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A posteriori error estimates of mixed finite element solutions for fourth order parabolic control problems

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Abstract
In this paper, a fourth order quadratic parabolic optimal control problem is analyzed. The state and co-state are discretized by the order k Raviart-Thomas mixed finite element spaces, and the control is approximated by piecewise polynomials of order k ($k \geq 0$). At last, the results of *a posteriori* error estimates are given in Lemma 2.1 by using mixed elliptic reconstruction methods.

Keywords: optimal control problems; fourth parabolic equation; mixed finite element methods; elliptic reconstruction

1 Introduction

It is known that optimal control problems governed by partial differential equations (PDEs, for short) play a great role in modern science, technology, engineering and so on. There has been extensive theoretical research for finite element approximation of various optimal control problems (see, *e.g.*, [1–12]), and some scholars have been paying much attention to the mixed finite element methods for PDEs (see, *e.g.*, [13–21]). As a matter of fact, the fourth PDEs of this method is always a hot special topic. For example, in 1978 (see [22]), Brezzi and Raviart studied fourth order elliptic equations by mixed element methods. In [13], Brezzi and Fortin presented some results on the application of the mixed finite element methods to linear elliptic problems. In [23], Li developed mixed finite element methods for solving fourth-order elliptic and parabolic problems by using RBFs and gave similar error estimates as classical mixed finite element methods. Several recent works have been devoted to the analysis of this field for the error estimates, for example, Cao and Yang got the *a priori* error estimates using Ciarlet-Raviart mixed finite element methods for the fourth order control problems governed by the first bi-harmonic equation (see [24]). Hou studied a class of fourth order quadratic elliptic optimal control problems, where the state and co-state are approximated by the order k Raviart-Thomas mixed finite element spaces and the control variable is approximated by piecewise polynomials of order k ($k \geq 1$), and he derived *a posteriori* error estimates for both the control and the state approximations (see [25]). Although the error analysis for the finite element discretization of optimal control problems for the fourth order PDEs is discussed in many publications, there are only a few published results on this topic for parabolic problems. Therefore, we will study the error estimates using mixed finite element for the fourth order parabolic optimal control problems.

This paper is organized as follows. Firstly, we discuss the semi-discrete mixed finite element approximation for the fourth order parabolic optimal control problem in Section 2. Next, *a posteriori* error estimates of mixed finite element approximation for the control problem are given in Section 3. Finally, we analyze the conclusion and future work in Section 4.

2 Mixed methods for optimal control problem

In the paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by

$$\|v\|_{m,p}^p = \sum_{|\alpha|\leq m} \|D^\alpha v\|_{L^p(\Omega)}^p,$$

a semi-norm $|\cdot|_{m,p}$ given by

$$|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p.$$

For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$.

For the sake of simplicity, we take $V = H(\operatorname{div};\Omega) = \{v \in (L^2(\Omega))^2, \operatorname{div} v \in L^2(\Omega)\}$ and $W = L^2(\Omega)$, the Hilbert space V is defined by the following norm:

$$\|v\|_{H(\operatorname{div})} = \left(\|v\|_{0,\Omega}^2 + \|\operatorname{div} v\|_{0,\Omega}^2\right)^{\frac{1}{2}}.$$

In this paper, the model problem that we shall investigate is the following two-dimensional optimal control problem:

$$\min_{u\in U_{\mathrm{ad}}} \left\{ \frac{1}{2} \int_0^T \left(\|\Delta y\|^2 + \|\nabla y\|^2 + \|y - y_d\|^2 + \|u\|^2\right) dt \right\} \tag{2.1}$$

subject to the state equations

$$y_t(x,t) + \Delta^2 y(x,t) = f(x,t) + u(x,t), \quad x \in \Omega, t \in (0,T], \tag{2.2}$$

$$y(x,t) = \Delta y(x,t) = 0, \quad x \in \partial\Omega, t \in [0,T], \tag{2.3}$$

$$y(x,0) = y_0(x), \quad x \in \Omega, \tag{2.4}$$

$$\Delta y(x,0) = y_1(x), \quad x \in \Omega, \tag{2.5}$$

where the bounded open set $\Omega \subset \mathbb{R}^2$ is a convex polygon with the bounded $\partial\Omega$, $J = [0,T]$. y_d is continuously differentiable with respect to t; moreover, $f, y_d \in L^2(J;W)$. We let U_{ad} denote the admissible set of the control variable, which is defined by

$$U_{\mathrm{ad}} = \left\{ u(x,t) \in L^2(J;W) : \int_0^T \int_\Omega u(x,t) \geq 0, x \in \Omega, \forall t \in J \right\}. \tag{2.6}$$

We denote by $L^s(0,T;W^{m,q}(\Omega))$ the Banach space of all L^s integrable functions from $(0,T)$ into $W^{m,q}(\Omega)$ with the norm

$$\|v\|_{L^s(0,T;W^{m,q}(\Omega))} = \left(\int_0^T \|v\|^s_{W^{m,q}(\Omega)}\,dt\right)^{\frac{1}{s}}, \quad \text{for } s \in [1,\infty),$$

and the standard modification for $s = \infty$. Similarly, one can define the spaces $H^k(0,T;W^{m,q}(\Omega))$ and $C^k(0,T;W^{m,q}(\Omega))$.

Throughout this paper, $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$, the form is as follows:

$$(u,v) = \int_\Omega uv, \quad \forall (u,v) \in W \times W.$$

Let $\tilde{p} = -\nabla y$ and $\tilde{y} = -\Delta y$, then we can rewrite (2.1)-(2.5) as

$$\min_{u\in U_{\text{ad}}} \left\{ \frac{1}{2}\int_0^T \left(\|\tilde{y}\|^2 + \|\tilde{p}\|^2 + \|y - y_d\|^2 + \|u\|^2\right) dt \right\} \tag{2.7}$$

subject to

$$\tilde{p} = -\nabla y, \quad x \in \Omega, t \in J, \tag{2.8}$$

$$\operatorname{div} \tilde{p} = \tilde{y}, \quad x \in \Omega, t \in J, \tag{2.9}$$

$$p = -\nabla \tilde{y}, \quad x \in \Omega, t \in J, \tag{2.10}$$

$$y_t + \operatorname{div} p = f + u, \quad x \in \Omega, t \in J, \tag{2.11}$$

$$y(x,t) = \tilde{y}(x,t) = 0, \quad x \in \partial\Omega, t \in J, \tag{2.12}$$

$$y(x,0) = y_0(x), \quad x \in \Omega, \tag{2.13}$$

$$\tilde{y}(x,0) = y_1(x), \quad x \in \Omega. \tag{2.14}$$

Then a possible weak formula for the state equation reads: find $(p,\tilde{y},\tilde{p},y,u) \in (V \times W)^2 \times U_{\text{ad}}$ such that

$$\min_{u\in U_{\text{ad}}} \left\{ \frac{1}{2}\int_0^T \left(\|\tilde{y}\|^2 + \|\tilde{p}\|^2 + \|y - y_d\|^2 + \|u\|^2\right) dt \right\} \tag{2.15}$$

subject to

$$(\tilde{p},v) - (y,\operatorname{div} v) = 0, \quad \forall v \in V, t \in J, \tag{2.16}$$

$$(\operatorname{div} \tilde{p}, w) = (\tilde{y}, w), \quad \forall w \in W, t \in J, \tag{2.17}$$

$$(p,v) - (\tilde{y},\operatorname{div} v) = 0, \quad \forall v \in V, t \in J, \tag{2.18}$$

$$(y_t, w) + (\operatorname{div} p, w) = (f + u, w), \quad \forall w \in W, t \in J, \tag{2.19}$$

$$y(x,0) = y_0(x), \quad x \in \Omega, \tag{2.20}$$

$$\tilde{y}(x,0) = y_1(x), \quad x \in \Omega. \tag{2.21}$$

It is well known (see [26]) that the above control problem has a unique solution $(p,\tilde{y},\tilde{p},y,u) \in (V \times W)^2 \times U_{\text{ad}}$, and that $(p,\tilde{y},\tilde{p},y,u)$ is the solution of (2.16)-(2.21) if and only if there exists a co-state $(q,\tilde{z},\tilde{q},z) \in (V \times W)^2$ such that $(p,\tilde{y},\tilde{p},y,q,\tilde{z},\tilde{q},z,u)$ satisfies the following optimal conditions for $t \in J$:

$$(\tilde{p}, v) - (y, \operatorname{div} v), \quad \forall v \in V, t \in J, \tag{2.22}$$

$$(\operatorname{div} \tilde{p}, w) = (\tilde{y}, w), \quad \forall w \in W, t \in J, \tag{2.23}$$

$$(p, v) - (\tilde{y}, \operatorname{div} v) = 0, \quad \forall v \in V, t \in J, \tag{2.24}$$

$$(y_t, w) + (\operatorname{div} p, w) = (f + u, w), \quad \forall w \in W, t \in J, \tag{2.25}$$

$$y(x,0) = y_0(x), \quad x \in \Omega, \tag{2.26}$$

$$\tilde{y}(x,0) = y_1(x), \quad x \in \Omega, \tag{2.27}$$

$$(\tilde{q}, v) - (z, \operatorname{div} v) = 0, \quad \forall v \in V, t \in J, \tag{2.28}$$

$$(\operatorname{div} \tilde{q}, w) = (\tilde{z}, w) + (\tilde{y}, w), \quad \forall w \in W, t \in J, \tag{2.29}$$

$$(q, v) - (\tilde{z}, \operatorname{div} v) = -(\tilde{p}, v), \quad \forall v \in V, t \in J, \tag{2.30}$$

$$-(z_t, w) + (\operatorname{div} q, w) = (y - y_d, w), \quad \forall w \in W, t \in J, \tag{2.31}$$

$$z(x,T) = \tilde{z}(x,T) = 0, \quad x \in \Omega, \tag{2.32}$$

$$\int_0^T (u + z, \tilde{u} - u)\,dt \geq 0, \quad \forall \tilde{u} \in U_{\text{ad}}. \tag{2.33}$$

In order to derive our final aim, we now give the following important result (see [27]).

