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LQP method with a new optimal step size rule for nonlinear complementarity problems

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Abstract

Inspired and motivated by results of Bnouhachem *et al.* (Hacet. J. Math. Stat. 41(1):103-117, 2012), we propose a new modified LQP method by using a new optimal step size, where the underlying function F is co-coercive. Under some mild conditions, we show that the method is globally convergent. Some preliminary computational results are given to illustrate the efficiency of the proposed method.

Keywords: nonlinear complementarity problems; co-coercive operator; logarithmic-quadratic proximal method

1 Introduction

The nonlinear complementarity problem (NCP) is to determine a vector $x \in R^n$ such that

$$x \geq 0, \quad F(x) \geq 0 \quad \text{and} \quad x^T F(x) = 0, \tag{1.1}$$

where F is a nonlinear mapping from R^n into itself. Complementarity problems introduced by Lemke [1] and Cottle and Dantzig [2] in the early 1960s has attracted great attention of researchers (see, *e.g.*, [3, 4] and the references therein). On the one hand, there have been many theoretical results on the existence of solutions and their structural properties. On the other hand, many attempts have been made to develop implementable algorithms for the solution of NCP. A popular way to solve the NCP is to reformulate as finding the zero point of the operator $T(x) = F(x) + N_{R^n_+}(x)$, *i.e.*, find $x^* \in R^n_+$ such that $0 \in T(x^*)$, where $N_{R^n_+}(\cdot)$ is the normal cone operator to R^n_+ defined by

$$N_{R^n_+}(x) = \begin{cases} \{y \in R^n : y^T(v - x) \leq 0, \forall v \in R^n_+\} & \text{if } x \in R^n_+, \\ \emptyset & \text{otherwise.} \end{cases}$$

The proximal point algorithm (PPA) is recognized as a powerful and successful algorithm in finding a solution of maximal monotone operators, and it has been proposed by Martinet [5] and studied by Rockafellar [6]. Starting from any initial $x^0 \in R^n$ and for positive real $\beta_k \geq \beta > 0$, iteratively updating x^{k+1} conforming to the following problem:

$$0 \in \beta_k T(x) + \nabla_x q(x, x^k), \tag{1.2}$$

where

$$q(x,x^k) = \frac{1}{2}\|x - x^k\|^2, \tag{1.3}$$

is a quadratic function of x. In place of the usual quadratic term many researchers have used some nonlinear functions $r(x,x^k)$; see, for example, [6–9]. Auslender *et al.* [10, 11] proposed a new type of proximal interior method through replacing the second term of (1.2) by

$$x - (1-\mu)x^k - \mu X_k^2 x^{-1} \tag{1.4}$$

or

$$x - x^k + \mu X_k \log\left(\frac{x}{x^k}\right), \tag{1.5}$$

where $\mu \in (0,1)$ is a given constant, $X_k = \operatorname{diag}(x_1^k, x_2^k, \dots, x_n^k)$, and x^{-1} is an n-vector whose jth elements is $\frac{1}{x_j}$. It is easy to see that, at the kth iteration, solving (1.1) by the LQP method is equivalent to the following system of nonlinear equations:

$$\beta_k F(x) + x - (1-\mu)x^k - \mu X_k^2 x^{-1} = 0 \tag{1.6}$$

or

$$\beta_k F(x) + x - x^k + \mu X_k \log\left(\frac{x}{x^k}\right) = 0. \tag{1.7}$$

Solving the subproblem (1.6) or (1.7) exactly is typically hard demand in practice. To overcome this difficulty, He *et al.* [12], Bnouhachem [13, 14], Bnouhachem and Yuan [15], Bnouhachem and Noor [16, 17], Bnouhachem *et al.* [18, 19], Noor and Bnouhachem [20], and Xu *et al.* [21] introduced some LQP-based prediction-correction methods which do not suffer from the above difficulty and make the LQP method more practical. Each iteration of the above methods contain a prediction and a correction, the predictor is obtained via solving the LQP system approximately under significantly relaxed accuracy criterion and the new iterate is computed directly by an explicit formula derived from the original LQP method for [12], while the new iterate is computed by using the projection operator for [14, 20, 21]. Inspired and motivated by the above research, we suggest and analyze a new LQP method for solving nonlinear complementarity problems (1.1) by using a new step size α_k to Bnouhachem's LQP method [18]. We also study the global convergence of the proposed modified LQP method under some mild conditions.

