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Some sufficient efficiency conditions in semiinfinite multiobjective fractional programming based on exponential type invexities

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Abstract

In this paper, we first generalize the first order exponential Hanson-Antczak type $(\alpha,\beta,\gamma,\xi,\rho,\eta,\theta)$-invexities to the case of the $HA(\alpha,\beta,\gamma,\xi,\rho,\eta,h(\cdot,\cdot),\theta)$-V-invexities, which encompass most of the exponential type invexities as well as other various invexity variants in the literature. The obtained results are new and general in nature relevant to various applications arising in semiinfinite multiobjective fractional programming and optimization.

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Keywords: semiinfinite programming; multiobjective fractional programming; generalized $(\alpha,\beta,\gamma,\xi,\rho,\eta,h(\cdot,\cdot);\theta)$-invex functions; infinitely many equality and inequality constraints; parametric sufficient efficiency conditions

1 Introduction

Zalmai [1] introduced some multiparameter generalizations of the class of V-r-invex functions defined by Antczak [2], and then, using the new functions, proved a number of parametric sufficient efficiency results under various Hanson-Antczak types generalized $(\alpha,\beta,\gamma,\xi,\rho,\theta)$-V-invexity assumptions for the semiinfinite multiobjective fractional programming problems. Recently, Verma [3, 4] has investigated some results on the multiobjective fractional programming based on new ϵ-optimality conditions, and second-order (Φ,η,ρ,θ)-invexities for parameter-free ϵ-efficiency conditions. On the other hand, Verma [5] established a class of results for multiobjective fractional subset programming problems as well. Now we consider the following semiinfinite multiobjective fractional programming problem based on the first order exponential type $HA(\alpha,\beta,\gamma,\xi,\rho,\eta,h(\cdot,\cdot),\theta)$-V-invexity:

$$\text{(P)}\quad \text{Minimize}\, \varphi(x) = \big(\varphi_1(x),\ldots,\varphi_p(x)\big) = \left(\frac{f_1(x)}{g_1(x)},\ldots,\frac{f_p(x)}{g_p(x)}\right)$$

subject to

$$G_j(x,t) \leqq 0, \quad \text{for all } t \in T_j, j \in \underline{q},$$

$$H_k(x,s) = 0, \quad \text{for all } s \in S_k, k \in \underline{r},$$
$$x \in X,$$

where p, q, and r are positive integers, X is a nonempty open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), for each $j \in \underline{q} \equiv \{1, 2, \ldots, q\}$ and $k \in \underline{r}$, T_j and S_k are compact subsets of complete metric spaces, for each $i \in \underline{p}$, f_i and g_i are real-valued functions defined on X, for each $j \in \underline{q}$, $G_j(\cdot, t)$ is a real-valued function defined on X, for all $t \in T_j$, for each $k \in \underline{r}$, $H_k(\cdot, s)$ is a real-valued function defined on X, for all $s \in S_k$, for each $j \in \underline{q}$ and $k \in \underline{r}$, $G_j(x, \cdot)$ and $H_k(x, \cdot)$ are continuous real-valued functions defined, respectively, on T_j and S_k, for all $x \in X$, and for each $i \in \underline{p}$, $g_i(x) > 0$ for all x satisfying the constraints of (P).

Multiobjective programming problems of the form (P) but with a finite number of constraints (where the functions G_j are independent of t, and the functions H_k are independent of s), have been investigated for the past three decades. Several classes of static and dynamic optimization problems with multiple fractional objective functions have been studied leading to a number of sufficient efficiency and duality results currently available in the related literature. We observe that despite phenomenal research advances in several areas of multiobjective programming, the semiinfinite nonlinear multiobjective fractional programming problems have not received much attention in the general area of mathematical programming.

In this communication, we first present a generalization - the first order exponential type $HA(\alpha, \beta, \gamma, \xi, \rho, \eta, h(\cdot, \cdot), \theta)$-V-invexities, and then formulate a number of parametric sufficient efficiency results for problem (P) under various generalized $(\alpha, \beta, \gamma, \xi, \rho, \eta, h(\cdot, \cdot), \theta)$-invexity assumptions. A mathematical programming problem is generally categorized as the *semiinfinite programming problem* if it has a finite number of variables and infinitely many constraints, while problems of this type have been applied for the modeling and analysis of a wide range of theoretical as well as concrete, real-world problems. Furthermore, semiinfinite programming concepts and techniques have challenging applications in approximation theory, statistics, game theory, engineering design, boundary value problems, defect minimization for operator equations, geometry, random graphs, wavelet analysis, reliability testing, environmental protection planning, decision making under uncertainty, semidefinite programming, geometric programming, disjunctive programming, optimal control problems, robotics, and continuum mechanics. For more details, we refer the reader to [1–52].

This communication begins with an introductory section, while in Section 2, we introduce the first order exponential type $HA(\alpha, \beta, \gamma, \xi, \rho, \eta, h(\cdot, \cdot), \theta)$-V-invexities along with some auxiliary results which will be needed in the sequel. In Section 3, we discuss some sufficient efficiency conditions where we formulate and prove several sets of sufficiency criteria under a variety of the first order exponential type $HA(\alpha, \beta, \gamma, \xi, \rho, \eta, h(\cdot, \cdot), \theta)$-V-invexities that are placed on certain vector-valued functions whose entries consist of the individual as well as some combinations of the problem functions. Finally, Section 4 deals with several families of sufficient efficiency results under various first order exponential type $HA(\alpha, \beta, \gamma, \xi, \eta, h(\cdot, \cdot), \rho, \theta)$-V-invexity hypotheses imposed on certain vector functions whose components are formed by considering different combinations of the problem functions, which is accomplished by applying a certain type of partitioning scheme.

As a matter of fact, all the parametric sufficient efficiency results established in this paper regarding problem (P) can easily be modified and restated for each one of the following seven special classes of nonlinear programming problems.

(P1) $\underset{x\in\mathbb{F}}{\text{Minimize}}\big(f_1(x),\dots,f_p(x)\big);$

(P2) $\underset{x\in\mathbb{F}}{\text{Minimize}}\,\frac{f_1(x)}{g_1(x)};$

(P3) $\underset{x\in\mathbb{F}}{\text{Minimize}}\,f_1(x),$

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of (P), that is,

$$\mathbb{F}=\big\{x\in X:G_j(x,t)\leqq 0,\text{for all } t\in T_j, j\in\underline{q}, H_k(x,s)=0, \text{for all } s\in S_k, k\in\underline{r}\big\};$$

(P4) $\text{Minimize}\left(\frac{f_1(x)}{g_1(x)},\dots,\frac{f_p(x)}{g_p(x)}\right)$

subject to

$$\tilde{G}_j(x)\leqq 0,\quad j\in\underline{q},\qquad \tilde{H}_k(x)=0,\quad k\in\underline{r}, x\in X,$$

where f_i and g_i, $i\in\underline{p}$, are as defined in the description of (P), $\tilde{G}_j$, $j\in\underline{q}$, and $\tilde{H}_k$, $k\in\underline{r}$, are real-valued functions defined on X;

(P5) $\underset{x\in\mathbb{G}}{\text{Minimize}}\big(f_1(x),\dots,f_p(x)\big);$

(P6) $\underset{x\in\mathbb{G}}{\text{Minimize}}\,\frac{f_1(x)}{g_1(x)};$

(P7) $\underset{x\in\mathbb{G}}{\text{Minimize}}\,f_1(x),$

where $\mathbb{G}$ is the feasible set of (P4), that is,

$$\mathbb{G}=\big\{x\in X:\tilde{G}_j(x)\leqq 0, j\in\underline{q}, \tilde{H}_k(x)=0, k\in\underline{r}\big\}.$$

2 Preliminaries

In this section we first introduce the notion of the first order exponential type $HA(\alpha,\beta,\gamma,\xi,\rho,\eta,h(\cdot,\cdot),\theta)$-V-invexities, and then recall some other related auxiliary results instrumental to the problem at hand.

Definition 2.1 Let f be a differentiable real-valued function defined on $\mathbb{R}^n$. Then f is said to be η-invex (invex with respect to η) at y if there exists a function $\eta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ such that for each $x\in\mathbb{R}^n$,

$$f(x)-f(y)\geqq\big\langle\nabla f(y),\eta(x,y)\big\rangle,$$

where $\nabla f(y)=(\partial f(y)/\partial y_1,\partial f(y)/\partial y_2,\dots,\partial f(y)/\partial y_n)$ is the gradient of f at y, and $\langle a,b\rangle$ denotes the inner product of the vectors a and b; f is said to be η-invex on $\mathbb{R}^n$ if the above inequality holds for all $x,y\in\mathbb{R}^n$.

Hanson [21] showed (based on the role of the function η) that for a nonlinear programming problem of the form

$$\text{Minimize} f(x) \text{ subject to } g_i(x) \leqq 0, \quad i \in \underline{m}, x \in \mathbb{R}^n,$$

where the differentiable functions $f, g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \underline{m}$, are invex with respect to the function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the Karush-Kuhn-Tucker necessary optimality conditions are also sufficient.