Lemma 2.1 [27] *Let* $(p,\tilde{y},\tilde{p},y,q,\tilde{z},\tilde{q},z,u)$ *be the solution of* (2.22)-(2.33), *then we have the relation*

$$u = \max\{0, \bar{z}\} - z,$$

where $\bar{z} = \frac{\int_0^T \int_\Omega z\,dx\,dt}{\int_0^T \int_\Omega 1\,dx\,dt}$ *denotes the integral average on* $\Omega \times J$ *of the function* z.

In the following, we will consider the semi-discrete finite element for the problem.

Let $\mathcal{T}^h$ denote a regular triangulation of the polygonal domain Ω, $\mathcal{T}^h = \{T_i\}$, here h is the maximum diameter of the element T_i in $\mathcal{T}^h$. Moreover, let e_h denote the set of element sides of the triangulation $\mathcal{T}^h$ with $E_h = \bigcup e_h$. Furthermore, let $V_h \times W_h \subset V \times W$ be the Raviart-Thomas space (see [28]) associated with the triangulations $\mathcal{T}^h$ of Ω. P_k denotes the space of polynomials of total degree at most k ($k \geq 0$). Let $V(T_i) = \{v \in P_k^2(T_i) + x \cdot P_k(T_i)\}$, $W(T_i) = P_k(T_i)$, and we define

$$W_h := \{w_h \in W : \forall T_i \in \mathcal{T}^h, w_h|_{T_i \in W(T_i)}\},$$

$$V_h := \{v_h \in V : \forall T_i \in \mathcal{T}^h, v_h|_{T_i \in V(T_i)}\},$$

$$K_h := L^2(J; W_h) \cap U_{\text{ad}}.$$

The mixed finite element discretization of (2.15)-(2.21) is rewritten as follows: find $(p_h, \tilde{y}_h, \tilde{p}_h, y_h, u_h) \in (L^2(J; V_h) \times L^2(J; W_h))^2 \times K_h$ such that

$$\min_{u_h \in K_h} \left\{ \frac{1}{2} \int_0^T \left(\|\tilde{y}\|^2 + \|\tilde{p}\|^2 + \|y - y_d\|^2 + \|u\|^2 \right) dt \right\}, \tag{2.34}$$

$$(\tilde{p}_h, v_h) - (y_h, \operatorname{div} v_h) = 0, \quad \forall v_h \in V_h, t \in J, \tag{2.35}$$

$$(\operatorname{div} \tilde{p}_h, w_h) = (\tilde{y}_h, w_h), \quad \forall w_h \in W_h, t \in J, \tag{2.36}$$

$$(p_h, v_h) - (\tilde{y}_h, \operatorname{div} v_h) = 0, \quad \forall v_h \in V_h, t \in J, \tag{2.37}$$

$$(y_{h,t}, w_h) + (\operatorname{div} p_h, w_h) = (f + u_h, w_h), \quad \forall w_h \in W_h, t \in J, \tag{2.38}$$

$$y_h(x,0) = y_0^h(x), \quad x \in \Omega, \tag{2.39}$$

$$\tilde{y}_h(x,0) = y_1^h(x), \quad x \in \Omega, \tag{2.40}$$

where $y_0^h(x) \in W_h$ and $y_1^h(x) \in W_h$ are two approximations of y_0 and y_1. The above optimal control problem again has a unique solution $(p_h, \tilde{y}_h, \tilde{p}_h, y_h, u_h)$, and that $(p_h, \tilde{y}_h, \tilde{p}_h, y_h, u_h)$ is the solution of (2.35)-(2.40) if and only if there is a co-state $(q_h, \tilde{z}_h, \tilde{q}_h, z_h) \in (L^2(J; V_h) \times L^2(J; W_h))^2$ such that $(p_h, \tilde{y}_h, \tilde{p}_h, y_h, q_h, \tilde{z}_h, \tilde{q}_h, z_h)$ satisfies the following optimality conditions:

$$(\tilde{p}_h, v_h) - (y_h, \operatorname{div} v_h) = 0, \quad \forall v_h \in V_h, t \in J, \tag{2.41}$$

$$(\operatorname{div} \tilde{p}_h, w_h) = (\tilde{y}_h, w_h), \quad \forall w_h \in W_h, t \in J, \tag{2.42}$$

$$(p_h, v_h) - (\tilde{y}_h, \operatorname{div} v_h) = 0, \quad \forall v_h \in V_h, t \in J, \tag{2.43}$$

$$(y_{h,t}, w_h) + (\operatorname{div} p_h, w_h) = (f + u_h, w_h), \quad \forall w_h \in W_h, t \in J, \tag{2.44}$$

$$y_h(x,0) = y_0^h(x), \quad x \in \Omega, \tag{2.45}$$

$$\tilde{y}_h(x,0) = y_1^h(x), \quad x \in \Omega, \tag{2.46}$$

$$(\tilde{q}_h, v_h) - (z_h, \operatorname{div} v_h) = 0, \quad \forall v \in V, t \in J, \tag{2.47}$$

$$(\operatorname{div} \tilde{q}_h, w_h) = (\tilde{z}_h, w_h) + (\tilde{y}_h, w_h), \quad \forall w_h \in W_h, t \in J, \tag{2.48}$$

$$(q_h, v_h) - (\tilde{z}_h, \operatorname{div} v_h) = -(\tilde{p}_h, v_h), \quad \forall v_h \in V_h, t \in J, \tag{2.49}$$

$$-(z_{h,t}, w_h) + (\operatorname{div} q_h, w_h) = (y_h - y_d, w_h), \quad \forall w_h \in W_h, t \in J, \tag{2.50}$$

$$z_h(x,T) = \tilde{z}_h(x,T) = 0, \quad x \in \Omega, \tag{2.51}$$

$$\int_0^T (u_h + z_h, \tilde{u}_h - u_h)\, dt \geq 0, \quad \forall \tilde{u}_h \in K_h. \tag{2.52}$$

Similar to Lemma 2.1, we can get the relationship between the control approximation u_h and the co-state approximation z_h, which satisfies

$$u_h = \max\{0, \bar{z}_h\} - z_h,$$

where $\bar{z}_h = \frac{\int_0^T \int_\Omega z_h\, dx\, dt}{\int_0^T \int_\Omega 1\, dx\, dt}$ denotes the integral average on $\Omega \times J$ of the function z_h.

In order to continue our analysis, we shall introduce some intermediate variables. For any control function $u_h \in K_h$, we define the state solution $p(u_h)$, $\tilde{y}(u_h)$, $\tilde{p}(u_h)$, $y(u_h)$, $q(u_h)$, $\tilde{z}(u_h)$, $\tilde{q}(u_h)$, $z(u_h)$ satisfying

$$\big(\tilde{p}(u_h), v\big) - \big(y(u_h), \operatorname{div} v\big) = 0, \quad \forall v \in V, t \in J, \tag{2.53}$$

$$\big(\operatorname{div} \tilde{p}(u_h), w\big) = \big(\tilde{y}(u_h), w\big), \quad \forall w \in W, t \in J, \tag{2.54}$$

$$\big(p(u_h), v\big) - \big(\tilde{y}(u_h), \operatorname{div} v\big) = 0, \quad \forall v \in V, t \in J, \tag{2.55}$$

$$\big(y_t(u_h), w\big) + \big(\operatorname{div} p(u_h), w\big) = (f + u_h, w), \quad \forall w \in W, t \in J, \tag{2.56}$$

$$y(u_h)(x, 0) = y_0(x), \quad x \in \Omega, \tag{2.57}$$

$$\tilde{y}(u_h)(x, 0) = y_1(x), \quad x \in \Omega, \tag{2.58}$$

$$\big(\tilde{q}(u_h), v\big) - \big(z(u_h), \operatorname{div} v\big) = 0, \quad \forall v \in V, t \in J, \tag{2.59}$$

$$\big(\operatorname{div} \tilde{q}(u_h), w\big) = \big(\tilde{z}(u_h), w\big) + \big(\tilde{y}(u_h), w\big), \quad \forall w \in W, t \in J, \tag{2.60}$$

$$\big(q(u_h), v\big) - \big(\tilde{z}(u_h), \operatorname{div} v\big) = -\big(\tilde{p}(u_h), v\big), \quad \forall v \in V, t \in J, \tag{2.61}$$

$$-\big(z_t(u_h), w\big) + \big(\operatorname{div} q(u_h), w\big) = \big(y(u_h) - y_d, w\big), \quad \forall w \in W, t \in J, \tag{2.62}$$

$$z(u_h)(x, T) = \tilde{z}(u_h)(x, T) = 0, \quad x \in \Omega, \tag{2.63}$$

where the exact solutions $y(u_h)$ and $z(u_h)$ satisfy the zero boundary condition.

