bigl[\mathcal{M}\bigl(\bigl(b-m_S(b)\bigr)f\chi_{6^N B\setminus\frac{6}{5}S}\bigr)\bigr]\bigr|\\
&\quad=B_1+B_2+B_3+B_4.
\end{aligned}$$

As in the estimate for the A_3, we have $B_1\le c\|f\|_{L^\infty(\mu)}$. To estimate B_2, for $y\in\mathcal{X}$, we get

$$\begin{aligned}
B_2 &\le \bigl|m_S\bigl[\mathcal{M}\bigl(\bigl(b-m_S(b)\bigr)f\chi_{\mathcal{X}\setminus 6^N B}\bigr)\bigr]-m_S\bigl[\mathcal{M}\bigl(\bigl(b-m_{\widetilde{B}}(b)\bigr)f\chi_{\mathcal{X}\setminus 6^N B}\bigr)\bigr]\bigr|\\
&\le cm_S\bigl|\bigl(m_S(b)-m_{\widetilde{B}}(b)\bigr)\mathcal{M}(f\chi_{\mathcal{X}\setminus 6^N B})\bigr|\\
&\le c\frac{K_{B,S}+K_{B,\widetilde{B}}}{\mu(S)}\int_S \mathcal{M}(f\chi_{\mathcal{X}\setminus 6^N B})(y)\,d\mu(y)\\
&\le c\frac{K_{B,S}}{\mu(S)}\mu(S)^{1/s'}\biggl(\int_S \mathcal{M}^s(f\chi_{\mathcal{X}\setminus 6^N B})(y)\,d\mu(y)\biggr)^{1/s}\\
&\le cK_{B,S}M_{s,6}\bigl[\mathcal{M}(f)\bigr].
\end{aligned}$$

For $y\in R$, we have

$$\begin{aligned}
B_4 &\le \bigl|m_S\bigl[\mathcal{M}\bigl(\bigl(b-m_S(b)\bigr)f\chi_{6^N B\setminus\frac{6}{5}S}\bigr)\bigr]\bigr|\\
&\le \int_{\mathcal{X}}\bigl|K(y,z)\bigr|\bigl|b(z)-m_S(b)\bigr|\bigl|f(z)\chi_{6^N B\setminus\frac{6}{5}S}\bigr|\biggl(\int_{d(y,z)<t}\frac{dt}{t^3}\biggr)^{1/2}d\mu(z)\\
&\le c\int_{\mathcal{X}}\frac{|b(z)-m_S(b)|}{\lambda(y,d(y,z))}\bigl|f(z)\chi_{6^N B\setminus\frac{6}{5}S}\bigr|\,d\mu(z)\\
&\le c\|f\|_{L^\infty(\mu)}\int_{6^N B}\frac{|b(z)-m_S(b)|}{\lambda(c_B,6^N r_B)}\,d\mu\\
&\le c\|f\|_{L^\infty(\mu)}\frac{1}{\lambda(c_B,6^N r_B)}\int_{6^N B}\bigl|m_S(b)-m_{\widetilde{6S}}(b)\bigr|+\bigl|b(z)-m_{\widetilde{6S}}\bigr|\,d\mu(z)\\
&\le c\|f\|_{L^\infty(\mu)}\frac{1}{\mu(6^2S)}\int_{6S}\bigl|b(z)-m_{\widetilde{6S}}\bigr|\,d\mu(z)+c\|f\|_{L^\infty(\mu)}\bigl|m_S(b)
\end{aligned}$$

$$
\begin{aligned}
&\quad - m_{\widetilde{6S}}(b)\Big| \frac{\mu(6^N B)}{\lambda(c_B, 6^N r_B)} \\
&\leq c\|f\|_{L^\infty(\mu)},
\end{aligned}
$$

where we have used $|m_S(b) - m_{\widetilde{6S}}(b)| \leq c(K_{S,6S} + K_{6S,\widetilde{6S}}) \leq c$.