Let the function $F = (F_1, F_2, \ldots, F_N) : \mathbb{R}^n \to \mathbb{R}^N$ be differentiable at x^*. The following generalizations of the notions of invexity, pseudoinvexity, and quasiinvexity for vector-valued functions were originally proposed in [28].

Definition 2.2 The function F is said to be (α, η)-*V-invex at* x^* if there exist functions $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \backslash \{0\} \equiv (0, \infty)$, $i \in \underline{N}$, and $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$ and $i \in \underline{N}$,

$$F_i(x) - F_i(x^*) \geqq \langle \alpha_i(x, x^*) \nabla F_i(x^*), \eta(x, x^*) \rangle.$$

Definition 2.3 The function F is said to be (β, η)-*V-pseudoinvex at* x^* if there exist functions $\beta_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \backslash \{0\}$, $i \in \underline{N}$, and $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$,

$$\left\langle \sum_{i=1}^N \nabla F_i(x^*), \eta(x, x^*) \right\rangle \geqq 0 \quad \Rightarrow \quad \sum_{i=1}^N \beta_i(x, x^*) F_i(x) \geqq \sum_{i=1}^N \beta_i(x, x^*) F_i(x^*).$$

Definition 2.4 The function F is said to be (γ, η)-*V-quasiinvex at* x^* if there exist functions $\gamma_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \backslash \{0\}$, $i \in \underline{N}$, and $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$,

$$\sum_{i=1}^N \gamma_i(x, x^*) F_i(x) \leqq \sum_{i=1}^N \gamma_i(x, x^*) F_i(x^*) \quad \Rightarrow \quad \left\langle \sum_{i=1}^N \nabla F_i(x^*), \eta(x, x^*) \right\rangle \leqq 0.$$

Recently, Antczak [2] introduced the following variant of the class of V-invex functions.

Definition 2.5 A differentiable function $f : X \to \mathbb{R}^k$ is called (strictly) ζ_i-$\tilde{r}$-invex with respect to η at $u \in X$ if there exist functions $\eta : X \times X \to \mathbb{R}^n$ and $\zeta_i : X \times X \to \mathbb{R}_+ \backslash \{0\}$, $i \in \underline{k}$, such for each $x \in X$,

$$\frac{1}{\tilde{r}} e^{\tilde{r} f_i(x)} (>) \geqq \frac{1}{\tilde{r}} e^{\tilde{r} f_i(u)} \big[1 + \tilde{r} \zeta_i(x, u) \langle \nabla f_i(u), \eta(x, u) \rangle \big] \quad \text{for } \tilde{r} \neq 0,$$
$$f_i(x) - f_i(u) \geqq \zeta_i(x, u) \langle \nabla f_i(u), \eta(x, u) \rangle \quad \text{for } \tilde{r} = 0.$$

This class of functions was considered in [2] for establishing some sufficiency and duality results for a nonlinear programming problem with differentiable functions, and their nonsmooth analogues were discussed in [6]. Recently, Zalmai [1] introduced the Hanson-Antczak type generalized $HA(\alpha, \beta, \gamma, \xi, \eta, \rho, \theta)$-V-invexity, an exponential type framework, and then he applied to a set of problems on fractional programming. As a result, he further envisioned a vast array of interesting and significant classes of generalized convex functions. Now we present first order exponential type $HA(\alpha,\beta,\gamma,\xi,\eta,h(\cdot,\cdot),\rho,\theta)$-V-invexities that generalize and encompass most of the existing notions available in the current literature. Let the function $F=(F_1,F_2,\dots,F_p):X\to\mathbb{R}^p$ be differentiable at x^*.

Definition 2.6 The function F is said to be (strictly) $HA(\alpha,\beta,\gamma,h(\cdot,\cdot),\xi,\eta,\rho,\theta)$-invex at $x^*\in X$ if there exist functions $\alpha:X\times X\to\mathbb{R}$, $\beta:X\times X\to\mathbb{R}$, $\gamma_i:X\times X\to\mathbb{R}_+$, $\xi_i:X\times X\to\mathbb{R}_+\backslash\{0\}$, $i\in\underline{p}$, $z\in\mathbb{R}^n$, $\eta:X\times X\to\mathbb{R}^n$, $\rho_i:X\times X\to\mathbb{R}$, $i\in\underline{p}$, and $\theta:X\times X\to\mathbb{R}^n$ such that, for all $x\in X$ $(x\neq x^*)$ and $i\in\underline{p}$,

$$\begin{aligned}
&\frac{1}{\alpha(x,x^*)}\gamma_i(x,x^*)\bigl(e^{\alpha(x,x^*)[F_i(x)-F_i(x^*)]}-1\bigr)\\
&\quad(>)\geqq\frac{1}{\beta(x,x^*)}\bigl\langle\xi_i(x,x^*)\nabla_z h_i(x^*,z),e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\bigr\rangle\\
&\qquad+\rho_i(x,x^*)\bigl\|\theta(x,x^*)\bigr\|^2\quad\text{if }\alpha(x,x^*)\neq 0\text{ and }\beta(x,x^*)\neq 0,\text{for all }x\in X,\\
&\frac{1}{\alpha(x,x^*)}\gamma_i(x,x^*)\bigl(e^{\alpha(x,x^*)[F_i(x)-F_i(x^*)]}-1\bigr)\\
&\quad(>)\geqq\bigl\langle\xi_i(x,x^*)\nabla_z h_i(x^*,z),\eta(x,x^*)\bigr\rangle\\
&\qquad+\rho_i(x,x^*)\bigl\|\theta(x,x^*)\bigr\|^2\quad\text{if }\alpha(x,x^*)\neq 0\text{ and }\beta(x,x^*)\to 0,\text{for all }x\in X,\\
&\gamma_i(x,x^*)\bigl[F_i(x)-F_i(x^*)\bigr]\\
&\quad(>)\geqq\frac{1}{\beta(x,x^*)}\bigl\langle\xi_i(x,x^*)\nabla_z h_i(x^*,z),e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\bigr\rangle\\
&\qquad+\rho_i(x,x^*)\bigl\|\theta(x,x^*)\bigr\|^2\quad\text{if }\alpha(x,x^*)\to 0\text{ and }\beta(x,x^*)\neq 0,\text{for all }x\in X,\\
&\gamma_i(x,x^*)\bigl[F_i(x)-F_i(x^*)\bigr]\\
&\quad(>)\geqq\bigl\langle\xi_i(x,x^*)\nabla_z h_i(x^*,z),\eta(x,x^*)\bigr\rangle+\rho_i(x,x^*)\bigl\|\theta(x,x^*)\bigr\|^2\\
&\qquad\text{if }\alpha(x,x^*)\to 0\text{ and }\beta(x,x^*)\to 0,\text{for all }x\in X,
\end{aligned}$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$ and

$$\bigl(e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\bigr)\equiv\bigl(e^{\beta(x,x^*)\eta_1(x,x^*)}-1,\dots,e^{\beta(x,x^*)\eta_n(x,x^*)}-1\bigr),$$

with $h:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ differentiable.

Definition 2.7 The function F is said to be (strictly) $HA(\alpha,\beta,\gamma,\xi,\eta,\rho,h(\cdot,\cdot),\theta)$-V-pseudoinvex at $x^*\in X$ if there exist functions $\alpha:X\times X\to\mathbb{R}$, $\beta:X\times X\to\mathbb{R}$, $\gamma:X\times X\to\mathbb{R}_+$, $\xi_i:X\times X\to\mathbb{R}_+\backslash\{0\}$, $i\in\underline{p}$, $z\in\mathbb{R}^n$, $\eta:X\times X\to\mathbb{R}^n$, $\rho:X\times X\to\mathbb{R}$, and $\theta:X\times X\to\mathbb{R}^n$ such that, for all $x\in X$ $(x\neq x^*)$,

$$\begin{aligned}
&\frac{1}{\beta(x,x^*)}\Biggl\langle\sum_{i=1}^{p}\nabla_z h_i(x^*,z),e^{\beta(x,x^*)\eta(x,x^*))}-\mathbf{1}\Biggr\rangle\geqq-\rho(x,x^*)\bigl\|\theta(x,x^*)\bigr\|^2\\
&\quad\Rightarrow\quad\frac{1}{\alpha(x,x^*)}\gamma(x,x^*)\bigl(e^{\alpha(x,x^*)\sum_{i=1}^{p}\xi_i(x,x^*)[F_i(x)-F_i(x^*)]}-1\bigr)(>)\geqq 0\\
&\qquad\qquad\text{if }\alpha(x,x^*)\neq 0\text{ and }\beta(x,x^*)\neq 0,\text{for all }x\in X,
\end{aligned}$$