Now we define the following errors:

$$e_y = y(u_h) - y_h, \qquad e_{\tilde{y}} = \tilde{y}(u_h) - \tilde{y}_h, \qquad e_p = p(u_h) - p_h, \qquad e_{\tilde{p}} = \tilde{p}(u_h) - \tilde{p}_h,$$

$$e_q = q(u_h) - q_h, \qquad e_{\tilde{q}} = \tilde{q}(u_h) - \tilde{q}_h, \qquad e_z = z(u_h) - z_h, \qquad e_{\tilde{z}} = \tilde{z}(u_h) - \tilde{z}_h.$$

Next, from (2.41)-(2.50), (2.53)-(2.62), we can get the error equations as follows:

$$(e_{\tilde{p}}, v) - (e_y, \operatorname{div} v) = -r_1(v), \quad \forall v \in V, t \in J, \tag{2.64}$$

$$(\operatorname{div} e_{\tilde{p}}, w) - (e_{\tilde{y}}, w) = -r_2(w), \quad \forall w \in W, t \in J, \tag{2.65}$$

$$(e_p, v) - (e_{\tilde{y}}, \operatorname{div} v) = -r_3(v), \quad \forall v \in V, t \in J, \tag{2.66}$$

$$(e_{y,t}, w) - (\operatorname{div} e_p, w) = -r_4(w), \quad \forall w \in W, t \in J, \tag{2.67}$$

$$(e_{\tilde{q}}, v) - (e_z, \operatorname{div} v) = -r_5(v), \quad \forall v \in V, t \in J, \tag{2.68}$$

$$(\operatorname{div} e_{\tilde{q}}, w) - (e_{\tilde{z}}, w) - (e_{\tilde{y}}, w) = -r_6(w), \quad \forall w \in W, t \in J, \tag{2.69}$$

$$(e_q, v) - (e_{\tilde{z}}, \operatorname{div} v) + (e_{\tilde{p}}, v) = -r_7(v), \quad \forall v \in V, t \in J, \tag{2.70}$$

$$-(e_{z,t}, w) + (\operatorname{div} e_q, w) = (e_y, w) - r_8(w), \quad \forall w \in W, t \in J, \tag{2.71}$$

where r_1-r_8 are given as follows:

$$r_1(v) := (\tilde{p}_h, v) - (y_h, \operatorname{div} v), \qquad r_2(w) := (\operatorname{div} \tilde{p}_h, w) - (\tilde{y}_h, w),$$

$$r_3(v) := (p_h, v) - (\tilde{y}_h, \operatorname{div} v), \qquad r_4(w) := (y_{h,t}, w) + (\operatorname{div} p_h, w) - (f + u_h, w),$$

$$r_5(v) := (\tilde{q}_h, v) - (z_h, \operatorname{div} v), \qquad r_6(w) := (\operatorname{div} \tilde{q}_h, w) - (\tilde{z}_h, w) - (\tilde{y}_h, w),$$

$$r_7(v) := (q_h, v) - (\tilde{z}_h, \operatorname{div} v) + (\tilde{p}_h, v), \qquad r_8(w) := (\operatorname{div} q_h, w) - (z_{h,t}, w) - (y_h - y_d, w).$$

Then, we introduce mixed elliptic reconstructions $\check{y}(t), \hat{y}(t), \check{z}(t), \hat{z}(t) \in H_0^1(\Omega)$ and $\check{p}(t), \hat{p}(t), \check{p}(t), \hat{p}(t) \in V$ of $\tilde{y}_h(t)$, $y_h(t)$, $\tilde{z}_h(t)$, $z_h(t)$ and $\tilde{p}_h(t)$, $p_h(t)$, $\tilde{q}_h(t)$, $q_h(t)$ for $t \in J$, respectively. For given functions $\tilde{y}_h$, y_h, $\tilde{z}_h$, z_h, $\tilde{p}_h$, p_h, $\tilde{q}_h$, q_h, let $\check{y}(t), \hat{y}(t), \check{z}(t), \hat{z}(t) \in H_0^1(\Omega)$

and $\check{p}(t), \hat{p}(t), \check{p}(t), \hat{p}(t) \in V$ satisfy the following equations:

$$(\check{p} - \tilde{p}_h, v) - (\hat{y} - y_h, \operatorname{div} v) = -r_1(v), \quad \forall v \in V, t \in J, \tag{2.72}$$

$$\bigl(\operatorname{div}(\check{p} - \tilde{p}_h), w\bigr) - (\check{y} - \tilde{y}_h, w) = -r_2(w), \quad \forall w \in W, t \in J, \tag{2.73}$$

$$(\hat{p} - p_h, v) - (\check{y} - \tilde{y}_h, \operatorname{div} v) = -r_3(v), \quad \forall v \in V, t \in J, \tag{2.74}$$

$$\bigl(\operatorname{div}(\hat{p} - p_h), w\bigr) = -r_4(w), \quad \forall w \in W, t \in J, \tag{2.75}$$

$$(\check{q} - \tilde{q}_h, v) - (\hat{z} - z_h, \operatorname{div} v) = -r_5(v), \quad \forall v \in V, t \in J, \tag{2.76}$$

$$\bigl(\operatorname{div}(\check{q} - \tilde{q}_h), w\bigr) - (\check{z} - \tilde{z}_h, w) - (\check{y} - \tilde{y}_h, w) = -r_6(w), \quad \forall w \in W, t \in J, \tag{2.77}$$

$$(\hat{q} - q_h, v) - (\check{z} - \tilde{z}_h, \operatorname{div} v) = -(\check{p} - \tilde{p}_h, v) - r_7(v), \quad \forall v \in V, t \in J, \tag{2.78}$$

$$\bigl(\operatorname{div}(\hat{q} - q_h), w\bigr) = (\hat{y} - y_h, w) - r_8(w), \quad \forall w \in W, t \in J. \tag{2.79}$$

We can derive $r_1(v_h) = r_3(v_h) = r_5(v_h) = r_7(v_h)$, $\forall v_h \in V_h$, and $r_2(w_h) = r_4(w_h) = r_6(w_h) = r_8(w_h)$, $\forall w_h \in W_h$, we note that p_h, $\tilde{y}_h$, $\tilde{p}_h$, y_h are standard mixed elliptic projections of $\hat{p}$, $\check{y}$, $\check{p}$, $\hat{y}$, respectively, q_h, $\tilde{z}_h$, $\tilde{q}_h$, z_h are nonstandard mixed elliptic projections of $\hat{q}$, $\check{z}$, $\check{q}$, $\hat{z}$.

We can define as follows by mixed elliptic reconstructions:

$$\begin{aligned}
e_y &= (\hat{y} - y_h) - \bigl(\hat{y} - y(u_h)\bigr) := \eta_y - \xi_y, & e_{\tilde{y}} &= (\check{y} - \tilde{y}_h) - \bigl(\check{y} - \tilde{y}(u_h)\bigr) := \eta_{\tilde{y}} - \xi_{\tilde{y}}, \\
e_p &= (\hat{p} - p_h) - \bigl(\hat{p} - p(u_h)\bigr) := \eta_p - \xi_p, & e_{\tilde{p}} &= (\check{p} - \tilde{p}_h) - \bigl(\check{p} - \tilde{p}(u_h)\bigr) := \eta_{\tilde{p}} - \xi_{\tilde{p}}, \\
e_q &= (\hat{q} - q_h) - \bigl(\hat{q} - q(u_h)\bigr) := \eta_q - \xi_q, & e_{\tilde{q}} &= (\check{q} - \tilde{q}_h) - \bigl(\check{q} - \tilde{q}(u_h)\bigr) := \eta_{\tilde{q}} - \xi_{\tilde{q}}, \\
e_z &= (\hat{z} - z_h) - \bigl(\hat{z} - z(u_h)\bigr) := \eta_z - \xi_z, & e_{\tilde{z}} &= (\check{z} - \tilde{z}_h) - \bigl(\check{z} - \tilde{z}(u_h)\bigr) := \eta_{\tilde{z}} - \xi_{\tilde{z}}.
\end{aligned}$$

Next, we will give some preliminary results about the intermediate solution. We define the standard L^2-orthogonal projection $P_h : W \to W_h$ which satisfies: for any $w \in W$,

$$(w - P_h w, w_h) = 0, \quad \forall w_h \in W_h, \tag{2.80}$$

$$\|P_h w - w\|_{0,q} \leq C\|w\|_{t,q} h^t, \quad 0 \leq t \leq k+1, \text{if } w \in W \cap W^{t,q}(\Omega), \tag{2.81}$$

$$\|P_h w - w\|_{-r} \leq C\|w\|_t h^{r+t}, \quad 0 \leq r, t \leq k+1, \text{if } w \in H^t(\Omega). \tag{2.82}$$

Recall the Fortin projection (see [22] and [28]) $\Pi_h : V \to V_h$, which satisfies: for any $v \in V$,

$$\bigl(\operatorname{div}(v - \Pi_h v), w_h\bigr) = 0, \quad \forall w_h \in W_h, \tag{2.83}$$

$$\|v - \Pi_h v\|_{0,q} \leq Ch^r \|v\|_{r,q}, \quad 1/q < r < k+1, \forall v \in V \cap \bigl(W^{r,q}(\Omega)\bigr)^2, \tag{2.84}$$

$$\bigl\|\operatorname{div}(v - \Pi_h v)\bigr\|_0 \leq Ch^r \|\operatorname{div} v\|_r, \quad 0 \leq r \leq k+1, \forall \operatorname{div} v \in H^r(\Omega). \tag{2.85}$$

We have the commuting properties

$$\operatorname{div} \circ \Pi_h = P_h \circ \operatorname{div} V \to W_h \quad \text{and} \quad \operatorname{div}(I - \Pi_h)V \perp W_h, \tag{2.86}$$

where I denotes an identity operator.