Throughout this paper we assume that F is co-coercive with modulus $c > 0$, that is, $\langle F(x) - F(y), x - y\rangle \geq c\|F(x) - F(y)\|^2$, $\forall x, y \in R_+^n$ and the solution set of (1.1), denoted by Ω^*, is nonempty.

2 The proposed method and some properties

In this section, we suggest and analyze the new modified LQP method for solving NCP (1.1). For given $x^k > 0$ and $\beta_k > 0$, each iteration of the proposed method consists of two steps, the first step offers a predictor $\tilde{x}^k$ and the second step produces the new iterate x^{k+1}.

Prediction step: Find an approximate solution $\tilde{x}^k$ of (1.6), called predictor, such that

$$0 \approx \beta_k F(\tilde{x}^k) + \tilde{x}^k - (1-\mu)x^k - \mu X_k^2 (\tilde{x}^k)^{-1} = \xi^k := \beta_k \bigl(F(\tilde{x}^k) - F(x^k)\bigr) \tag{2.1}$$

and ξ^k which satisfies

$$\|\xi^k\| \le \eta \|x^k - \tilde{x}^k\|, \quad 0 < \eta < 1. \tag{2.2}$$

Correction step: For $0 < \rho < 1$, the new iterate $x^{k+1}(\alpha_k)$ is defined by

$$x^{k+1}(\alpha_k) = \rho x^k + (1-\rho) P_{R_+^n}\bigl[x^k - \alpha_k d(x^k, \beta_k)\bigr], \tag{2.3}$$

where

$$d(x^k, \beta_k) := (x^k - \tilde{x}^k) + \frac{\beta_k}{1+\mu} F(\tilde{x}^k) \tag{2.4}$$

and α_k is a positive scalar. How to choose a suitable α_k we will discuss later.

Remark 2.1 Equation (2.1) can be written as

$$\beta_k F(x^k) + \tilde{x}^k - (1-\mu)x^k - \mu X_k^2 (\tilde{x}^k)^{-1} = 0, \tag{2.5}$$

and the solution of (2.5) can be componentwise obtained by

$$\tilde{x}_j^k = \frac{(1-\mu)x_j^k - \beta_k F_j(x^k) + \sqrt{[(1-\mu)x_j^k - \beta_k F_j(x^k)]^2 + 4\mu (x_j^k)^2}}{2}. \tag{2.6}$$

Moreover, for any $x^k > 0$ we have always $\tilde{x}^k > 0$.

We now consider the criterion for α_k, which ensures that $x^{k+1}(\alpha_k)$ is closer to the solution set than x^k. For this purpose, we define

$$\Theta(\alpha_k) = \|x^k - x^*\|^2 - \|x^{k+1}(\alpha_k) - x^*\|^2. \tag{2.7}$$

Theorem 2.1 [18] *Let $x^* \in \Omega^*$, $x^{k+1}(\alpha_k)$ be defined by* (2.3), *then we have*

$$\begin{aligned}\Theta(\alpha_k) \ge (1-\rho)\Bigl\{ & 2\alpha_k (x^k - \tilde{x}^k)^T D(x^k, \beta_k) - \alpha_k^2 \bigl(\|D(x^k, \beta_k)\|^2 + 2D(x^k, \beta_k)^T (x^k - \tilde{x}^k)\bigr) \\ & + \left(\frac{2\alpha_k}{1+\mu}\left(1 - \mu - \frac{\beta_k}{4c}\right) - \alpha_k^2\right) \|x^k - \tilde{x}^k\|^2 \Bigr\},\end{aligned} \tag{2.8}$$