$$\left\langle \sum_{i=1}^{p} \nabla_z h_i(x^*, z), \eta(x, x^*) \right\rangle \geqq -\rho(x, x^*) \left\| \theta(x, x^*) \right\|^2$$
$$\Rightarrow \quad \frac{1}{\alpha(x, x^*)} \gamma(x, x^*) \left(e^{\alpha(x,x^*) \sum_{i=1}^{p} \xi_i(x,x^*)[F_i(x)-F_i(x^*)]} - 1\right)(>) \geqq 0$$
$$\text{if } \alpha(x, x^*) \neq 0 \text{ and } \beta(x, x^*) \to 0, \text{for all } x \in X,$$
$$\frac{1}{\beta(x, x^*)} \left\langle \sum_{i=1}^{p} \nabla_z h_i(x^*, z), e^{\beta(x,x^*)\eta(x,x^*)} - \mathbf{1} \right\rangle \geqq -\rho(x, x^*) \left\| \theta(x, x^*) \right\|^2$$
$$\Rightarrow \quad \gamma(x, x^*) \sum_{i=1}^{p} \xi_i(x, x^*) \left[F_i(x) - F_i(x^*)\right](>) \geqq 0$$
$$\text{if } \alpha(x, x^*) \to 0 \text{ and } \beta(x, x^*) \neq 0, \text{for all } x \in X,$$
$$\left\langle \sum_{i=1}^{p} \nabla_z h_i(x^*, z), \eta(x, x^*) \right\rangle \geqq -\rho(x, x^*) \left\| \theta(x, x^*) \right\|^2$$
$$\Rightarrow \quad \gamma(x, x^*) \sum_{i=1}^{p} \xi_i(x, x^*) \left[F_i(x) - F_i(x^*)\right](>) \geqq 0$$
$$\text{if } \alpha(x, x^*) \to 0 \text{ and } \beta(x, x^*) \to 0, \text{for all } x \in X.$$

The function F is said to be (strictly) $HA(\alpha, \beta, \gamma, \xi, \eta, \rho, h(\cdot, \cdot), \theta)$-V-pseudoinvex on X if it is (strictly) $HA(\alpha, \beta, \gamma, \xi, \eta, \rho, h(\cdot, \cdot), \theta)$-V-pseudoinvex at each point $x^* \in X$.

Definition 2.8 The function F is said to be (prestrictly) $(\alpha, \beta, \gamma, \xi, \eta, \rho, h(\cdot, \cdot), \theta)$-quasi-invex at $x^* \in X$ if there exist functions $\alpha : X \times X \to \mathbb{R}$, $\beta : X \times X \to \mathbb{R}$, $\gamma : X \times X \to \mathbb{R}_+$, $\xi_i : X \times X \to \mathbb{R}_+ \backslash \{0\}$, $i \in \underline{p}$, $\eta : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$ such that, for all $x \in X$,

$$\frac{1}{\alpha(x, x^*)} \gamma(x, x^*) \left(e^{\alpha(x,x^*) \sum_{i=1}^{p} \xi_i(x,x^*)[F_i(x)-F_i(x^*)]} - 1\right)(<) \leqq 0$$
$$\Rightarrow \quad \frac{1}{\beta(x, x^*)} \left\langle \sum_{i=1}^{p} \nabla_z h_i(x^*, z), e^{\beta(x,x^*)\eta(x,x^*)} - \mathbf{1} \right\rangle \leqq -\rho(x, x^*) \left\| \theta(x, x^*) \right\|^2$$
$$\text{if } \alpha(x, x^*) \neq 0 \text{ and } \beta(x, x^*) \neq 0, \text{for all } x \in X,$$
$$\frac{1}{\alpha(x, x^*)} \gamma(x, x^*) \left(e^{\alpha(x,x^*) \sum_{i=1}^{p} \xi_i(x,x^*)[F_i(x)-F_i(x^*)]} - 1\right)(<) \leqq 0$$
$$\Rightarrow \quad \left\langle \sum_{i=1}^{p} \nabla_z h_i(x^*, z), \eta(x, x^*) \right\rangle \leqq -\rho(x, x^*) \left\| \theta(x, x^*) \right\|^2$$
$$\text{if } \alpha(x, x^*) \neq 0 \text{ and } \beta(x, x^*) \to 0, \text{for all } x \in X,$$
$$\gamma(x, x^*) \sum_{i=1}^{p} \xi_i(x, x^*) \left[F_i(x) - F_i(x^*)\right](<) \leqq 0$$
$$\Rightarrow \quad \frac{1}{\beta(x, x^*)} \left\langle \sum_{i=1}^{p} \nabla_z h_i(x^*, z), e^{\beta(x,x^*)\eta(x,x^*)} - \mathbf{1} \right\rangle \leqq -\rho(x, x^*) \left\| \theta(x, x^*) \right\|^2$$
$$\text{if } \alpha(x, x^*) \to 0 \text{ and } \beta(x, x^*) \neq 0, \text{for all } x \in X,$$

$$\gamma(x,x^*)\sum_{i=1}^{p}\xi_i(x,x^*)\big[F_i(x)-F_i(x^*)\big](<)\leqq 0$$
$$\Rightarrow\quad \left\langle \sum_{i=1}^{p}\nabla_z h_i(x^*,z),\eta(x,x^*)\right\rangle \leqq -\rho(x,x^*)\big\|\theta(x,x^*)\big\|^2$$
$$\text{if } \alpha(x,x^*)\to 0 \text{ and } \beta(x,x^*)\to 0, \text{for all } x\in X.$$

We also noticed that, for the proofs of the sufficient efficiency theorems, sometimes it may be more appropriate to apply certain alternative but equivalent forms of the above definitions based on considering the contrapositive statements. For example, the exponential type $HA(\alpha,\beta,\gamma,\xi,\eta,\rho,h(\cdot,\cdot),\theta)$-V-quasiinvexity (when $\alpha(x,x^*)\neq 0$ and $\beta(x,x^*)\neq 0$, for all $x\in X$) can be defined in the following equivalent way:

The function F is an exponential type $HA(\alpha,\beta,\gamma,\xi,\eta,\rho,h(\cdot,\cdot),\theta)$-V-quasiinvex at $x^*\in X$ if there exist functions $\alpha: X\times X\to\mathbb{R}$, $\beta: X\times X\to\mathbb{R}$, $\gamma: X\times X\to\mathbb{R}_+$, $\xi_i: X\times X\to\mathbb{R}_+\backslash\{0\}$, $i\in\underline{p}$, $\eta: X\times X\to\mathbb{R}^n$, $\rho: X\times X\to\mathbb{R}$, and $\theta: X\times X\to\mathbb{R}^n$ such that, for all $x\in X$,

$$\frac{1}{\beta(x,x^*)}\left\langle \sum_{i=1}^{p}\nabla_z h_i(x^*,z),e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\right\rangle > -\rho(x,x^*)\big\|\theta(x,x^*)\big\|^2$$
$$\Rightarrow\quad \frac{1}{\alpha(x,x^*)}\gamma(x,x^*)\big(e^{\alpha(x,x^*)\sum_{i=1}^{p}\xi_i(x,x^*)[F_i(x)-F_i(x^*)]}-1\big)>0,$$

where $h:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is differentiable.

Example 2.1 In this example, we note that the exponential type invexity notion does not reduce to Definition 2.6. Furthermore to the best our knowledge, there is no such general notion is available in the current literature. The function F is said to be (strictly) $HA(\alpha,\beta,\gamma,\xi,\eta,\zeta,\rho,\theta)$-invex at $x^*\in X$ if there exist functions $\alpha: X\times X\to\mathbb{R}$, $\beta: X\times X\to\mathbb{R}$, $\gamma_i: X\times X\to\mathbb{R}_+$, $\xi_i: X\times X\to\mathbb{R}_+\backslash\{0\}$, $i\in\underline{p}$, $z\in\mathbb{R}^n$, $\eta,\zeta: X\times X\to\mathbb{R}^n$, $\rho_i: X\times X\to\mathbb{R}$, $i\in\underline{p}$, and $\theta: X\times X\to\mathbb{R}^n$ such that, for all $x\in X$ ($x\neq x^*$) and $i\in\underline{p}$,

$$\frac{1}{\alpha(x,x^*)}\gamma_i(x,x^*)\big(e^{\alpha(x,x^*)[F_i(x)-F_i(x^*)+\langle\nabla_z h_i(x^*,z),e^{\zeta(x,x^*)}\rangle]}-1\big)$$
$$(>)\geqq \frac{1}{\beta(x,x^*)}\big\langle\xi_i(x,x^*)\nabla_z h_i(x^*,z),e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\big\rangle$$
$$+\rho_i(x,x^*)\big\|\theta(x,x^*)\big\|^2\quad \text{if } \alpha(x,x^*)\neq 0 \text{ and } \beta(x,x^*)\neq 0, \text{for all } x\in X.$$

In the sequel, we shall also need a consistent notation for vector inequalities. For $a,b\in\mathbb{R}^m$, the following order notation will be used: $a\geqq b$ if and only if $a_i\geqq b_i$, for all $i\in\underline{m}$; $a\geq b$ if and only if $a_i\geqq b_i$, for all $i\in\underline{m}$, but $a\neq b$; $a>b$ if and only if $a_i>b_i$, for all $i\in\underline{m}$; and $a\not\geq b$ is the negation of $a\geq b$.