3 *A posteriori* error estimates

In this section, we give some lemmas to prepare for our results, and then we give *a posteriori* estimates for the mixed finite element approximation to the fourth order parabolic optimal control problems. Let $(p,\tilde{y},\tilde{p},y,q,\tilde{z},\tilde{q},z,u)$ and $(p_h,\tilde{y}_h,\tilde{p}_h,y_h,q_h,\tilde{z}_h,\tilde{q}_h,z_h,u_h)$ be the solutions of (2.22)-(2.33) and (2.41)-(2.52), respectively. Now we decompose the errors as the following forms:

$$
\begin{aligned}
&p-p_h = p-p(u_h)+p(u_h)-p_h := r_p+e_p,\\
&\tilde{y}-\tilde{y}_h = \tilde{y}-\tilde{y}(u_h)+\tilde{y}(u_h)-\tilde{y}_h := r_{\tilde{y}}+e_{\tilde{y}},\\
&\tilde{p}-\tilde{p}_h = \tilde{p}-\tilde{p}(u_h)+\tilde{p}(u_h)-\tilde{p}_h := r_{\tilde{p}}+e_{\tilde{p}},\\
&y-y_h = y-y(u_h)+y(u_h)-y_h := r_y+e_y,\\
&q-q_h = q-q(u_h)+q(u_h)-q_h := r_q+e_q,\\
&\tilde{z}-\tilde{z}_h = \tilde{z}-\tilde{z}(u_h)+\tilde{z}(u_h)-\tilde{z}_h := r_{\tilde{z}}+e_{\tilde{z}},\\
&\tilde{q}-\tilde{q}_h = \tilde{q}-\tilde{q}(u_h)+\tilde{q}(u_h)-\tilde{q}_h := r_{\tilde{q}}+e_{\tilde{q}},\\
&z-z_h = z-z(u_h)+z(u_h)-z_h := r_z+e_z.
\end{aligned}
$$

From (2.22)-(2.25), (2.53)-(2.56) and (2.59)-(2.62), we can get the error equations as follows:

$$(r_{\tilde{p}},v)-(r_y,\operatorname{div} v)=0,\quad \forall v\in V, \tag{3.1}$$

$$(\operatorname{div} r_{\tilde{p}},w)=(r_{\tilde{y}},w),\quad \forall w\in W, \tag{3.2}$$

$$(r_p,v)-(r_{\tilde{y}},\operatorname{div} v)=0,\quad \forall v\in V, \tag{3.3}$$

$$(r_{y,t},w)+(\operatorname{div} r_p,w)=(u-u_h,w),\quad \forall w\in W, \tag{3.4}$$

$$(r_{\tilde{q}},v)-(r_z,\operatorname{div} v)=0,\quad \forall v\in V, \tag{3.5}$$

$$(\operatorname{div} r_{\tilde{q}},w)=(r_{\tilde{z}},w)+(r_{\tilde{y}},w),\quad \forall w\in W, \tag{3.6}$$

$$(r_q,v)-(r_{\tilde{z}},\operatorname{div} v)=-(r_{\tilde{p}},v),\quad \forall v\in V, \tag{3.7}$$

$$-(r_{z,t},w)+(\operatorname{div} r_q,w)=(r_y,w),\quad \forall w\in W. \tag{3.8}$$

Lemma 3.1 *Let* $r_p, r_{\tilde{y}}, r_{\tilde{p}}, r_y, r_q, r_{\tilde{z}}, r_{\tilde{q}}, r_z$ *satisfy* (3.1)-(3.8), *then there exists a constant* $C>0$ *independent of h such that*

$$\|r_y\|_{L^\infty(J;W)}+\|r_{\tilde{y}}\|_{L^\infty(J;W)}+\|r_p\|_{L^2(J;W)}+\|r_{\tilde{p}}\|_{L^2(J;W)}\le C\|u-u_h\|_{L^2(J;W)}, \tag{3.9}$$

$$\|r_z\|_{L^\infty(J;W)}+\|r_{\tilde{z}}\|_{L^2(J;W)}+\|r_q\|_{L^2(J;W)}+\|r_{\tilde{q}}\|_{L^2(J;W)}\le C\|u-u_h\|_{L^2(J;W)}. \tag{3.10}$$

Proof Part I. Let $t=0$ and $v=r_{\tilde{p}}(0)$ in (3.1), since $r_y(0)=0$, so we find that $r_{\tilde{p}}=0$. Differentiate (3.1) with respect to t, and set $v=r_p$ as the test function, then we have

$$(r_{\tilde{p},t},r_p)=(r_{y,t},\operatorname{div} r_p). \tag{3.11}$$

Then, let $v=r_{\tilde{p},t}$ in (3.3), and from $\operatorname{div} r_{\tilde{p}}=r_{\tilde{y}}$, we get that

$$(r_p,r_{\tilde{p},t})=(r_{\tilde{y}},\operatorname{div} r_{\tilde{p},t})=(r_{\tilde{y}},r_{\tilde{y},t}). \tag{3.12}$$

Now we set $w = \operatorname{div} r_p$ in (3.4), we derive that

$$(r_{y,t}, \operatorname{div} r_p) + (\operatorname{div} r_p, \operatorname{div} r_p) = (u - u_h, \operatorname{div} r_p). \tag{3.13}$$

From (3.11)-(3.13), we have

$$(r_{\tilde{y}}, r_{\tilde{y},t}) + (\operatorname{div} r_p, \operatorname{div} r_p) = (u - u_h, \operatorname{div} r_p),$$

the above equation can be rewritten as follows:

$$\frac{1}{2}\frac{d}{dt}\|r_{\tilde{y}}\|^2 + \|\operatorname{div} r_p\|^2 \leq C\big(\|u - u_h\|^2 + \|\operatorname{div} r_p\|^2\big).$$

We integrate the above inequality from 0 to T and note $r_{\tilde{y}}(0) = 0$, then we get

$$\|r_{\tilde{y}}\|_{L^\infty(J;W)} + \|\operatorname{div} r_p\|_{L^2(J;W)} \leq C\|u - u_h\|_{L^2(J;W)}. \tag{3.14}$$

Set $v = r_p$ and $v = r_{\tilde{p}}$ as the test functions in (3.1) and (3.3), respectively, we have

$$(r_y, \operatorname{div} r_p) = (r_{\tilde{y}}, \operatorname{div} r_{\tilde{p}}) = (r_{\tilde{y}}, r_{\tilde{y}}). \tag{3.15}$$

Then, let $w = r_y$ in (3.4) and combine with (3.15), we derive that

$$(r_{y,t}, r_y) + (r_{\tilde{y}}, r_{\tilde{y}}) = (u - u_h, r_y),$$

which leads to

$$\frac{1}{2}\frac{d}{dt}\|r_y\|^2 + \|r_{\tilde{y}}\|^2 \leq C\big(\|u - u_h\|^2 + \|r_y\|^2\big).$$

On integrating the above inequality from 0 to t, using Gronwall's lemma and noting $r_y(0) = 0$, we can get

$$\|r_y\|_{L^\infty(J;W)} + \|r_{\tilde{y}}\|_{L^2(J;W)} \leq C\|u - u_h\|_{L^2(J;W)}. \tag{3.16}$$

In (3.13), let $v = r_p$, we have

$$(r_p, r_p) = (r_{\tilde{y}}, \operatorname{div} r_p).$$

Integrating the above equation with respect to time from 0 to T, combining with (3.14) and (3.16), we arrive at

$$\|r_p\|_{L^2(J;W)} \leq \|r_{\tilde{y}}\|_{L^2(J;W)} + \|\operatorname{div} r_p\|_{L^2(J;W)} \leq C\|u - u_h\|_{L^2(J;W)}. \tag{3.17}$$

Let $v = r_{\tilde{p}}$, so we have

$$(r_{\tilde{p}}, r_{\tilde{p}}) = (r_y, \operatorname{div} r_{\tilde{p}}) = (r_y, r_{\tilde{y}}),$$

we can get the following inequality from the above equation:

$$\|r_{\tilde{p}}\|^2 \leq C\big(\|r_y\|^2 + \|r_{\tilde{y}}\|^2\big).$$

Integrating the above inequality again from 0 to T and noticing (3.16), we can obtain

$$\|r_{\tilde{p}}\|_{L^2(J;W)} \leq C\|u - u_h\|_{L^2(J;W)}. \tag{3.18}$$

By (3.14), (3.16)-(3.18), we derive (3.9).