In order to estimate B_3, for $y \in B$, we get

$$
\begin{aligned}
&\big|m_B\big[\mathcal{M}\big((b - m_{\widetilde{B}}(b))f\chi_{6^N B\setminus \frac{6}{5}B}\big)\big](y)\big| \\
&\leq \big|m_B\big[\mathcal{M}\big((b - m_{\widetilde{B}}(b))f\chi_{6^N B\setminus 6B}\big)\big](y) - m_B\big[\mathcal{M}\big((b - m_{\widetilde{B}}(b))f\chi_{6B\setminus \frac{6}{5}B}\big)\big](y)\big| \\
&\leq \int_{6B\setminus \frac{6}{5}B} \big|K(y,z)\big|\big|b(z) - m_{\widetilde{B}}(b)\big|\big|f(z)\big| \bigg(\int_{d(y,z)<t} \frac{dt}{t^3}\bigg)^{1/2} d\mu(z) \\
&\quad + \int_{6^N B\setminus 6B} \big|K(y,z)\big|\big|b(z) - m_{\widetilde{B}}(b)\big|\big|f(z)\big| \bigg(\int_{d(y,z)<t} \frac{dt}{t^3}\bigg)^{1/2} d\mu(z) \\
&\leq c\|f\|_{L^\infty(\mu)} \int_{6B\setminus \frac{6}{5}B} \frac{|b(z) - m_{\widetilde{B}}(b)|}{\lambda(y, d(y,z))} d\mu(z) + c\|f\|_{L^\infty(\mu)} \int_{6^N B\setminus 6B} \frac{|b(z) - m_{\widetilde{B}}(b)|}{\lambda(y, d(y,z))} d\mu(z) \\
&\leq c\|f\|_{L^\infty(\mu)} \frac{\mu(6^2 B)}{\lambda(c_B, 6r_B)} \frac{1}{\mu(6^2 B)} \int_{6B} \big|b(z) - m_{\widetilde{B}}(b)\big| d\mu(z) \\
&\quad + c\|f\|_{L^\infty(\mu)} \sum_{k=1}^{N-1} \int_{6^{k+1}B\setminus 6^k B} \frac{|b(z) - m_{\widetilde{B}}(b)|}{\lambda(y, d(y,z))} d\mu(z) \\
&\leq c\|f\|_{L^\infty(\mu)} + c\|f\|_{L^\infty(\mu)} \sum_{k=1}^{N-1} \frac{\mu(6^{k+2}B)}{\lambda(c_B, 6^{k+1}r_B)} \frac{1}{\mu(6^{k+2}B)} \int_{6^{k+1}B} \big|b(z) - m_{\widetilde{6^{k+1}B}}(b)\big| d\mu(z) \\
&\quad + c\|f\|_{L^\infty(\mu)} \sum_{k=1}^{N-1} \frac{\mu(6^{k+2}B)}{\lambda(c_B, 6^{k+1}r_B)} \frac{1}{\mu(6^{k+2}B)} \int_{6^{k+1}B} \big|m_{\widetilde{6^{k+1}B}}(b) - m_{\widetilde{B}}(b)\big| d\mu(z) \\
&\leq c\|f\|_{L^\infty(\mu)} + c\|f\|_{L^\infty(\mu)} \sum_{k=1}^{N-1} \frac{\mu(6^{k+2}B)}{\lambda(c_B, 6^{k+1}r_B)} (cK_{B,6^{k+1}B} + 1) \\
&\leq cK_{B,S}^2 \|f\|_{L^\infty(\mu)}.
\end{aligned}
$$

That is to say, $B_3 \leq cK_{B,S}^2 \|f\|_{L^\infty(\mu)}$.