Consider the multiobjective problem

$$(\mathrm{P}^*)\quad \underset{x\in\mathbb{F}}{\text{Minimize}}\, F(x)=\big(F_1(x),\ldots,F_p(x)\big),$$

where F_i, $i\in\underline{p}$, are real-valued functions defined on $\mathbb{R}^n$.

An element $x^\circ\in\mathbb{F}$ is said to be an *efficient (Pareto optimal, nondominated, noninferior)* solution of (P^*) if there exists no $x\in\mathbb{F}$ such that $F(x)\leq F(x^\circ)$. In the area of multiobjective programming, there exist several versions of the notion of efficiency most of which are discussed in [4, 32, 49, 51]. However, throughout this paper, we shall deal exclusively with the efficient solutions of (P) in the sense defined above.

For the purpose of comparison with the sufficient efficiency conditions that will be proposed and discussed in this paper, we next recall a set of necessary efficiency conditions for (P).

Theorem 2.9 ([1]) *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*)$, for each $i \in \underline{p}$, let f_i and g_i be continuously differentiable at x^*, for each $j \in \underline{q}$, let the function $G_j(\cdot, t)$ be continuously differentiable at x^*, for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $H_k(\cdot, s)$ be continuously differentiable at x^*, for all $s \in S_k$. If x^* is an efficient solution of (P), if the generalized Guignard constraint qualification holds at x^*, and if for each $i_0 \in \underline{p}$, the set* $\operatorname{cone}(\{\nabla G_j(x^*, t) : t \in \hat{T}_j(x^*), j \in \underline{q}\} \cup \{\nabla f_i(x^*) - \lambda_i^* \nabla g_i(x^*) : i \in \underline{p}, i \neq i_0\}) + \operatorname{span}(\{\nabla H_k(x^*, s) : s \in S_k, k \in \underline{r}\})$ *is closed, then there exist $u^* \in U$ and integers ν_0^* and ν^*, with $0 \leqq \nu_0^* \leqq \nu^* \leqq n + 1$, such that there exist ν_0^* indices j_m, with $1 \leqq j_m \leqq q$, together with ν_0^* points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0^*}$, $\nu^* - \nu_0^*$ indices k_m, with $1 \leqq k_m \leqq r$, together with $\nu^* - \nu_0^*$ points $s^m \in S_{k_m}$ for $m \in \underline{\nu^*} \backslash \underline{\nu_0^*}$, and ν^* real numbers v_m^*, with $v_m^* > 0$ for $m \in \underline{\nu_0^*}$, with the property that*

$$\sum_{i=1}^{p} u_i^* \bigl[\nabla f_i(x^*) - \lambda_i^* \nabla g_i(x^*)\bigr] + \sum_{m=1}^{\nu_0^*} v_m^* \nabla G_{j_m}(x^*, t^m) + \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla H_{k_m}(x^*, s^m) = 0,$$

where $\operatorname{cone}(V)$ *is the conic hull of the set $V \subset \mathbb{R}^n$ (i.e., the smallest convex cone containing V),* $\operatorname{span}(V)$ *is the linear hull of V (i.e., the smallest subspace containing V), $\hat{T}_j(x^*) = \{t \in T_j : G_j(x^*, t) = 0\}$, $U = \{u \in \mathbb{R}^p : u > 0, \sum_{i=1}^{p} u_i = 1\}$, and $\underline{\nu^*} \backslash \underline{\nu_0^*}$ is the complement of the set $\underline{\nu_0^*}$ relative to the set $\underline{\nu^*}$.*

3 Sufficient efficiency conditions

In this section, we present several sets of sufficiency results in which various generalized exponential type $HA(\alpha, \beta, \gamma, \xi, \eta, \rho, h(\cdot,\cdot), \theta)$-V-invexity assumptions are imposed on certain vector functions whose components are the individual as well as some combinations of the problem functions.

Let the function $\mathcal{E}_i(\cdot, \lambda, u) : X \to \mathbb{R}$ be defined, for fixed λ and u, on X by

$$\mathcal{E}_i(z, \lambda, u) = u_i\bigl[f_i(z) - \lambda_i g_i(z)\bigr], \quad i \in \underline{p}.$$

Theorem 3.1 *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*)$, let the functions f_i, g_i, $i \in \underline{p}$, $G_j(\cdot, t)$, and $H_k(\cdot, s)$ be differentiable at x^*, for all $t \in T_j$ and $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$, and assume that there exist $u^* \in U$ and integers ν_0 and ν, with $0 \leqq \nu_0 \leqq \nu \leqq n + 1$, such that there exist ν_0 indices j_m, with $1 \leqq j_m \leqq q$, together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m, with $1 \leqq k_m \leqq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \underline{\nu} \backslash \underline{\nu_0}$, and ν real numbers v_m^*, with $v_m^* > 0$ for $m \in \underline{\nu_0}$, with the property that*

$$\sum_{i=1}^{p} u_i^* \bigl[\nabla_z h_i(x^*, z) - \lambda_i^* \nabla_z \kappa_i(x^*, z)\bigr] + \sum_{m=1}^{\nu_0} v_m^* \nabla_z \omega_{j_m}(x^*, t^m, z)$$
$$+ \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla_z \varpi_{k_m}(x^*, s^m, z) = 0. \tag{3.1}$$

Assume, furthermore, that either one of the following two sets of conditions holds:

(a) (i) f_i *is exponential type* $HA(\alpha,\beta,\bar{\gamma},\xi,\eta,\bar{\rho},h(\cdot,\cdot),\theta)$-*V-invex at* x^*, g_i *is exponential type* $HA(\alpha,\beta,\bar{\gamma},\xi,\eta,\bar{\rho},\kappa(\cdot,\cdot),\theta)$-*V-invex at* x^*, *and* $\bar{\gamma}(x,x^*)>0$, *for all* $x\in\mathbb{F}$;

(ii) $(v_1^*G_{j_1}(\cdot,t^1),\ldots,v_{\nu_0}^*G_{j_{\nu_0}}(\cdot,t^{\nu_0}))$ *is exponential type* $HA(\alpha,\beta,\hat{\gamma},\pi,\eta,\hat{\rho},\omega(\cdot,\cdot),\theta)$-*V-invex at* x^*;

(iii) $(v_{\nu_0+1}^*H_{k_{\nu_0+1}}(\cdot,s^{\nu_0+1}),\ldots,v_\nu^*H_{k_\nu}(\cdot,s^\nu))$ *is exponential type* $HA(\alpha,\beta,\breve{\gamma},\delta,\eta,\breve{\rho},\varpi(\cdot,\cdot),\theta)$-*V-invex at* x^*;

(iv) $\xi_i=\pi_k=\delta_l=\sigma$, *for all* $i\in\underline{p}$, $k\in\underline{\nu_0}$, *and* $l\in\underline{\nu}\backslash\underline{\nu_0}$;

(v) $\sum_{i=1}^p u_i^*\bar{\rho}_i(x,x^*)+\sum_{m=1}^{\nu_0}\hat{\rho}_m(x,x^*)+\sum_{m=\nu_0+1}^{\nu}\breve{\rho}_m(x,x^*)\geqq 0$, *for all* $x\in\mathbb{F}$;

(b) *the function* $(L_1(\cdot,u^*,v^*,\lambda^*,\bar{t},\bar{s}),\ldots,L_p(\cdot,u^*,v^*,\lambda^*,\bar{t},\bar{s}))$ *is exponential type* $HA(\alpha,\beta,\gamma,\xi,0,h(\cdot,\cdot),\kappa(\cdot,\cdot),\omega(\cdot,\cdot),\varpi(\cdot,\cdot),\theta)$-*V-pseudoinvex at* x^* *and* $\gamma(x,x^*)>0$, *for all* $x\in\mathbb{F}$, *where*

$$L_i\big(z,u^*,v^*,\lambda^*,\bar{t},\bar{s}\big) = u_i^*\Bigg[f_i(z)-\lambda_i^*g_i(z)+\sum_{m=1}^{\nu_0}v_m^*G_{j_m}\big(z,t^m\big)+\sum_{m=\nu_0+1}^{\nu}v_m^*H_{k_m}\big(z,s^m\big)\Bigg],\quad i\in\underline{p}.$$

Then x^* *is an efficient solution of* (P).