Part II. Choosing $v = r_q$ in (3.5) and $v = r_{\tilde{q}}$ in (3.7), respectively, we obtain

$$(\operatorname{div} r_q, r_z) = (r_{\tilde{z}}, \operatorname{div} r_{\tilde{q}}) - (r_{\tilde{p}}, r_{\tilde{q}}). \tag{3.19}$$

Let $v = r_{\tilde{q}}$ in (3.1), we have

$$(r_{\tilde{p}}, r_{\tilde{q}}) = (r_y, \operatorname{div} r_{\tilde{q}}). \tag{3.20}$$

Set $w = r_z$ in (3.8), we arrive at

$$-(r_{z,t}, r_z) + (\operatorname{div} r_q, r_z) = (r_y, r_z). \tag{3.21}$$

Now from (3.19)-(3.21) we can get that

$$-(r_{z,t}, r_z) + (r_{\tilde{z}}, \operatorname{div} r_{\tilde{q}}) - (r_y, \operatorname{div} r_{\tilde{q}}) = (r_y, r_z). \tag{3.22}$$

Then, set $w = \operatorname{div} r_{\tilde{q}}$ in (3.6) and combine with (3.22), we obtain

$$-(r_{z,t}, r_z) + (\operatorname{div} r_{\tilde{q}}, \operatorname{div} r_{\tilde{q}}) = (r_y, \operatorname{div} r_{\tilde{q}}) + (r_{\tilde{y}}, \operatorname{div} r_{\tilde{q}}) + (r_y, r_z),$$

which leads to

$$-\frac{1}{2}\frac{d}{dt}\|r_z\|^2 + \|\operatorname{div} r_{\tilde{q}}\|^2 \leq C\big(\|r_y\|^2 + \|r_{\tilde{y}}\|^2 + \|\operatorname{div} r_{\tilde{q}}\|^2 + \|r_z\|^2\big).$$

Integrating the above equation with respect to time from t to T, using Gronwall's lemma and (3.9), noting $r_z(T) = 0$, we can derive that

$$\|r_z\|^2_{L^\infty(J;W)} + \|\operatorname{div} r_{\tilde{q}}\|^2_{L^2(J;W)} \leq C\big(\|u - u_h\|^2_{L^2(J;W)}\big). \tag{3.23}$$

Let $v = r_{\tilde{q}}$ in (3.5), we get

$$(r_{\tilde{q}}, r_{\tilde{q}}) = (r_z, \operatorname{div} r_{\tilde{q}}).$$

We integrate the above equation from 0 to T and notice (3.14), then we can obtain that

$$\|r_{\tilde{q}}\|_{L^2(J;W)} \leq C\big(\|u - u_h\|_{L^2(J;W)}\big). \tag{3.24}$$

Let $w = r_{\tilde{z}}$ in (3.6), so we arrive at

$$(r_{\tilde{z}}, r_{\tilde{z}}) = (\operatorname{div} r_{\tilde{q}}, r_{\tilde{z}}) - (r_{\tilde{y}}, r_{\tilde{z}}),$$

the above equation can be rewritten as follows:

$$\|r_{\tilde{z}}\|^2 \le C\bigl(\|\operatorname{div} r_{\tilde{q}}\|^2 + \|r_{\tilde{z}}\|^2 + \|r_{\tilde{y}}\|^2\bigr).$$

We integrate the above equation from 0 to T, and from (3.9), (3.23), we have

$$\|r_{\tilde{z}}\|_{L^2(J;W)} \le C\bigl(\|u - u_h\|_{L^2(J;W)}\bigr). \tag{3.25}$$

Set $v = r_q$ in (3.7), it yields that

$$(r_q, r_q) = (r_{\tilde{z}}, \operatorname{div} r_q) - (r_{\tilde{p}}, r_q),$$

then we have

$$\|r_q\|^2 \le C\bigl(\|r_{\tilde{z}}\|^2 + \|r_{\tilde{p}}\|^2 + \|\operatorname{div} r_q\|^2\bigr). \tag{3.26}$$

Let $w = \operatorname{div} r_q$ in (3.8), we have

$$(\operatorname{div} r_q, \operatorname{div} r_q) = (r_{z,t}, \operatorname{div} r_q) + (r_y, \operatorname{div} r_q),$$

it also can be restated as

$$\|\operatorname{div} r_q\|^2 \le C\bigl(\|r_{z,t}\|^2 + \|\operatorname{div} r_q\|^2 + \|r_y\|^2\bigr),$$

where it leads to

$$\|\operatorname{div} r_q\|^2 \le C\bigl(\|r_{z,t}\|^2 + \|u - u_h\|^2\bigr). \tag{3.27}$$

Now set $w = r_{z,t}$ in (3.8), we get that

$$-(r_{z,t}, r_{z,t}) + (\operatorname{div} r_q, r_{z,t}) = (r_y, r_{z,t}),$$

we can rewrite the above equation as follows:

$$\begin{aligned}\|r_{z,t}\|^2_{L^2(J;W)} &\le C\bigl(\|r_y\|^2_{L^\infty(J;W)} + \|\operatorname{div} r_q\|^2_{L^2(J;W)}\bigr)\\ &\le C\bigl(\|u - u_h\|^2_{L^2(J;W)} + \|\operatorname{div} r_q\|^2_{L^2(J;W)}\bigr).\end{aligned} \tag{3.28}$$

From (3.26)-(3.28), we can obtain that

$$\|r_q\|^2_{L^2(J;W)} \le C\|u - u_h\|^2_{L^2(J;W)}. \tag{3.29}$$

Combining (3.29) with (3.23)-(3.25), we complete the result of (3.10). □

Lemma 3.2 *Let* $(p, \tilde{y}, \tilde{p}, y, q, \tilde{z}, \tilde{q}, z, u)$ *and* $(p_h, \tilde{y}_h, \tilde{p}_h, y_h, q_h, \tilde{z}_h, \tilde{q}_h, z_h, u_h)$ *be the solutions of* (2.22)-(2.33) *and* (2.41)-(2.52), *respectively. Suppose* $(u_h + z_h)|_{T_i} \in H^1(T_i)$ *and that there exists* $w \in K_h$ *such that*

$$\left|\int_0^T (u_h + z_h, w - u)\,dt\right| \le C\int_0^T \sum_{T_i} h_{T_i}|u_h + z_h|_{H^1(T_i)}\|u - u_h\|_{L^2(T_i)}\,dt.$$

Then we have

$$\|u - u_h\|_{L^2(J;W)} \le C\eta_u + C\big\|z_h - z(u_h)\big\|_{L^2(J;W)}, \tag{3.30}$$

where $\eta_u = (\int_0^T \sum_{T_i} h_{T_i}^2 |u_h + z_h|_{H^1(T_i)}^2\,dt)^{\frac{1}{2}}$.

Proof From (2.33), (2.52) and (3.10), we derive that

$$\begin{aligned}
\|u - u_h\|_{L^2(J;W)}^2 &= \int_0^T (u - u_h, u - u_h)\,dt \\
&= \int_0^T (u + z, u - u_h)\,dt + \int_0^T (z_h + u_h, u_h - u)\,dt \\
&\quad + \int_0^T \big(z_h - z(u_h), u - u_h\big)\,dt + \int_0^T \big(z(u_h) - z, u - u_h\big)\,dt \\
&\le \int_0^T (z_h + u_h, w - u)\,dt + \int_0^T \big(z_h - z(u_h), u - u_h\big)\,dt \\
&\quad + \int_0^T \big(z(u_h) - z, u - u_h\big)\,dt \\
&\le C(\delta)\eta_u^2 + \delta\|u - u_h\|_{L^2(J;W)}^2 + C\big\|z_h - z(u_h)\big\|_1^2 + \|r_z\|_{L^2(J;W)}^2 \\
&\le C(\delta)\eta_u^2 + \delta\|u - u_h\|_{L^2(J;W)}^2 + C\big\|z_h - z(u_h)\big\|_1^2,
\end{aligned} \tag{3.31}$$

where δ denotes an arbitrary small positive number, $C(\delta)$ is dependent on δ^{-1}. By using (3.31), we can easily obtain (3.30). □

Lemma 3.3 *Let* $\check{y}(t)$, $\hat{y}(t)$, $\check{z}(t)$, $\hat{z}(t)$, $\check{p}(t)$, $\hat{p}(t)$, $\check{q}(t)$, $\hat{q}(t)$ *satisfy* (2.72)-(2.79). *Then we can derive the following properties*:

$$\check{p} = -\nabla\hat{y}, \qquad \check{p} = \check{y}, \qquad \hat{p} = -\nabla\check{y}, \qquad \check{q} = -\nabla\hat{z}, \qquad \check{q} = \check{z} + \check{y}, \qquad \hat{q} + \check{p} = -\nabla\check{z}.$$