Combining the estimates through B_1 to B_4 establishes (2.5), which completes the proof of Lemma 2.6. □

3 Proof of Theorem 1.10

In this section, we prove Theorem 1.10. Let $0 < r < 1$, we prove that, for any $p \in (1, \infty)$, $b \in L^\infty(\mu)$, and all bounded functions f with compact support,

$$
\mu\big(\big\{x \in \mathcal{X} : M_r^{\#}\big[\mathcal{M}_b(f)\big](x) > \lambda\big\}\big) \leq c\lambda^{-p}\|b\|_{\mathrm{RBMO}(\mu)}^p \|f\|_{L^p(\mu)}^p. \tag{3.1}
$$

Once (3.1) is established, it follows from the Marcinkiewicz interpolation theorem that

$$
\big\|M_r^{\#}\big[\mathcal{M}_b(f)\big]\big\|_{L^p(\mu)} \leq c\|b\|_{\mathrm{RBMO}(\mu)} \|f\|_{L^p(\mu)}. \tag{3.2}
$$

This via Theorem 4.2 in [25] states that, for any $p \in (1,\infty)$, $b \in L^\infty(\mu)$, and all bounded functions f with compact support and integral zero,

$$\left\|\mathcal{M}_b(f)]\right\|_{L^p(\mu)} \le c\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{L^p(\mu)}. \tag{3.3}$$

In [25], Theorem 6.4, the authors show the density in $L^p(\mu)$ of bounded functions with compact support and integral zero. Similar to [18], Lemma 3.3, using the truncation argument, a routine argument leads to (3.3) for all $b \in \mathrm{RBMO}(\mu)$ and $f \in L^p(\mu)$.

Now we prove (3.1). Without loss of generality, we assume that $\rho = 6$ in Lemma 2.1 and $\|b\|_{\mathrm{RBMO}(\mu)} = 1$. For each fixed $t > 0$ and bounded function f with compact support, applying the Calderón-Zygmund decomposition to $|f|^p$ at level t^p as Lemma 2.2, we decompose $f(x) = g(x) + h(x)$, where

$$g(x) = f(x)\chi_{\mathcal{X}\setminus\bigcup_i 6B_i}(x) + \sum_i \varphi_i(x), h(x) = \sum_i \bigl[\omega_i(x)f(x) - \varphi_i(x)\bigr] = \sum_i h_i(x).$$

It is obvious that $\|g\|_{L^\infty(\mu)} \le ct$. Using Lemma 2.1(2) we have

$$\begin{aligned}\left\|\sum_i \varphi_i\right\|_{L^p(\mu)}^p &\le \left\|\sum_i |\varphi_i|\right\|_{L^\infty(\mu)}^{p-1}\left\|\sum_i \varphi_i\right\|_{L^1(\mu)}\\ &\le ct^{p-1}\sum_i \left(\int_{R_i}\left|\varphi_i(x)\right|^p\,d\mu(x)\right)^{1/p}\mu(S_i)^{1/p'}\\ &\le c\sum_i \int_{6B_i}\left\|f(x)\right\|^p\,d\mu(x) \le c\|f\|_p^p.\end{aligned}$$

That is to say, $\|g\|_{L^p(\mu)} \le c\|f\|_{L^p(\mu)}$. Using (3.2) and Lemma 2.6 we have

$$\begin{aligned}&\mu\bigl(\bigl\{x \in \mathcal{X} : M_r^\#\bigl(\mathcal{M}_b(g)\bigr)(x) > 2ct\bigr\}\bigr)\\ &\quad\le c\mu\bigl(\bigl\{x \in \mathcal{X} : M_{s,6}\bigl(\mathcal{M}(g)\bigr)(x) > t\bigr\}\bigr)\\ &\quad\le ct^{-p}\left\|M_{s,6}\bigl(\mathcal{M}(g)\bigr)\right\|_{L^p(\mu)}^p \le ct^{-p}\|f\|_{L^p(\mu)}^p,\end{aligned}$$

where $1 < s < p$.