Proof (a): In view of our assumptions in (i)-(iv), we have

$$\frac{1}{\alpha(x,x^*)}\bar{\gamma}_i\big(x,x^*\big)\big(e^{\alpha(x,x^*)\{f_i(x)-\lambda_i^*g_i(x)-[f_i(x^*)-\lambda_i^*g_i(x^*)]\}}-1\big) \geqq \frac{1}{\beta(x,x^*)}\big\langle\sigma\big(x,x^*\big)\big[\nabla_z h_i\big(x^*,z\big)-\lambda_i^*\nabla_z\kappa_i\big(x^*,z\big)\big],e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\big\rangle + \bar{\rho}_i\big(x,x^*\big)\big\|\theta\big(x,x^*\big)\big\|^2,\quad i\in\underline{p},\tag{3.2}$$

$$\frac{1}{\alpha(x,x^*)}\hat{\gamma}_m\big(x,x^*\big)\big(e^{\alpha(x,x^*)[v_m^*G_{j_m}(x,t^m)-v_m^*G_{j_m}(x^*,t^m)]}-1\big) \geqq \frac{1}{\beta(x,x^*)}\big\langle\sigma\big(x,x^*\big)v_m^*\nabla_z\omega_{j_m}\big(x^*,t^m,z\big),e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\big\rangle + \hat{\rho}_m\big(x,x^*\big)\big\|\theta\big(x,x^*\big)\big\|^2,\quad m\in\underline{\nu_0},\tag{3.3}$$

$$\frac{1}{\alpha(x,x^*)}\breve{\gamma}_m\big(x,x^*\big)\big(e^{\alpha(x,x^*)[v_m^*H_{k_m}(x,s^m)-v_m^*H_{k_m}(x^*,s^m)]}-1\big) \geqq \frac{1}{\beta(x,x^*)}\big\langle\sigma\big(x,x^*\big)v_m^*\nabla_z\varpi_{k_m}\big(x^*,s^m,z\big),e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\big\rangle + \breve{\rho}_m\big(x,x^*\big)\big\|\theta\big(x,x^*\big)\big\|^2,\quad m\in\underline{\nu}\backslash\underline{\nu_0}.\tag{3.4}$$

Multiplying (3.2) by u_i^* and then summing over $i\in\underline{p}$, summing (3.3) over $m\in\underline{\nu_0}$, and summing (3.4) over $m\in\underline{\nu}\backslash\underline{\nu_0}$, and finally adding the resulting inequalities, we get

$$\frac{1}{\alpha(x,x^*)}\Bigg\{\sum_{i=1}^p u_i^*\bar{\gamma}_i\big(x,x^*\big)\big(e^{\alpha(x,x^*)\{f_i(x)-\lambda_i^*g_i(x)-[f_i(x^*)-\lambda_i^*g_i(x^*)]\}}-1\big) + \sum_{m=1}^{\nu_0}\hat{\gamma}_m\big(x,x^*\big)\big(e^{\alpha(x,x^*)[v_m^*G_{j_m}(x,t^m)-v_m^*G_{j_m}(x^*,t^m)]}-1\big)$$

$$+ \sum_{m=\nu_0+1}^{\nu} \breve{\gamma}_m(x,x^*)\big(e^{\alpha(x,x^*)[\nu_m^* H_{k_m}(x,s^m)-\nu_m^* H_{k_m}(x^*,s^m)]}-1\big)\Bigg\}$$

$$\geqq \frac{1}{\beta(x,x^*)}\sigma(x,x^*)\Bigg\langle \sum_{i=1}^{p} u_i^*\big[\nabla_z h_i(x^*,z)-\lambda_i^*\nabla_z \kappa_i(x^*,z)\big]+\sum_{m=1}^{\nu_0}\nu_m^*\nabla_z\omega_{j_m}(x^*,t^m,z)$$

$$+\sum_{m=\nu_0+1}^{\nu}\nu_m^*\nabla_z\varpi_{k_m}(x^*,s^m,z), e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\Bigg\rangle$$

$$+\Bigg[\sum_{i=1}^{p}u_i^*\bar{\rho}_i(x,x^*)+\sum_{m=1}^{\nu_0}\hat{\rho}_m(x,x^*)+\sum_{m=\nu_0+1}^{\nu}\breve{\rho}_m(x,x^*)\Bigg]\big\|\theta(x,x^*)\big\|^2.$$

Now using (3.1) and (v), and noticing that $\sigma(x,x^*)>0$, $\varphi(x^*)=\lambda^*$; $x,x^*\in\mathbb{F}$, and $G_{j_m}(x^*,t^m)=0$, for all $m\in\underline{\nu_0}$, the above inequality reduces to

$$\frac{1}{\alpha(x,x^*)}\sum_{i=1}^{p}u_i^*\bar{\gamma}_i(x,x^*)\big(e^{\alpha(x,x^*)[f_i(x)-\lambda_i^* g_i(x)]}-1\big)\geqq 0.$$

Since $\bar{\gamma}(x,x^*)>0$, even if we consider the both cases $\alpha(x,x^*)>0$ and $\alpha(x,x^*)<0$, it follows from the above inequality

$$\sum_{i=1}^{p}u_i^*\big[f_i(x)-\lambda_i^* g_i(x)\big]\geqq 0. \tag{3.5}$$

Therefore, we conclude that x^* is an efficient solution of (P).

(b): Let x be an arbitrary feasible solution of (P). From (3.1) we observe

$$\frac{1}{\beta(x,x^*)}\Bigg\langle \sum_{i=1}^{p}u_i^*\big[\nabla_z h_i(x^*,z)-\lambda_i^*\nabla_z\kappa_i(x^*,z)\big]+\sum_{m=1}^{\nu_0}\nu_m^*\nabla_z\omega_{j_m}(x^*,t^m,z)$$

$$+\sum_{m=\nu_0+1}^{\nu}\nu_m^*\nabla_z\varpi_{k_m}(x^*,s^m,z), e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\Bigg\rangle=0, \tag{3.6}$$

which in view of our $(\alpha,\beta,\gamma,\xi,0,h(\cdot,\cdot),\kappa(\cdot,\cdot),\omega(\cdot,\cdot),\varpi(\cdot,\cdot),\theta)$-pseudoinvexity assumption implies that

$$\frac{1}{\alpha(x,x^*)}\gamma(x,x^*)\big(e^{\alpha(x,x^*)\sum_{i=1}^{p}\xi_i(x,x^*)[L_i(x,u^*,\nu^*,\lambda^*,\bar{t},\bar{s})-L_i(x^*,u^*,\nu^*,\lambda^*,\bar{t},\bar{s})]}-1\big)\geqq 0.$$

We need to examine the two cases: $\alpha(x,x^*)>0$ and $\alpha(x,x^*)<0$. If we assume that $\alpha(x,x^*)>0$ and recall that $\gamma(x,x^*)>0$, then the above inequality becomes

$$e^{\alpha(x,x^*)\sum_{i=1}^{p}\xi_i(x,x^*)[L_i(x,u^*,\nu^*,\lambda^*,\bar{t},\bar{s})-L_i(x^*,u^*,\nu^*,\lambda^*,\bar{t},\bar{s})]}\geqq 1,$$

which implies that

$$\sum_{i=1}^{p}\xi_i(x,x^*)L_i(x,u^*,\nu^*,\lambda^*,\bar{t},\bar{s})\geqq\sum_{i=1}^{p}\xi_i(x,x^*)L_i(x^*,u^*,\nu^*,\lambda^*,\bar{t},\bar{s}).$$

Because $x^* \in \mathbb{F}$, $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, and $\lambda_i^* = \varphi_i(x^*)$, $i \in \underline{p}$, the right-hand side of the above inequality is equal to zero, and hence we have $L(x, u^*, v^*, \lambda^*, \bar{t}, \bar{s}) \geqq 0$. Next, as $x \in \mathbb{F}$, and $v_m^* > 0$, $m \in \underline{\nu_0}$, this inequality simplifies to

$$\sum_{i=1}^{p} u_i^* \xi_i(x, x^*) [f_i(x) - \lambda_i^* g_i(x)] \geqq 0. \tag{3.7}$$

Since $u^* > 0$ and $\xi_i(x, x^*) > 0$, $i \in \underline{p}$, the above inequality implies that

$$(f_1(x) - \lambda_1^* g_1(x), \dots, f_p(x) - \lambda_p^* g_p(x)) \nleq (0, \dots, 0),$$

which in turn implies that

$$\left(\frac{f_1(x)}{g_1(x)}, \dots, \frac{f_p(x)}{g_p(x)}\right) \nleq (\lambda_1^*, \dots, \lambda_p^*) = \varphi(x^*).$$

Since $x \in \mathbb{F}$ was arbitrary, we conclude from this inequality that x^* is an efficient solution of (P). On the other hand, we arrive at the same conclusion if we assume that $\alpha(x, x^*) < 0$. □

Remark We observe that the proof for solutions of Theorem 3.1 can be achieved using the method of contradictions as well.