Using (2.72)-(2.79) in (2.64)-(2.71), we obtain the following error equations:

$$(\xi_{\tilde{p}}, v) - (\xi_y, \operatorname{div} v) = 0, \quad \forall v \in V, t \in J, \tag{3.32}$$

$$(\operatorname{div}\xi_{\tilde{p}}, w) - (\xi_{\tilde{y}}, w) = 0, \quad \forall w \in W, t \in J, \tag{3.33}$$

$$(\xi_p, v) - (\xi_{\tilde{y}}, \operatorname{div} v) = 0, \quad \forall v \in V, t \in J, \tag{3.34}$$

$$(\xi_{y,t}, w) + (\operatorname{div}\xi_p, w) = (\eta_{y,t}, w), \quad \forall w \in W, t \in J, \tag{3.35}$$

$$(\xi_{\tilde{q}}, v) - (\xi_z, \operatorname{div} v) = 0, \quad \forall v \in V, t \in J, \tag{3.36}$$

$$(\operatorname{div}\xi_{\tilde{q}}, w) - (\xi_{\tilde{z}}, w) - (\xi_{\tilde{y}}, w) = 0, \quad \forall w \in W, t \in J, \tag{3.37}$$

$$(\xi_q, v) - (\xi_{\tilde{z}}, \operatorname{div} v) = -(\xi_{\tilde{p}}, v), \quad \forall v \in V, t \in J, \tag{3.38}$$

$$-(\xi_{z,t}, w) + (\operatorname{div}\xi_q, w) = (\xi_y, w) + (\eta_{z,t}, w), \quad \forall w \in W, t \in J. \tag{3.39}$$

Lemma 3.4 *Let ξ_y, $\xi_{\tilde{y}}$, ξ_p, $\xi_{\tilde{p}}$, ξ_z, $\xi_{\tilde{z}}$, ξ_q, $\xi_{\tilde{q}}$ satisfy* (3.32)-(3.39). *Then we have the error estimates as follows*:

$$\|\xi_y\|_{L^\infty(J;W)} + \|\xi_{\tilde{y}}\|_{L^2(J;W)} + \|\xi_{\tilde{p}}\|_{L^2(J;W)} \\ \le C\big(\|\eta_y(0)\| + \|y_0 - y_0^h\| + \|\eta_{y,t}\|_{L^2(J;W)}\big), \tag{3.40}$$

$$\|\xi_p\|_{L^2(J;W)} \le C\big(\|\eta_y(0)\| + \|y_0 - y_0^h\| + \|\eta_{\tilde{y}}(0)\| + \|y_1 - y_1^h\| + \|\eta_{y,t}\|_{L^2(J;W)}\big), \tag{3.41}$$

$$\|\xi_{\tilde{y}}\|_{L^\infty(J;W)} \le C\big(\|\eta_{\tilde{y}}(0)\| + \|y_1 - y_1^h\| + \|\eta_{y,t}\|_{L^2(J;W)}\big), \tag{3.42}$$

$$\|\xi_z\|_{L^\infty(J;W)} + \|\xi_{\tilde{z}}\|_{L^2(J;W)} + \|\xi_q\|_{L^2(J;W)} + \|\xi_{\tilde{q}}\|_{L^2(J;W)} \\ \le C\big(\|\eta_y(0)\| + \|y_0 - y_0^h\| + \|\eta_{y,t}\|_{L^2(J;W)} + \|\eta_z(T)\|\big). \tag{3.43}$$

Proof First of all, we differentiate equation (3.32) with respect to t and derive

$$(\xi_{\tilde{p},t}, v) - (\xi_{y,t}, \operatorname{div} v) = 0, \quad \forall v \in V, t \in J. \tag{3.44}$$

Set $v = \xi_{\tilde{p},t}$ in (3.34) as the test function, and from $\operatorname{div}\xi_{\tilde{p}} = \xi_{\tilde{y}}$, we obtain

$$(\xi_p, \xi_{\tilde{p},t}) = (\xi_{\tilde{y}}, \operatorname{div}\xi_{\tilde{p},t}) = (\xi_{\tilde{y}}, \xi_{\tilde{y},t}). \tag{3.45}$$

Choose $w = \operatorname{div}\xi_p$ in (3.35) as the test function, we have

$$(\xi_{y,t}, \operatorname{div}\xi_p) + (\operatorname{div}\xi_p, \operatorname{div}\xi_p) = (\eta_{y,t}, \operatorname{div}\xi_p). \tag{3.46}$$

From (3.44)-(3.46), we derive

$$(\xi_{\tilde{y}}, \xi_{\tilde{y},t}) + (\operatorname{div}\xi_p, \operatorname{div}\xi_p) = (\eta_{y,t}, \operatorname{div}\xi_p),$$

it can also be read as

$$\frac{1}{2}\frac{d}{dt}\|\xi_{\tilde{y}}\|^2 + \|\operatorname{div}\xi_p\|^2 \le C\big(\|\eta_{y,t}\|^2 + \|\operatorname{div}\xi_p\|^2\big).$$

Integrating the above equation with respect to time from 0 to t, we have

$$\|\xi_{\tilde{y}}\|_{L^\infty(J;W)} + \|\operatorname{div}\xi_p\|_{L^2(J;W)} \le C\big(\|\xi_{\tilde{y}}(0)\| + \|\eta_{y,t}\|_{L^2(J;W)}\big). \tag{3.47}$$

Set $v = \xi_p$ and $v = \xi_{\tilde{p}}$ as the test functions in (3.32) and (3.34), respectively, and note that $\operatorname{div}\xi_{\tilde{p}} = \xi_{\tilde{y}}$, we have

$$(\xi_y, \operatorname{div}\xi_p) = (\xi_{\tilde{y}}, \operatorname{div}\xi_{\tilde{p}}) = (\xi_{\tilde{y}}, \xi_{\tilde{y}}). \tag{3.48}$$

Choose $w = \xi_y$ in (3.35), by using (3.48), we obtain

$$(\xi_{y,t}, \xi_y) + (\xi_{\tilde{y}}, \xi_{\tilde{y}}) = (\eta_{y,t}, \xi_y),$$

it can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\|\xi_y\|^2 + \|\xi_{\tilde{y}}\|^2 \le C\bigl(\|\eta_{y,t}\|^2 + \|\xi_y\|^2\bigr).$$

On integrating the above inequality with respect from 0 to t and using Gronwall's lemma, it reduces to

$$\|\xi_y\|_{L^\infty(J;W)} + \|\xi_{\tilde{y}}\|_{L^2(J;W)} \le C\bigl(\|\eta_{y,t}\|_{L^2(J;W)} + \bigl\|\xi_y(0)\bigr\|\bigr). \tag{3.49}$$

Let $v = \xi_p$ in (3.34), we have

$$(\xi_p, \xi_p) = (\xi_{\tilde{y}}, \operatorname{div} \xi_p),$$

integrate it from 0 to T, and from (3.47)-(3.49), we get

$$\begin{aligned}\|\xi_p\|^2_{L^2(J;W)} &\le \|\xi_{\tilde{y}}\|^2_{L^2(J;W)} + \|\operatorname{div} \xi_p\|^2_{L^2(J;W)} \\ &\le C\bigl(\bigl\|\xi_{\tilde{y}}(0)\bigr\| + \bigl\|\xi_y(0)\bigr\| + \|\eta_{y,t}\|_{L^2(J;W)}\bigr)^2,\end{aligned}$$

it also means that

$$\|\xi_p\|_{L^2(J;W)} \le C\bigl(\bigl\|\xi_{\tilde{y}}(0)\bigr\| + \bigl\|\xi_y(0)\bigr\| + \|\eta_{y,t}\|_{L^2(J;W)}\bigr). \tag{3.50}$$

Choose $v = \xi_{\tilde{p}}$ in (3.32), we derive

$$(\xi_{\tilde{p}}, \xi_{\tilde{p}}) = (\xi_y, \operatorname{div} \xi_{\tilde{p}}) = (\xi_y, \xi_{\tilde{y}}),$$

which leads to

$$\|\xi_{\tilde{p}}\|^2 \le C\bigl(\|\xi_y\|^2 + \|\xi_{\tilde{y}}\|^2\bigr).$$

Integrate the above inequality from 0 to T, using (3.49), we can see that

$$\|\xi_{\tilde{p}}\|_{L^2(J;W)} \le C\bigl(\bigl\|\xi_y(0)\bigr\| + \|\eta_{y,t}\|_{L^2(J;W)}\bigr), \tag{3.51}$$

and notice that

$$\bigl\|\xi_y(0)\bigr\| \le C\bigl(\bigl\|\eta_y(0)\bigr\| + \bigl\|y_0 - y_0^h\bigr\|\bigr), \qquad \bigl\|\xi_{\tilde{y}}(0)\bigr\| \le C\bigl(\bigl\|\eta_{\tilde{y}}(0)\bigr\| + \bigl\|y_1 - y_1^h\bigr\|\bigr). \tag{3.52}$$

From (3.47) and (3.49)-(3.51), then (3.40)-(3.42) is proved.