Similar to [25], Section 4.1, we have, for any f,

$$M_r^\# f(x) \le M_{r,6}(f)(x) + 3N_r(f)(x) \le cM_{r,6}(f)(x).$$

From this we write

$$\begin{aligned}&\mu\bigl(\bigl\{x \in \mathcal{X} : M_r^\#\bigl[\mathcal{M}_b(h)\bigr](x) > t\bigr\}\bigr)\\ &\quad\le \mu\left(\left\{x \in \mathcal{X} : M_{r,6}\left[\mathcal{M}\left(\sum_i (b - m_{\widetilde{6B_i}}(b))h_i\right)\right](x) > ct\right\}\right)\\ &\qquad+ \mu\left(\left\{x \in \mathcal{X} : M_{r,6}\left[\sum_i \left|b - m_{\widetilde{6B_i}}(b)\right|\mathcal{M}(h_i)\right](x) > t/2\right\}\right)\\ &\quad= D_1 + D_2.\end{aligned}$$

According to the weak type 1-1 estimate for M_ρ, we have, for any $\lambda > 0$,

$$\lambda\mu\big(\{x \in \mathcal{X} : M_{r,6}(g)(x) > \lambda\}\big) \le c \sup_{\delta > c\lambda} \delta\mu\big(\{x \in \mathcal{X} : |g(x)| > c\delta\}\big).$$

Taking $1 < p_1 < p$, it follows that

$$\begin{aligned} D_1 &\le t^{-1} \sup_{\delta > ct} \delta\mu\left(\left\{x \in \mathcal{X} : \mathcal{M}\left(\sum_i (b - m_{\widetilde{B}}(b))h_i\right)(x) > c\delta\right\}\right) \\ &\le ct^{-p_1} \left\| \sum_i (b - m_{\widetilde{6B_i}}(b))h_i \right\|_{L^{p_1}(\mu)}^{p_1} \\ &\le ct^{-p_1} \left\| \sum_i (b - m_{\widetilde{6B_i}}(b))f\omega_i \right\|_{L^{p_1}(\mu)}^{p_1} + ct^{-p_1} \left\| \sum_i (b - m_{\widetilde{6B_i}}(b))\varphi_i \right\|_{L^{p_1}(\mu)}^{p_1} \\ &\le D_{11} + D_{12}. \end{aligned}$$

For D_{11}, it follows that

$$\begin{aligned} D_{11} &\le ct^{-p_1} \sum_i \left[\int_{6B_i} |f(x)|^p \, d\mu(x)\right]^{p_1/p} \left[\int_{6B_i} \left|b(x) - m_{\widetilde{6B_i}}(b)\right|^{p_1(p/p_1)'} \, d\mu(x)\right]^{1-p_1/p} \\ &\le ct^{-p_1} \sum_i \left[\int_{6B_i} |f(x)|^p \, d\mu(x)\right]^{p_1/p} \mu\big(6^2 B_i\big)^{1-p_1/p} \\ &\le ct^{-p_1} \sum_i \int_{6B_i} |f(x)|^p \, d\mu(x) \left[\int_{6B_i} |f(x)|^p \, d\mu(x)\right]^{p_1/p-1} \mu\big(6^3 B_i\big)^{1-p_1/p} \\ &\le ct^{-p} \|f\|_{L^p(\mu)}^p, \end{aligned}$$

where we use Lemma 2.2(1). To estimate D_{12}, by the fact $\sum_i \varphi_i \le ct$, we have