Theorem 3.2 *Let $x^* \in \mathbb{F}$, $\lambda^* = \varphi(x^*)$, the functions f_i, g_i, $i \in \underline{p}$, $G_j(\cdot, t)$, and $H_k(\cdot, s)$ be differentiable at x^*, for all $t \in T_j$ and $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$, and assume that there exist $u^* \in U$ and integers ν_0 and ν, with $0 \leqq \nu_0 \leqq \nu \leqq n+1$, such that there exist ν_0 indices j_m, with $1 \leqq j_m \leqq q$, together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m, with $1 \leqq k_m \leqq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \underline{\nu} \backslash \underline{\nu_0}$, and ν real numbers v_m^*, with $v_m^* > 0$ for $m \in \underline{\nu_0}$, such that* (3.1) *holds.*

In addition, assume that any one of the following four sets of hypotheses is satisfied:

(a) (i) $(\mathcal{E}_1(\cdot, \lambda^*, u^*), \dots, \mathcal{E}_p(\cdot, \lambda^*, u^*))$ *is exponential type* $HA(\alpha, \beta, \bar{\gamma}, \xi, h(\cdot,\cdot), \kappa(\cdot,\cdot), \bar{\rho}, \eta, \theta)$*-V-pseudoinvex at* x^* *and* $\bar{\gamma}(x, x^*) > 0$, *for all* $x \in \mathbb{F}$;
 (ii) $(v_1^* G_{j_1}(\cdot, t^1), \dots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot, t^{\nu_0}))$ *is exponential type* $HA(\alpha, \beta, \hat{\gamma}, \pi, \omega(\cdot,\cdot), \hat{\rho}, \eta, \theta)$*-V-quasiinvex at* x^*;
 (iii) $(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \dots, v_\nu^* H_{k_\nu}(\cdot, s^\nu))$ *is exponential type* $HA(\alpha, \beta, \breve{\gamma}, \delta, \varpi(\cdot,\cdot), \breve{\rho}, \eta, \theta)$*-V-quasiinvex at* x^*;
 (iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geqq 0$, *for all* $x \in \mathbb{F}$;

(b) (i) $(\mathcal{E}_1(\cdot, \lambda^*, u^*), \dots, \mathcal{E}_p(\cdot, \lambda^*, u^*))$ *is exponential type prestrictly* $HA(\alpha, \beta, \bar{\gamma}, \xi, h(\cdot,\cdot), \kappa(\cdot,\cdot), \bar{\rho}, \eta, \theta)$*-quasiinvex at* x^* *and* $\bar{\gamma}(x, x^*) > 0$, *for all* $x \in \mathbb{F}$;
 (ii) $(v_1^* G_{j_1}(\cdot, t^1), \dots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot, t^{\nu_0}))$ *is exponential type* $HA(\alpha, \beta, \hat{\gamma}, \pi, \omega(\cdot,\cdot), \hat{\rho}, \eta, \theta)$*-V-quasiinvex at* x^*;
 (iii) $(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \dots, v_\nu^* H_{k_\nu}(\cdot, s^\nu))$ *is exponential type* $HA(\alpha, \beta, \breve{\gamma}, \delta, \varpi(\cdot,\cdot), \breve{\rho}, \eta, \theta)$*-V-quasiinvex at* x^*;
 (iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) > 0$, *for all* $x \in \mathbb{F}$;

(c) (i) $(\mathcal{E}_1(\cdot, \lambda^*, u^*), \dots, \mathcal{E}_p(\cdot, \lambda^*, u^*))$ *is prestrictly exponential type* $HA(\alpha, \beta, \bar{\gamma}, \xi, h(\cdot,\cdot), \kappa(\cdot,\cdot), \bar{\rho}, \eta, \theta)$*-V-quasiinvex at* x^* *and* $\bar{\gamma}(x, x^*) > 0$, *for all* $x \in \mathbb{F}$;

(ii) $(v_1^* G_{j_1}(\cdot, t^1), \ldots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot, t^{\nu_0}))$ *is strictly exponential type* $HA(\alpha, \beta, \hat{\gamma}, \pi, \omega(\cdot,\cdot), \hat{\rho}, \eta, \theta)$*-V-pseudoinvex at* x^*;

(iii) $(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \ldots, v_\nu^* H_{k_\nu}(\cdot, s^\nu))$ *is exponential type* $HA(\alpha, \beta, \breve{\gamma}, \delta, \varpi(\cdot,\cdot), \breve{\rho}, \eta, \theta)$*-V-quasiinvex at* x^*;

(iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geqq 0$, *for all* $x \in \mathbb{F}$;

(d) (i) $(\mathcal{E}_1(\cdot, \lambda^*, u^*), \ldots, \mathcal{E}_p(\cdot, \lambda^*, u^*))$ *is prestrictly exponential type* $HA(\alpha, \beta, \bar{\gamma}, \xi, h(\cdot,\cdot), \kappa(\cdot,\cdot), \bar{\rho}, \eta, \theta)$*-V-quasiinvex at* x^* *and* $\bar{\gamma}(x, x^*) > 0$, *for all* $x \in \mathbb{F}$;

(ii) $(v_1^* G_{j_1}(\cdot, t^1), \ldots, v_{\nu_0}^* G_{j_{\nu_0}}(\cdot, t^{\nu_0}))$ *is exponential type* $HA(\alpha, \beta, \hat{\gamma}, \pi, \omega(\cdot,\cdot), \hat{\rho}, \eta, \theta)$*-V-quasiinvex at* x^*;

(iii) $(v_{\nu_0+1}^* H_{k_{\nu_0+1}}(\cdot, s^{\nu_0+1}), \ldots, v_\nu^* H_{k_\nu}(\cdot, s^\nu))$ *is strictly exponential type* $HA(\alpha, \beta, \breve{\gamma}, \delta, \varpi(\cdot,\cdot), \breve{\rho}, \eta, \theta)$*-V-pseudoinvex at* x^*;

(iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geqq 0$, *for all* $x \in \mathbb{F}$.

Then x^* *is an efficient solution of* (P).

Proof (a): Let x be an arbitrary feasible solution to (P). Since $G_{j_m}(x, t^m) \leqq 0 = G_{j_m}(x^*, t^m)$, it follows that

$$\sum_{m=1}^{\nu_0} v_m^* \pi_m(x, x^*) G_{j_m}(x, t^m) \leqq \sum_{m=1}^{\nu_0} v_m^* \pi_m(x, x^*) G_{j_m}(x^*, t^m),$$

and so

$$\frac{1}{\alpha(x,x^*)} \hat{\gamma}(x, x^*) \bigl(e^{\alpha(x,x^*) \sum_{m=1}^{\nu_0} \pi_m(x,x^*)[v_m^* G_{j_m}(x,t^m) - v_m^* G_{j_m}(x^*,t^m)]} - 1\bigr) \leqq 0$$

by using $\alpha(x, x^*) \neq 0$ and $\hat{\gamma}(x, x^*) \geqq 0$. In light of (ii), this inequality implies that

$$\frac{1}{\beta(x,x^*)} \Biggl\langle \sum_{m=1}^{\nu_0} v_m^* \nabla \omega_{j_m}(x^*, t^m, z), e^{\beta(x,x^*)\eta(x,x^*)} - \mathbf{1} \Biggr\rangle \leqq -\hat{\rho}(x, x^*) \bigl\| \theta(x, x^*) \bigr\|^2. \tag{3.8}$$

Similarly, the assumptions in (iii) lead to the following inequality:

$$\frac{1}{\beta(x,x^*)} \Biggl\langle \sum_{m=\nu_0+1}^{\nu} v_m^* \nabla \varpi_{k_m}(x^*, s^m, z), e^{\beta(x,x^*)\eta(x,x^*)} - \mathbf{1} \Biggr\rangle \leqq -\breve{\rho}(x, x^*) \bigl\| \theta(x, x^*) \bigr\|^2. \tag{3.9}$$

Now combining (3.1), (3.8), and (3.9), and using (iv), we obtain

$$\frac{1}{\beta(x,x^*)} \Biggl\langle \sum_{i=1}^{p} u_i^* \bigl[\nabla_z h_i(x^*, z) - \lambda_i^* \nabla_z \kappa_i(x^*, z)\bigr], e^{\beta(x,x^*)\eta(x,x^*)} - \mathbf{1} \Biggr\rangle \geqq -\bar{\rho}(x, x^*) \bigl\| \theta(x, x^*) \bigr\|^2,$$

which in view of (i) implies that

$$\frac{1}{\alpha(x,x^*)} \bar{\gamma}(x, x^*) \bigl(e^{\alpha(x,x^*) \sum_{i=1}^{p} u_i^* \xi_i(x,x^*)\{f_i(x) - \lambda_i^* g_i(x) - [f_i(x^*) - \lambda_i^* g_i(x^*)]\}} - 1\bigr) \geqq 0.$$

Since $\bar{\gamma}(x, x^*) > 0$ and $\varphi(x^*) = \lambda^*$, this inequality implies that

$$\sum_{i=1}^{p} u_i^* \xi_i(x, x^*) \bigl[f_i(x) - \lambda_i^* g_i(x)\bigr] \geqq 0.$$

In the proof of Theorem 3.1, it was shown that this inequality leads to the conclusion that x^* is an efficient solution of (P).

(b)-(e): The proofs are similar to that of part (a). □

Now we briefly discuss some modifications of Theorems 3.1 and 3.2 based on replacing (3.1) with an inequality.