Choose $v = \xi_q$ and $v = \xi_{\tilde{q}}$ as the test functions in (3.36) and (3.38), respectively, we get

$$(\operatorname{div} \xi_q, \xi_z) = (\xi_{\tilde{z}}, \operatorname{div} \xi_{\tilde{q}}) - (\xi_{\tilde{p}}, \xi_{\tilde{q}}). \tag{3.53}$$

Let $v = \xi_{\tilde{q}}$ in (3.32), we have

$$(\xi_{\tilde{p}}, \xi_{\tilde{q}}) = (\xi_y, \operatorname{div} \xi_{\tilde{q}}). \tag{3.54}$$

Set $w = \xi_z$ in (3.39), we obtain

$$-(\xi_{z,t}, \xi_z) + (\operatorname{div} \xi_q, \xi_z) = (\xi_y, \xi_z) + (\eta_{z,t}, \xi_z). \tag{3.55}$$

From (3.53)-(3.55), we derive

$$-(\xi_{z,t}, \xi_z) + (\xi_{\tilde{z}}, \operatorname{div} \xi_{\tilde{q}}) - (\xi_y, \operatorname{div} \xi_{\tilde{q}}) = (\xi_y, \xi_z) + (\eta_{z,t}, \xi_z). \tag{3.56}$$

Set $w = \operatorname{div} \xi_{\tilde{q}}$ in (3.37) and combine with (3.56), we can find that

$$-(\xi_{z,t}, \xi_z) + (\operatorname{div} \xi_{\tilde{q}}, \operatorname{div} \xi_{\tilde{q}}) = (\xi_y, \operatorname{div} \xi_{\tilde{q}}) + (\xi_{\tilde{y}}, \operatorname{div} \xi_{\tilde{q}}) + (\xi_y, \xi_z) + (\eta_{z,t}, \operatorname{div} \xi_{\tilde{q}}),$$

the above equality is equivalent to

$$-\frac{1}{2}\frac{d}{dt}\|\xi_z\|^2 + \|\operatorname{div} \xi_{\tilde{q}}\|^2 \le C\left(\|\xi_y\|^2 + \|\xi_{\tilde{y}}\|^2 + \|\operatorname{div} \xi_{\tilde{q}}\|^2 + \|\xi_z\|^2 + \|\eta_{y,t}\|^2\right).$$

Integrating this inequality from t to T and using Gronwall's lemma, we have

$$\begin{aligned} &\|\xi_z\|^2_{L^\infty(J;W)} + \|\operatorname{div} \xi_{\tilde{q}}\|^2_{L^2(J;W)} \\ &\quad \le C\left(\|\xi_y\|^2_{L^2(J;W)} + \|\xi_{\tilde{y}}\|^2_{L^2(J;W)} + \|\eta_{y,t}\|^2_{L^2(J;W)} + \left\|\xi_z(T)\right\|^2\right) \\ &\quad \le C\left(\left\|\eta_y(0)\right\|^2 + \left\|y_0 - y_0^h\right\|^2 + \|\eta_{y,t}\|^2_{L^2(J;W)} + \left\|\xi_z(T)\right\|^2\right). \end{aligned} \tag{3.57}$$

Choose $v = \xi_{\tilde{q}}$ in (3.36), we get

$$(\xi_{\tilde{q}}, \xi_{\tilde{q}}) = (\xi_z, \operatorname{div} \xi_{\tilde{q}}),$$

integrating the two sides from 0 to T and using (3.57), we obtain

$$\|\xi_{\tilde{q}}\|^2_{L^2(J;W)} \le C\left(\left\|\eta_y(0)\right\|^2 + \left\|y_0 - y_0^h\right\|^2 + \|\eta_{y,t}\|^2_{L^2(J;W)} + \left\|\xi_z(T)\right\|^2\right). \tag{3.58}$$

Let $w = \xi_{\tilde{z}}$ in (3.37), we derive

$$(\xi_{\tilde{z}}, \xi_{\tilde{z}}) = (\operatorname{div} \xi_{\tilde{q}}, \tilde{z}) - (\xi_{\tilde{y}}, \xi_{\tilde{z}}),$$

namely,

$$\|\xi_{\tilde{z}}\|^2 \le C\left(\|\operatorname{div} \xi_{\tilde{q}}\|^2 + \|\xi_{\tilde{z}}\|^2 + \|\xi_{\tilde{y}}\|^2\right).$$

Integrating the two sides from 0 to T again and using (3.40) and (3.57), we get

$$\|\xi_{\tilde{z}}\|_{L^2(J;W)} \le C\left(\left\|\eta_y(0)\right\| + \left\|y_0 - y_0^h\right\| + \|\eta_{y,t}\|_{L^2(J;W)} + \left\|\xi_z(T)\right\|\right). \tag{3.59}$$

Let $v = \xi_q$ in (3.38), we get

$$(\xi_q, \xi_q) = (\xi_{\tilde{z}}, \operatorname{div} \xi_q) - (\xi_{\tilde{p}}, \xi_q),$$

it can be read as

$$\|\xi_q\|^2 \le C\big(\|\xi_{\tilde{z}}\|^2 + \|\xi_{\tilde{p}}\|^2 + \|\operatorname{div} \xi_q\|^2\big). \tag{3.60}$$

Set $w = \operatorname{div} \xi_q$ in (3.39), we can see that

$$(\operatorname{div} \xi_q, \operatorname{div} \xi_q) = (\xi_{z,t}, \operatorname{div} \xi_q) + (\xi_y, \operatorname{div} \xi_q) + (\eta_{z,t}, \operatorname{div} \xi_q),$$

which equals

$$\|\operatorname{div} \xi_q\|^2 \le C\big(\|\xi_{z,t}\|^2 + \|\xi_y\|^2 + \|\eta_{y,t}\|^2\big). \tag{3.61}$$

Choose $w = \xi_{z,t}$ in (3.39), we have

$$-(\xi_{z,t}, \xi_{z,t}) + (\operatorname{div} \xi_q, \xi_{z,t}) = (\xi_y, \xi_{z,t}) + (\eta_{z,t}, \xi_{z,t}).$$

From inequality (3.60), we deduce that

$$\|\xi_{z,t}\|^2_{L^2(J;W)} \le C\big(\|\xi_{z,t}\|^2_{L^2(J;W)} + \|\xi_y\|^2_{L^\infty(J;W)} + \|\eta_{y,t}\|^2_{L^2(J;W)}\big). \tag{3.62}$$

Due to (3.51), (3.59)-(3.62), we can give that

$$\|\xi_q\|_{L^2(J;W)} \le C\big(\|\eta_y(0)\| + \|y_0 - y_0^h\| + \|\eta_{y,t}\|_{L^2(J;W)} + \|\xi_z(T)\|\big). \tag{3.63}$$

Note that $e_z + \xi_z = \eta_z$, from (3.57)-(3.59) and (3.63), we obtain the results (3.42) and (3.43). □

Lemma 3.5 *Considering Raviart-Thomas elements, there exists a positive constant C, which is in relation to the domain* Ω*, polynomial degree k and the shape regularity of the elements, such that*

$$\|\eta_y\|^2 \le C\Big(\big\|h^{1+\min\{1,k\}}(\operatorname{div} \tilde{p}_h + \tilde{y}_h)\big\|^2 + \|\eta_{\tilde{y}}\|^2 + \min_{w_h \in W_h} \big\|h(\tilde{p}_h - \nabla_h w_h)\big\|^2\Big), \tag{3.64}$$

$$\|\eta_{y,t}\|^2 \le C\Big(\big\|h^{1+\min\{1,k\}}(\operatorname{div} \tilde{p}_h + \tilde{y}_h)_t\big\|^2 + \|\eta_{\tilde{y}}\|^2 + \min_{w_h \in W_h} \big\|h(\tilde{p}_h - \nabla_h w_h)\big\|^2\Big), \tag{3.65}$$

$$\|\eta_{\tilde{y}}\|^2 \le C\Big(\big\|h^{1+\min\{1,k\}}(y_{h,t} + \operatorname{div} p_h - f - u_h)\big\|^2 + \min_{w_h \in W_h} \big\|h(\tilde{p}_h - \nabla_h w_h)\big\|^2\Big), \tag{3.66}$$

$$\|\eta_p\|^2 \le C\big(\big\|h(\operatorname{div} \tilde{p}_h + \tilde{y}_h)\big\|^2 + \|\eta_{\tilde{y}}\|^2 + \big\|h^{\frac{1}{2}} J(\tilde{p}_h \cdot t)\big\|^2 + \big\|h \cdot \operatorname{curl}_h(\tilde{p}_h)\big\|^2\big), \tag{3.67}$$

$$\|\eta_{\tilde{p}}\|^2 \le C\big(\big\|h(y_{h,t} + \operatorname{div} p_h - f - u_h)\big\|^2 + \big\|h^{\frac{1}{2}} J(p_h \cdot t)\big\|^2 + \big\|h \cdot \operatorname{curl}_h(p_h)\big\|^2\big), \tag{3.68}$$

$$\|\eta_q\|^2 \le C\big(\|\eta_{\tilde{z}}\|^2 + \|\eta_{\tilde{y}}\|^2 + \big\|h^{\frac{1}{2}} J(\tilde{q}_h \cdot t)\big\|^2 + \big\|h \cdot \operatorname{curl}_h(\tilde{q}_h)\big\|^2\big), \tag{3.69}$$

$$\|\eta_{\tilde{q}}\|^2 \le C\big(\big\|h(z_{h,t} + \operatorname{div} q_h + y_h - y_d)\big\|^2 + \|\eta_y\|^2 + \|\eta_{\tilde{p}}\|^2$$

$$+ \|h^{\frac{1}{2}} J((\tilde{p}_h + q_h)\cdot t)\|_{0,E_h}^2 + \|h \cdot \operatorname{curl}_h(\tilde{p}_h + q_h)\|^2\Big), \tag{3.70}$$

$$\|\eta_z\|^2 \le C\Big(\|\eta_{\tilde{z}}\|^2 + \|\eta_{\tilde{y}}\|^2 + \min_{w_h\in W_h}\|h(\tilde{q}_h - \nabla_h w_h)\|^2\Big), \tag{3.71}$$

$$\|\eta_{\tilde{z}}\|^2 \le C\Big(\|h^{1+\min\{1,k\}}(z_{h,t} + \operatorname{div} q_h + y_h - y_d)\|^2 + \|\eta_y\|^2 + \|\eta_{\tilde{p}}\|^2 + \min_{w_h\in W_h}\|h(\tilde{p}_h + q_h - \nabla_h w_h)\|^2\Big), \tag{3.72}$$

where $J(v\cdot t)$ expresses the jump of $v\cdot t$ across the element edge Γ with the time t being the tangential unit vector along the edge $\Gamma \in E_h$ for all $v \in V$.