$$\begin{aligned} D_{12} &\le c \left\| \sum_i (b - m_{\widetilde{6B_i}}(b))\varphi_i t^{-1} \right\|_{L^{p_1}(\mu)}^{p_1} \\ &\le c \left\| \left[\sum_i t^{-1}|\varphi_i| \left|b - m_{\widetilde{6B_i}}(b)\right|^{p_1}\right]^{1/p_1} \left[\sum_i \left|t^{-1}\varphi_i\right|\right]^{1/p_1'} \right\|_{L^{p_1}(\mu)}^{p_1} \\ &\le c \left\| \left[\sum_i t^{-1}|\varphi_i| \left|b - m_{\widetilde{6B_i}}(b)\right|^{p_1}\right]^{1/p_1} \right\|_{L^{p_1}(\mu)}^{p_1} \\ &\le ct^{-1} \sum_i \int_{R_i} \left|\varphi_i(x)\right| \left|b(x) - m_{\widetilde{6B_i}}(b)\right|^{p_1} \, d\mu(x) \\ &\le ct^{-1} \sum_i \left(\int_{R_i} \left|\varphi_i(x)\right|^p \, d\mu(x)\right)^{1/p} \left(\int_{R_i} \left|b(x) - m_{\widetilde{R_i}}(b)\right|^{p_1 p'} \, d\mu(x)\right)^{1/p'} \\ &\quad + ct^{-1} \sum_i \int_{R_i} \left|\varphi_i(x)\right| \left|m_{\widetilde{R_i}}(b) - m_{\widetilde{6B_i}}(b)\right|^{p_1} \, d\mu(x) \\ &\le ct^{-1} \sum_i \left(\int_{R_i} \left|\varphi_i(x)\right|^p \, d\mu(x)\right)^{1/p} \mu(R_i)^{1/p'} + ct^{-1} \sum_i \int_{R_i} \left|\varphi_i(x)\right| \, d\mu(x) \end{aligned}$$

$$\le ct^{-p}\sum_i \int_{6B_i} |f(x)|^p\,d\mu(x) + ct^{-1}\sum_i \left(\int_{R_i} |\varphi_i(x)|^p\,d\mu(x)\right)^{1/p} \mu(S_i)^{1/p'}$$
$$\le ct^{-p}\|f\|^p_{L^p(\mu)}.$$

In order to estimate D_2, we write

$$\begin{aligned} D_2 &\le ct^{-1}\sum_i \int_{\mathcal{X}\setminus 6S_i} |b(x) - m_{\widetilde{6B_i}}(b)|\mathcal{M}(h_i)(x)\,d\mu(x) \\ &\quad + ct^{-1}\sum_i \int_{6S_i} |b(x) - m_{\widetilde{6B_i}}(b)|\mathcal{M}(\varphi_i)(x)\,d\mu(x) \\ &\quad + ct^{-1}\sum_i \int_{\frac{6}{5}6B_i} |b(x) - m_{\widetilde{6B_i}}(b)|\mathcal{M}(\omega_i f)(x)\,d\mu(x) \\ &\quad + ct^{-1}\sum_i \int_{6S_i\setminus\frac{6}{5}6B_i} |b(x) - m_{\widetilde{6B_i}}(b)|\mathcal{M}(\omega_i f)(x)\,d\mu(x) \\ &= D_{21} + D_{22} + D_{23} + D_{24}. \end{aligned}$$

For each i, we have

$$\begin{aligned} &\int_{\mathcal{X}\setminus 6S_i} |b(x) - m_{\widetilde{6B_i}}(b)|\mathcal{M}(h_i)(x)\,d\mu(x) \\ &\quad\le \int_{\mathcal{X}\setminus 6S_i} |b(x) - m_{\widetilde{6B_i}}(b)| \left[\int_0^{d(x,c_{S_i})+r_{6S_i}} \left|\int_{d(x,y)<t} K(x,y)h_i(y)\,d\mu(y)\right|^2 \frac{dt}{t^3}\right]^{1/2} d\mu(x) \\ &\quad\quad + \int_{\mathcal{X}\setminus 6S_i} |b(x) - m_{\widetilde{6B_i}}(b)| \left[\int_{d(x,c_{S_i})+r_{6S_i}}^{\infty} \left|\int_{d(x,y)<t} K(x,y)h_i(y)\,d\mu(y)\right|^2 \frac{dt}{t^3}\right]^{1/2} d\mu(x) \\ &\quad= D^1_{21} + D^2_{21}. \end{aligned}$$