Theorem 3.3 *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*)$, let the functions f_i, g_i, $i \in \underline{p}$, $G_j(\cdot, t)$, and $H_k(\cdot, s)$ be differentiable at x^*, for all $t \in T_j$ and $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$, and assume that there exist $u^* \in U$ and integers ν_0 and ν, with $0 \leqq \nu_0 \leqq \nu \leqq n+1$, such that there exist ν_0 indices j_m, with $1 \leqq j_m \leqq q$, together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m, with $1 \leqq k_m \leqq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \underline{\nu}\backslash\underline{\nu_0}$, and ν real numbers v_m^*, with $v_m^* > 0$ for $m \in \underline{\nu_0}$, such that the following inequality holds*:

$$\begin{aligned}\frac{1}{\beta(x,x^*)}\Bigg\langle &\sum_{i=1}^{p} u_i^*\big[\nabla_z h_i(x^*,z) - \lambda_i^*\nabla_z \kappa_i(x^*,z)\big] + \sum_{m=1}^{\nu_0} v_m^*\nabla_z \omega_{j_m}(x^*,t^m,z) \\ &+ \sum_{m=\nu_0+1}^{\nu} v_m^*\nabla_z \varpi_{k_m}(x^*,s^m,z), e^{\beta(x,x^*)\eta(x,x^*)} - \mathbf{1}\Bigg\rangle \geq 0, \end{aligned} \tag{3.10}$$

where $\beta : X \times X \to \mathbb{R}$ and $z \in \mathbb{R}^n$. Furthermore, assume that either one of the two sets of conditions specified in Theorem 3.1 *is satisfied. Then x^* is an efficient solution of* (P).

We observe that any solution of (3.1) is also a solution of (3.10), but the converse may not be true.

4 Generalized sufficiency criteria

In this section, we discuss several families of sufficient efficiency results under various exponential type $HA(\alpha,\beta,\gamma,\xi,\eta,h(\cdot,\cdot),\kappa(\cdot,\cdot),\omega(\cdot,\cdot),\varpi(\cdot,\cdot),\rho,\theta)$-V-invexity hypotheses imposed on certain vector functions whose components are formed by considering different combinations of the problem functions. This is accomplished by applying a certain type of partitioning scheme. Let ν_0 and ν be integers, with $1 \leqq \nu_0 \leqq \nu \leqq n+1$, and let $\{J_0, J_1, \dots, J_M\}$ and $\{K_0, K_1, \dots, K_M\}$ be partitions of the sets $\underline{\nu_0}$ and $\underline{\nu}\backslash\underline{\nu_0}$, respectively; thus, $J_i \subseteq \underline{\nu_0}$ for each $i \in \underline{M} \cup \{0\}$, $J_i \cap J_j = \emptyset$ for each $i, j \in \underline{M} \cup \{0\}$ with $i \neq j$, and $\bigcup_{i=0}^{M} J_i = \underline{\nu_0}$. Obviously, similar properties hold for $\{K_0, K_1, \dots, K_M\}$. Moreover, if m_1 and m_2 are the numbers of the partitioning sets of $\underline{\nu_0}$ and $\underline{\nu}\backslash\underline{\nu_0}$, respectively, then $M = \max\{m_1, m_2\}$ and $J_i = \emptyset$ or $K_i = \emptyset$ for $i > \min\{m_1, m_2\}$.

In addition, we use the real-valued functions $\Phi_i(\cdot, u, v, \lambda, \bar{t}, \bar{s})$ and $\Lambda_\tau(\cdot, v, \bar{t}, \bar{s})$, $\tau \in \underline{M}$, defined, for fixed u, v, λ, $\bar{t} \equiv (t^1, t^2, \dots, t^{\nu_0})$, and $\bar{s} \equiv (s^{\nu_0+1}, s^{\nu_0+2}, \dots, s^{\nu})$, on X as follows:

$$\Phi_i(z,u,v,\lambda,\bar{t},\bar{s}) = u_i\Bigg[f_i(z) - \lambda_i g_i(z) + \sum_{m\in J_0} v_m G_{j_m}(z,t^m) + \sum_{m\in K_0} v_m H_{k_m}(z,s^m)\Bigg], \quad i \in \underline{p},$$

$$\Lambda_\tau(z,v,\bar{t},\bar{s}) = \sum_{m\in J_\tau} v_m G_{j_m}(z,t^m) + \sum_{m\in K_\tau} v_m H_{k_m}(z,s^m), \quad \tau \in \underline{M}.$$

Making use of the sets and functions defined above, we can now formulate our first collection of generalized sufficiency results for (P) as follows.

Theorem 4.1 *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*)$, let the functions f_i, g_i, $i \in \underline{p}$, $G_j(\cdot, t)$, and $H_k(\cdot, s)$ be differentiable at x^*, for all $t \in T_j$ and $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$, and assume that there exist $u^* \in U$ and integers ν_0 and ν, with $0 \leqq \nu_0 \leqq \nu \leqq n+1$, such that there exist ν_0 indices j_m, with $1 \leqq j_m \leqq q$, together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m, with $1 \leqq k_m \leqq r$, together with $\nu - \nu_0$ points $s^m \in S_{k_m}$, $m \in \underline{\nu} \backslash \underline{\nu_0}$, and ν real numbers v_m^*, with $v_m^* > 0$ for $m \in \underline{\nu_0}$, such that* (3.1) *holds. Assume, furthermore, that any one of the following three sets of hypotheses is satisfied*:

(a) (i) $(\Phi_1(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s}), \dots, \Phi_p(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s}))$ *is exponential type* $HA(\alpha, \beta, \bar{\gamma}, \xi, \eta, h(\cdot,\cdot), \kappa(\cdot,\cdot), \omega(\cdot,\cdot), \varpi(\cdot,\cdot), \bar{\rho}, \theta)$*-V-pseudoinvex at* x^* *and* $\bar{\gamma}(x, x^*) > 0$, *for all* $x \in \mathbb{F}$;

(ii) $(\Lambda_1(\cdot, v^*, \bar{t}, \bar{s}), \dots, \Lambda_M(\cdot, v^*, \bar{t}, \bar{s}))$ *is exponential type* $HA(\alpha, \beta, \hat{\gamma}, \pi, \eta, \omega(\cdot,\cdot), \varpi(\cdot,\cdot), \hat{\rho}, \theta)$*-V-quasiinvex at* x^*;

(iii) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) \geqq 0$;

(b) (i) $(\Phi_1(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s}), \dots, \Phi_p(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s}))$ *is prestrictly exponential type* $HA(\alpha, \beta, \bar{\gamma}, \xi, \eta, h(\cdot,\cdot), \kappa(\cdot,\cdot), \omega(\cdot,\cdot), \varpi(\cdot,\cdot), \bar{\rho}, \theta)$*-V-quasiinvex at* x^* *and* $\bar{\gamma}(x, x^*) > 0$, *for all* $x \in \mathbb{F}$;

(ii) $(\Lambda_1(\cdot, v^*, \bar{t}, \bar{s}), \dots, \Lambda_M(\cdot, v^*, \bar{t}, \bar{s}))$ *is exponential type* $HA(\alpha, \beta, \hat{\gamma}, \pi, \eta, \omega(\cdot,\cdot), \varpi(\cdot,\cdot), \hat{\rho}, \theta)$*-V-quasiinvex at* x^*;

(iii) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) > 0$;

(c) (i) $(\Phi_1(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s}), \dots, \Phi_p(\cdot, u^*, v^*, \lambda^*, \bar{t}, \bar{s}))$ *is prestrictly exponential type* $HA(\alpha, \beta, \bar{\gamma}, \xi, \eta, h(\cdot,\cdot), \kappa(\cdot,\cdot), \omega(\cdot,\cdot), \varpi(\cdot,\cdot), \bar{\rho}, \theta)$*-V-quasiinvex at* x^* *and* $\bar{\gamma}(x, x^*) > 0$, *for all* $x \in \mathbb{F}$;

(ii) $(\Lambda_1(\cdot, v^*, \bar{t}, \bar{s}), \dots, \Lambda_M(\cdot, v^*, \bar{t}, \bar{s}))$ *is strictly exponential type* $HA(\alpha, \beta, \hat{\gamma}, \pi, \eta, \omega(\cdot,\cdot), \varpi(\cdot,\cdot), \hat{\rho}, \theta)$*-V-pseudoinvex at* x^*;

(iii) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) \geqq 0$.

Then x^* is an efficient solution of (P).

Proof Let x be an arbitrary feasible solution of (P).