Proof First of all, we must refer to [29] and [30], based on which we can obtain *a posteriori* error estimates for η_y, $\eta_{y,t}$, $\eta_{\tilde{y}}$, η_p, $\eta_{\tilde{p}}$, η_q, $\eta_{\tilde{q}}$, η_z, $\eta_{\tilde{z}}$. We only give the proof of L^2-norm estimate of $\eta_{\tilde{z}}$ for simplicity. Now, with the help of Aubin-Nitsche duality arguments, we think about $\Phi \in H_0^1(\Omega)\cap H^2(\Omega)$ as the following elliptic problem:

$$-\operatorname{div}(A\nabla\Phi) = \Psi, \quad \text{in } \Omega, \tag{3.73}$$

which satisfies the elliptic regularity result as follows:

$$\|\Phi\|_2 \le C\|\Psi\|. \tag{3.74}$$

Exploiting (2.49), (2.79), (3.73) and the definition of Π_h, furthermore, noting that $\hat{q} + \check{p} = -\nabla\check{z}$ in Lemma 3.3 and the equation of $(\nabla_h w_h, (I - \Pi_h)(\Delta\Phi)) = 0$, we gain

$$\begin{aligned}
(\eta_{\tilde{z}}, \Psi) &= (\eta_{\tilde{z}}, -\operatorname{div}(\nabla\Phi)) \\
&= (\check{z}, -\operatorname{div}(\nabla\Phi)) + (\tilde{z}_h, -\operatorname{div}(\nabla\Phi)) \\
&= (\nabla\check{z}, \nabla\Phi) + (\tilde{z}_h, \operatorname{div}(\nabla\Phi)) \\
&= (-\hat{q} - \check{p}, \nabla\Phi) + (\tilde{z}_h, \operatorname{div}(\Pi_h(\nabla\Phi))) \\
&= -(\eta_q, \nabla\Phi) - (\eta_{\tilde{p}}, \nabla\Phi) - (\tilde{p}_h + q_h, \nabla\Phi) + (\tilde{p}_h + q_h, \Pi_h(\nabla\Phi)) \\
&= (\operatorname{div}\eta_q, \Phi - P_h\Phi) - (\eta_y, \Phi - P_h\Phi) + (\eta_y, \Phi) \\
&\quad - (\eta_{\tilde{p}}, \nabla\Phi) - (\tilde{p}_h + q_h, (I - \Pi_h)\nabla\Phi) \\
&= (\operatorname{div}(\hat{q} - q_h) - (\hat{y} - y_h), \Phi - P_h\Phi) + (\eta_y, \Phi) \\
&\quad - (\eta_{\tilde{p}}, \nabla\Phi) - (\tilde{p}_h + q_h, (I - \Pi_h)\nabla\Phi) \\
&= (\operatorname{div}(\hat{q} - q_h) - (\hat{y} - y_h), \Phi - P_h\Phi) + (\eta_y, \Phi) \\
&\quad - (\eta_{\tilde{p}}, \nabla\Phi) - (\tilde{p}_h + q_h, (I - \Pi_h)\nabla\Phi) \\
&= (z_{h,t} - \operatorname{div} q_h + y_h - y_d, \Phi - P_h\Phi) + (\eta_y, \Phi) \\
&\quad - (\eta_{\tilde{p}}, \nabla\Phi) - (\tilde{p}_h + q_h - \nabla_h w_h, (I - \Pi_h)\nabla\Phi) \\
&\le C\big(\|h^{1+\min\{1,k\}}(z_{h,t} - \operatorname{div} q_h + y_h - y_d)\| \cdot \|\Phi\|_2 + \|\eta_y\| \cdot \|\Phi\| \\
&\quad + \|\eta_{\tilde{p}}\| \cdot \|\nabla\Phi\| + \|h(\tilde{p}_h + q_h - \nabla_h w_h)\| \cdot \|\nabla\Phi\|\big) \\
&\le C\big(\|h^{1+\min\{1,k\}}(z_{h,t} - \operatorname{div} q_h + y_h - y_d)\| + \|\eta_y\| + \|\eta_{\tilde{p}}\| \\
&\quad + \|h(\tilde{p}_h + q_h - \nabla_h w_h)\|\big)\|\Phi\|_2.
\end{aligned} \tag{3.75}$$

Combining (3.75) with (3.74), we can derive that

$$\frac{(\eta_{\tilde{z}},\Psi)}{\|\Psi\|}\leq C\Big(\big\|h^{1+\min\{1,k\}}(z_{h,t}-\operatorname{div}q_h+y_h-y_d)\big\| \\ +\|\eta_y\|+\|\eta_{\tilde{p}}\|+\min_{w_h\in W_h}\big\|h(\tilde{p}_h+q_h-\nabla_h w_h)\big\|\Big). \tag{3.76}$$

Next, taking supremum over Ψ, we should get estimate (3.72). Using a similar method, we can obtain the other estimates of Lemma 3.5 at last. □

Now, by the aid of Lemmas 3.1-3.5, we can obtain the final result.

Theorem 3.1 *Let* $(p,\tilde{y},\tilde{p},y,q,\tilde{z},\tilde{q},z,u)$ *and* $(p_h,\tilde{y}_h,\tilde{p}_h,y_h,q_h,\tilde{z}_h,\tilde{q}_h,z_h,u_h)$ *be the solutions of* (2.22)-(2.33) *and* (2.41)-(2.52), *respectively. Then, for* $\forall t\in J$, *the following a posteriori estimates hold true*:

$$\|u-u_h\|_{L^2(J;W)}\leq C\big(\eta_u+\big\|\eta_y(0)\big\|+\big\|y_0-y_0^h\big\|+\|\eta_{y,t}\|_{L^2(J;W)} \\ +\big\|\eta_z(T)\big\|+\|\eta_z\|_{L^2(J;W)}\big), \tag{3.77}$$

$$\|y-y_h\|_{L^\infty(J;W)}\leq C\big(\|u-u_h\|_{L^2(J;W)}+\|\eta_y\|_{L^2(J;W)}\big), \tag{3.78}$$

$$\|\tilde{y}-\tilde{y}_h\|_{L^\infty(J;W)}\leq C\big(\|u-u_h\|_{L^2(J;W)}+\|\eta_{\tilde{y}}\|_{L^2(J;W)}\big), \tag{3.79}$$

$$\|p-p_h\|_{L^\infty(J;W)}\leq C\big(\|u-u_h\|_{L^2(J;W)}+\|\eta_p\|_{L^2(J;W)}\big), \tag{3.80}$$

$$\|\tilde{p}-\tilde{p}_h\|_{L^\infty(J;W)}\leq C\big(\|u-u_h\|_{L^2(J;W)}+\|\eta_{\tilde{p}}\|_{L^2(J;W)}\big), \tag{3.81}$$

$$\|z-z_h\|_{L^\infty(J;W)}\leq C\big(\|u-u_h\|_{L^2(J;W)}+\|\eta_z\|_{L^2(J;W)}\big), \tag{3.82}$$

$$\|\tilde{z}-\tilde{z}_h\|_{L^\infty(J;W)}\leq C\big(\|u-u_h\|_{L^2(J;W)}+\|\eta_{\tilde{z}}\|_{L^2(J;W)}\big), \tag{3.83}$$

$$\|q-q_h\|_{L^\infty(J;W)}\leq C\big(\|u-u_h\|_{L^2(J;W)}+\|\eta_q\|_{L^2(J;W)}\big), \tag{3.84}$$

$$\|\tilde{q}-\tilde{q}_h\|_{L^\infty(J;W)}\leq C\big(\|u-u_h\|_{L^2(J;W)}+\|\eta_{\tilde{q}}\|_{L^2(J;W)}\big), \tag{3.85}$$

where η_u *is introduced in Lemma* 3.2, *and* η_y, $\eta_{y,t}$, $\eta_{\tilde{y}}$, η_p, $\eta_{\tilde{p}}$, η_q, $\eta_{\tilde{q}}$, η_z, $\eta_{\tilde{z}}$ *are given in Lemma* 3.5.

4 Conclusion and future works

In this paper we discuss the semi-discrete mixed finite element methods of the fourth order quadratic parabolic optimal control problems. We have established *a posteriori* error estimates for both the state, the co-state and the control variables. The *a posteriori* error estimates for those problems by finite element methods seem to be new.

In our future work, we shall use the mixed finite element method to deal with fourth order hyperbolic optimal control problems. Furthermore, we shall consider *a posteriori* error estimates and superconvergence of mixed finite element solution for fourth order hyperbolic optimal control problems.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CH, YC and ZL participated in the sequence alignment and drafted the manuscript. All authors read and approved the final manuscript.

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