Using

$$|m_{\widetilde{6^{k+1}6S_i}}(b) - m_{\widetilde{6B_i}}(x)| \le c(K_{6B_i,\widetilde{6B_i}} + K_{6B_i,6S_i} + K_{6S_i,6^{k+1}6S_i} + K_{6^{k+1}6S_i,\widetilde{6^{k+1}6S_i}}) \le ck,$$

we get

$$\begin{aligned} D^1_{21} &\le \int_{\mathcal{X}\setminus 6S_i} |b(x) - m_{\widetilde{6B_i}}(b)| \int_{\mathcal{X}} |K(x,y)h_i(y)| \left(\int_{d(x,y)}^{d(x,c_{S_i})+r_{6S_i}} \frac{dt}{t^3}\right)^{1/2} d\mu(y)\,d\mu(x) \\ &\le c\int_{\mathcal{X}\setminus 6S_i} |b(x) - m_{\widetilde{6B_i}}(b)| \int_{\mathcal{X}} |h_i(y)| \frac{r^{1/2}_{6S_i}}{d(x,c_{S_i})^{1/2}\lambda(x,d(x,c_{S_i}))}\,d\mu(y)\,d\mu(x) \\ &= c\|h_i\|_{L^1(\mu)} \sum_j \int_{6^{j+1}6S_i\setminus 6^j 6S_i} |b(x) - m_{\widetilde{6B_i}}(b)| \frac{r^{1/2}_{6S_i}}{d(x,c_{S_i})^{1/2}\lambda(x,d(x,c_{S_i}))}\,d\mu(x) \\ &\le c\|h_i\|_{L^1(\mu)} \sum_j \int_{6^{j+1}6S_i\setminus 6^j 6S_i} |b(x) - m_{\widetilde{6^{j+1}6S_i}}(b)| \frac{r^{1/2}_{6R_i}}{d(x,c_{S_i})^{1/2}\lambda(x,d(x,c_{S_i}))}\,d\mu(x) \\ &\quad + c\|h_i\|_{L^1(\mu)} \sum_j |m_{\widetilde{6^{j+1}6S_i}}(b) - m_{\widetilde{6B_i}}(b)| \end{aligned}$$

$$\times \int_{6^{j+1}6S_i\setminus 6^j 6S_i} \frac{r_{6S_i}^{1/2}}{d(x,c_{S_i})^{1/2}\lambda(x,d(x,c_{S_i}))}\,d\mu(x)$$
$$\le c\|h_i\|_{L^1(\mu)} \sum_j (j+1) \frac{r_{6S_i}^{1/2}\mu(6^{j+2}6S_i)}{(6^j r_{6S_i})^{1/2}\lambda(c_{S_i},6^j r_{6S_i})}$$
$$\le c\|h_i\|_{L^1(\mu)}.$$

By the vanishing moment of h_i, it follows that

$$\begin{aligned}
D_{21}^2 &\le \int_{\mathcal{X}\setminus 6S_i} \bigl|b(x)-m_{\widetilde{6B_i}}(b)\bigr| \int_{\mathcal{X}} \bigl|\bigl[K(x,y)-K(x,c_{R_i})\bigr]h_i(y)\bigr| \\
&\quad \times \biggl(\int_{d(x,c_{S_i})+r_{6S_i}}^{\infty} \frac{dt}{t^3}\biggr)^{1/2} d\mu(y)\,d\mu(x) \\
&\le c\int_{\mathcal{X}\setminus 6S_i} \bigl|b(x)-m_{\widetilde{6B_i}}(b)\bigr| \int_{\mathcal{X}} \bigl|\bigl[K(x,y)-K(x,c_{R_i})\bigr]h_i(y)\bigr| \frac{1}{d(x,c_{S_i})}\,d\mu(y)\,d\mu(x) \\
&\le \int_{\mathcal{X}} h_i(y)\biggl[\sum_j \int_{6^{j+1}6S_i\setminus 6^j 6S_i} \frac{|[K(x,y)-K(x,c_{S_i})|}{d(x,c_{S_i})} \\
&\quad \times \bigl(\bigl|b(x)-m_{6^{j+1}6S_i}\bigr| + |m_{6^{j+1}6S_i}-m_{\widetilde{6B_i}}|\bigr)\,d\mu(x)\biggr]\,d\mu(y) \\
&\le c\|h_i\|_{L^1(\mu)}.
\end{aligned}$$