(a): It is clear that (3.1) can be expressed as follows:

$$\sum_{i=1}^{p} u_i^* \bigl[\nabla h_i(x^*, z) - \lambda_i^* \nabla \kappa_i(x^*, z)\bigr] + \sum_{m \in J_0} v_m^* \nabla \omega_{j_m}(x^*, t^m, z) + \sum_{m \in K_0} v_m^* \nabla \varpi_{k_m}(x^*, s^m, z)$$
$$+ \sum_{\tau=1}^{M} \Biggl[\sum_{m \in J_\tau} v_m^* \nabla \omega_{j_m}(x^*, t^m, z) + \sum_{m \in K_\tau} v_m^* \nabla \varpi_{k_m}(x^*, s^m, z)\Biggr] = 0. \tag{4.1}$$

Since $x, x^* \in \mathbb{F}$, $v_m^* > 0$, and $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$, it follows that

$$\begin{aligned}\sum_{\tau=1}^{M} \pi_\tau(x, x^*) \Lambda_\tau(x, v^*, \bar{t}, \bar{s}) &= \sum_{\tau=1}^{M} \pi_\tau(x, x^*) \Biggl[\sum_{m \in J_\tau} v_m^* G_{j_m}(x, t^m) + \sum_{m \in K_\tau} v_m^* H_{k_m}(x, s^m)\Biggr] \\ &\leqq 0 \\ &= \sum_{\tau=1}^{M} \pi_\tau(x, x^*) \Biggl[\sum_{m \in J_\tau} v_m^* G_{j_m}(x^*, t^m) + \sum_{m \in K_\tau} v_m^* H_{k_m}(x^*, s^m)\Biggr] \\ &= \sum_{\tau=1}^{M} \pi_\tau(x, x^*) \Lambda_\tau(x^*, v^*, \bar{t}, \bar{s}),\end{aligned}$$

and hence

$$\frac{1}{\alpha(x,x^*)}\hat{\gamma}(x,x^*)\left(e^{\alpha(x,x^*)\sum_{\tau=1}^{M}\pi_\tau(x,x^*)[\Lambda_\tau(x,v^*,\bar{t},\bar{s})-\Lambda_\tau(x^*,v^*,\bar{t},\bar{s})]}-1\right)\leqq 0,$$

which using (ii) implies that

$$\begin{aligned}&\frac{1}{\beta(x,x^*)}\left\langle\sum_{\tau=1}^{M}\left[\sum_{m\in J_\tau}v_m^*\nabla\omega_{j_m}(x^*,t^m,z)+\sum_{m\in K_\tau}v_m^*\nabla\varpi_{k_m}(x^*,s^m,z)\right],e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\right\rangle\\&\quad\leqq-\hat{\rho}(x,x^*)\left\|\theta(x,x^*)\right\|^2.\end{aligned}\tag{4.2}$$

Combining (4.1) and (4.2), and using (iii) we get

$$\begin{aligned}&\frac{1}{\beta(x,x^*)}\left\langle\sum_{i=1}^{p}u_i^*\left[\nabla f_i(x^*)-\lambda_i^*\nabla g_i(x^*)\right]+\sum_{m\in J_0}v_m^*\nabla G_{j_m}(x^*,t^m)\right.\\&\qquad\left.+\sum_{m\in K_0}v_m^*\nabla H_{k_m}(x^*,s^m),e^{\beta(x,x^*)\eta(x,x^*)}-\mathbf{1}\right\rangle\\&\quad\geqq\hat{\rho}(x,x^*)\left\|\theta(x,x^*)\right\|^2\geqq-\bar{\rho}(x,x^*)\left\|\theta(x,x^*)\right\|^2,\end{aligned}$$

which by virtue of (i) implies that

$$\frac{1}{\alpha(x,x^*)}\bar{\gamma}(x,x^*)\left(e^{\alpha(x,x^*)\sum_{i=1}^{p}\xi_i(x,x^*)[\Phi_i(x,u^*,v^*,\lambda^*,\bar{t},\bar{s})-\Phi_i(x^*,u^*,v^*,\lambda^*,\bar{t},\bar{s})]}-1\right)\geqq 0.$$

Since $\bar{\gamma}(x,x^*)>0$, this inequality implies that

$$\sum_{i=1}^{p}\xi_i(x,x^*)\Phi_i(x,u^*,v^*,\lambda^*,\bar{t},\bar{s})\geqq\sum_{i=1}^{p}\xi_i(x,x^*)\Phi_i(x^*,u^*,v^*,\lambda^*,\bar{t},\bar{s})]=0,$$

where the equality follows from the fact that $\lambda_i^*=\varphi_i(x^*)$, $i\in\underline{p}$, $t^m\in\hat{T}_{j_m}(x^*)$, and $x^*\in\mathbb{F}$. Because $x\in\mathbb{F}$ and $v_m^*>0$ for each $m\in\underline{v_0}$, this inequality further reduces to

$$\sum_{i=1}^{p}u_i^*\xi_i(x,x^*)\left[f_i(x)-\lambda_i^*g_i(x)\right]\geqq 0.$$

Now it follows that x^* is an efficient solution to (P). The rest of the proofs follow from part (a), and this concludes the proof. □

Next, we present the dual problem (DI) (which is new) to primal problem (P) based on the parametric efficiency conditions for (P) as an example of a semiinfinite multiobjective fractional programming dual problem.

Example 4.1 Consider the dual problem (DI) to (P) as follows:

(DI) Maximize $\lambda=(\lambda_1,\ldots,\lambda_p)$

subject to

$$\sum_{i=1}^{p} u_i\left[\nabla_z h_i(y,z) - \lambda\nabla_z\kappa_i(y,z)\right] + \sum_{m=1}^{\nu_0}\nu_m\nabla_z\omega_{j_m}\left(y,t^m,z\right) + \sum_{m=\nu_0+1}^{\nu}\nu_m\nabla_z\varpi_{k_m}\left(y,s^m,z\right) = 0, \tag{4.3}$$

$$\sum_{i=1}^{p} u_i\left[f_i(y) - \lambda_i g_i(y)\right] + \sum_{m=1}^{\nu_0}\nu_m G_{j_m}\left(y,t^m\right) + \sum_{m=\nu_0+1}^{\nu}\nu_m H_{k_m}\left(y,s^m\right) \geqq 0. \tag{4.4}$$

It can be shown that (DI) is a dual problem to (P) by applying higher order exponential type hybrid invexity assumptions. Let x and y be arbitrary feasible solutions to (P) and (DI), respectively. Assume that the function $L(\cdot,u,\nu,\lambda,\bar{t},\bar{s}) : X \to \mathbb{R}^p$ defined by

$$L(\zeta,u,\nu,\lambda) = \left(L_1(\zeta,u,\nu,\lambda,\bar{t},\bar{s}),\dots,L_p(\zeta,u,\nu,\lambda,\bar{t},\bar{s})\right)$$

is higher order exponential type hybrid $(\alpha,\beta,\gamma,\eta,h(\cdot,\cdot),\kappa(\cdot,\cdot),\omega(\cdot,\cdot,\cdot),\varpi(\cdot,\cdot,\cdot),\rho,\theta)$-pseudoinvex at y for $\gamma(x,y) > 0$, where

$$L_i(\zeta,u,\nu,\lambda,\bar{t},\bar{s}) = u_i\left[f_i(\zeta) - \lambda_i g_i(\zeta) + \sum_{m=1}^{\nu_0}\nu_m G_{j_m}\left(\zeta,t^m\right) + \sum_{m=\nu_0+1}^{\nu}\nu_m H_{k_m}\left(\zeta,s^m\right)\right], \quad i \in \underline{p}.$$

Then from the pseudoinvexity assumption and (4.4) it follows that

$$\frac{1}{\alpha(x,y)}\gamma(x,y)\left(e^{\alpha(x,y)\sum_{i=1}^{p}[L_i(x,u,\nu,\lambda,\bar{t},\bar{s})-L_i(y,u,\nu,\lambda,\bar{t},\bar{s})} - 1\right) \geqq 0.$$

If we assume that $\alpha(x,y) > 0$ (while we arrive at the same conclusion for $\alpha(x,y) < 0$) and $\gamma(x,y) > 0$, then we have

$$e^{\alpha(x,y)\sum_{i=1}^{p}[L_i(x,u,\nu,\lambda,\bar{t},\bar{s})-L_i(y,u,\nu,\lambda,\bar{t},\bar{s})} \geqq 1.$$

This implies

$$\sum_{i=1}^{p} L_i(x,u,\nu,\lambda,\bar{t},\bar{s}) \geqq \sum_{i=1}^{p}[L_i(y,u,\nu,\lambda,\bar{t},\bar{s}) \geqq 0.$$

Since $x \in \mathbb{F}$ and $\nu_m > 0$, $m \in \underline{\nu_0}$, the above inequality reduces to

$$\sum_{i=1}^{p} u_i\left[f_i(x) - \lambda_i g_i(x)\right] \geqq 0. \tag{4.5}$$

Since $u > 0$, $i \in \underline{p}$, it further follows that

$$\left(f_1(x) - \lambda_1 g_1(x),\dots,f_p(x) - \lambda_p g_p(x)\right) \nleq (0,\dots,0),$$

which in turn implies that

$$\varphi(x) = \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\right) \nleq (\lambda_1, \ldots, \lambda_p) = \lambda.$$

This results in $\varphi(x) \nleq \lambda$, that is, (DI) is a dual problem to (P).

Furthermore, the dual problem (DI) generalizes most of the duality models, especially in the context of semiinfinite multiobjective fractional programming problems.

5 Concluding remarks

In this communication we established several results based on sufficient efficiency conditions for achieving efficient solutions to semiinfinite multiobjective fractional programming problems under the exponential type $HA(\alpha, \beta, \gamma, \xi, \eta, h(\cdot,\cdot,\cdot), \rho, \theta)$-V-invexity hypotheses and generalized sufficiency criteria, based on certain partitioning schemes imposed on certain vector functions. The obtained results can further be applied/generalized to a wide range of problems on higher order invexities.

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The authors declare that they have no competing interests.

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