But

$$\begin{aligned}
\|h_i\|_{L^1(\mu)} &\le c\biggl(\biggl(\int_{6B_i}\bigl|f(x)\bigr|^p\,d\mu(x)\biggr)^{1/p}\mu(6B_i)^{1/p'} + \biggl(\int_{6S_i}\bigl|\varphi(x)\bigr|^p d\mu(x)\biggr)^{1/p}\mu(6S_i)^{1/p'} \\
&\le ct^{1-p}\|f\|_{L^p(\mu)}^p,
\end{aligned}$$

so $D_{21} \le ct^{-1}t^{1-p}\|f\|_{L^p(\mu)}^p \le ct^{-p}\|f\|_{L^p(\mu)}^p$.

For D_{22}, it follows from the $L^p(\mu)$ boundedness of $\mathcal{M}$ that

$$\begin{aligned}
D_{22} &\le ct^{-1}\sum_i \int_{6S_i}\bigl|b(x)-m_{\widetilde{6B_i}}(b)\bigr|\mathcal{M}(\varphi_i)(x)\,d\mu(x) \\
&\quad + ct^{-1}\sum_i \bigl|m_{\widetilde{6B_i}}(b)-m_{\widetilde{6B_i}}(b)\bigr|\int_{6S_i}\mathcal{M}(\varphi_i)(x)\,d\mu(x) \\
&\le ct^{-1}\sum_i\biggl(\int_{6S_i}\bigl|b(x)-m_{\widetilde{6B_i}}(b)\bigr|^{p'}\,d\mu(x)\biggr)^{1/p'}\bigl\|\mathcal{M}(\varphi_i)\bigr\|_{L^p(\mu)} \\
&\quad + ct^{-1}\sum_i\bigl\|\mathcal{M}(\varphi_i)\bigr\|_{L^p(\mu)}\mu(6S_i)^{1/p'} \\
&\le ct^{-1}\sum_i\biggl(\int_{S_i}\varphi_i(x)\,d\mu(x)\biggr)^{1/p}\mu\bigl(6^2S_i\bigr)^{1/p'} \\
&\le ct^{-1}t^{1-p}\sum_i\int_{6B_i}\bigl|f(x)\bigr|^p\,d\mu(x) \le ct^{-p}\|f\|_{L^p(\mu)}^p.
\end{aligned}$$

Similar to D_{22}, we have

$$\begin{aligned} D_{23} &\le ct^{-1}\sum_i \biggl(\int_{\frac{6}{5}6B_i}\bigl|b(x)-m_{\widetilde{6B_i}}(b)\bigr|^{p'}\,d\mu(x)\biggr)^{1/p'}\bigl\|\mathcal{M}(\omega_i f)\bigr\|_{L^p(\mu)} \\ &\le ct^{-1}\sum_i \mu(S_i)^{1/p'}\|\omega_i f\|_{L^p(\mu)} \\ &\le ct^{-1}\sum_i t^{1-p}\|\omega_i f\|_{L^p(\mu)}\int_{6B_i}\bigl|f(x)\bigr|^p\,d\mu(x)\biggl(\int_{S_i}\bigl|\varphi_i(x)^p\bigr|\,d\mu(x)\biggr)^{-1} \\ &\le ct^{-p}\|f\|_{L^p(\mu)}. \end{aligned}$$

Next we estimate D_{24}. If $\operatorname{supp} f \subset B$ for some ball then, for any $\rho > 1$ and $x \in \mathcal{X} \setminus \rho B$, we have

$$\begin{aligned} \mathcal{M}(f)(x) &\le c\int_B \bigl|K(x,y)f(y)\bigr|\biggl(\int_{d(x,y)}^{\infty}\frac{dt}{t^3}\biggr)^{1/2}\,d\mu(y) \\ &\le c\int_B \frac{|f(y)|}{\lambda(x,d(x,y))} \le \frac{c}{\lambda(c_B,d(x,c_B))}\int_B \bigl|f(y)\bigr|\,d\mu(y). \end{aligned}$$

For any i we write $S_i = (3\times 6^2)^{k_i}B_i$. It follows that

$$\begin{aligned} D_{24} &\le t^{-1}\sum_i \int_{6S_i\setminus\frac{6}{5}6B_i}\frac{|b(x)-m_{\widetilde{6B_i}}(b)|}{\lambda(c_{B_i},d(x,c_{B_i}))}\int_{6B_i}\bigl|f(y)\omega_i(y)\bigr|\,d\mu(y)\,d\mu(x) \\ &\le ct^{-1}\sum_i\biggl(\int_{6B_i}\bigl|f(y)\bigr|\,d\mu(y)\biggr)\int_{6S_i\setminus S_i}\frac{|b(x)-m_{\widetilde{6B_i}}(b)|}{\lambda(c_{B_i},d(x,c_{B_i}))}\,d\mu(x) \\ &\quad + ct^{-1}\sum_i\biggl(\int_{6B_i}\bigl|f(y)\bigr|\,d\mu(y)\biggr)\int_{S_i\setminus\frac{6}{5}6B_i}\frac{|b(x)-m_{\widetilde{6B_i}}(b)|}{\lambda(c_{B_i},d(x,c_{B_i}))}\,d\mu(x) \\ &\le ct^{-1}\sum_i\biggl(\int_{6B_i}\bigl|f(y)\bigr|\,d\mu(y)\biggr)\int_{6S_i\setminus S_i}\frac{|b(x)-m_{\widetilde{6S_i}}(b)|+|m_{\widetilde{6S_i}}(b)-m_{\widetilde{6B_i}}(b)|}{\lambda(c_{B_i},d(x,c_{B_i}))}\,d\mu(x) \\ &\quad + ct^{-1}\sum_i\biggl(\int_{6B_i}\bigl|f(y)\bigr|\,d\mu(y)\biggr) \\ &\quad \times\sum_{j=0}^{k_i-1}\int_{6^{j+1}6B_i\setminus 6^j6B_i}\frac{|b(x)-m_{\widetilde{6S_i}}(b)|+|m_{\widetilde{6S_i}}(b)-m_{\widetilde{6B_i}}(b)|}{\lambda(c_{B_i},d(x,c_{B_i}))}\,d\mu(x) \\ &\le ct^{-1}\sum_i\biggl(\int_{6B_i}\bigl|f(y)\bigr|\,d\mu(y)\biggr)\biggl(1+\frac{\mu(6S_i)}{\lambda(c_{B_i},r_{S_i})}\biggr) \\ &\quad + ct^{-1}\sum_i\biggl(\int_{6B_i}\bigl|f(y)\bigr|\,d\mu(y)\biggr)\Biggl[1+\sum_{j=0}^{k_i-1}\frac{\mu(6^{j+2}6B_i)}{\lambda(c_{B_i},6^{j+1}r_{B_i})}K_{B_i,S_i}\Biggr] \\ &\le ct^{-1}\sum_i\biggl(\int_{6B_i}\bigl|f(y)\bigr|\,d\mu(y)\biggr) \le ct^{-1}\sum_i\biggl(\int_{6B_i}\bigl|f(y)\bigr|^p\,d\mu(y)\biggr)^{1/p}\mu(6B_i)^{1/p'} \\ &\le ct^{-p}\|f\|_{L^p(\mu)}^p. \end{aligned}$$

Combining these estimates for the term D_{21}, D_{22}, D_{23}, and D_{24} yields the desired estimate for D_2. So we complete the proof of Theorem 1.10.

Competing interests
The authors declare that they do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
Project (Grant No. 11261055, 11161044) supported by NSFC. The authors cordially thank the referees for their careful reading and helpful comments.

Received: 3 March 2015 Accepted: 13 August 2015 Published online: 27 August 2015

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