where

$$D(x^k, \beta_k) := (x^k - \tilde{x}^k) + \frac{1}{1+\mu} \xi^k.$$

Lemma 2.1 [18] *For given $x^k \in R^n_{++}$, let $\tilde{x}^k$ satisfy the condition* (2.2), *then we have the following*:

$$\left(x^k - \tilde{x}^k\right)^T D\left(x^k, \beta_k\right) \geq \left(\frac{1-\eta}{1+\mu}\right) \left\|x^k - \tilde{x}^k\right\|^2 \tag{2.9}$$

and

$$\left(x^k - \tilde{x}^k\right)^T D\left(x^k, \beta_k\right) \geq \frac{1}{2} \left\|D\left(x^k, \beta_k\right)\right\|^2. \tag{2.10}$$

From (2.8), we have

$$\begin{aligned}
\Theta(\alpha_k) \geq{}& (1-\rho)\Biggl\{2\alpha_k\left(x^k - \tilde{x}^k\right)^T D\left(x^k, \beta_k\right) - \alpha_k^2\left(\left\|D\left(x^k, \beta_k\right)\right\|^2 + 2D\left(x^k, \beta_k\right)^T\left(x^k - \tilde{x}^k\right)\right) \\
&+ \left(\frac{2\alpha_k}{1+\mu}\left(1 - \mu - \frac{\beta_k}{4c}\right) - \alpha_k^2\right)\left\|x^k - \tilde{x}^k\right\|^2\Biggr\} \\
={}& (1-\rho)\Biggl\{2\alpha_k\left(\left(x^k - \tilde{x}^k\right)^T D\left(x^k, \beta_k\right) + \frac{1}{1+\mu}\left(1 - \mu - \frac{\beta_k}{4c}\right)\left\|x^k - \tilde{x}^k\right\|^2\right) \\
&- \alpha_k^2\left(\left\|D\left(x^k, \beta_k\right)\right\|^2 + 2D\left(x^k, \beta_k\right)^T\left(x^k - \tilde{x}^k\right) + \left\|x^k - \tilde{x}^k\right\|^2\right)\Biggr\} \\
={}& (1-\rho)\Biggl\{2\alpha_k\left(\left(x^k - \tilde{x}^k\right)^T D\left(x^k, \beta_k\right) + \frac{1}{1+\mu}\left(1 - \mu - \frac{\beta_k}{4c}\right)\left\|x^k - \tilde{x}^k\right\|^2\right) \\
&- \alpha_k^2\left\|D\left(x^k, \beta_k\right) + x^k - \tilde{x}^k\right\|^2\Biggr\} \\
={}& (1-\rho)\Psi(\alpha_k),
\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}
\Psi(\alpha_k) :={}& 2\alpha_k\left(\left(x^k - \tilde{x}^k\right)^T D\left(x^k, \beta_k\right) + \frac{1}{1+\mu}\left(1 - \mu - \frac{\beta_k}{4c}\right)\left\|x^k - \tilde{x}^k\right\|^2\right) \\
&- \alpha_k^2\left\|D\left(x^k, \beta_k\right) + x^k - \tilde{x}^k\right\|^2.
\end{aligned} \tag{2.12}$$

3 Convergence analysis

In this section, we prove some useful results which will be used in the consequent analysis and then investigate the strategy of how to choose the new step size α_k.

Note that $\Psi(\alpha_k)$ is a quadratic function of α_k and it reaches its maximum at

$$\alpha_k^* = \frac{(x^k - \tilde{x}^k)^T D(x^k, \beta_k) + \frac{1}{1+\mu}(1 - \mu - \frac{\beta_k}{4c})\|x^k - \tilde{x}^k\|^2}{\|D(x^k, \beta_k) + x^k - \tilde{x}^k\|^2} \tag{3.1}$$

and

$$\Psi\left(\alpha_k^*\right) = \alpha_k^*\left(\left(x^k - \tilde{x}^k\right)^T D\left(x^k, \beta_k\right) + \frac{1}{1+\mu}\left(1 - \mu - \frac{\beta_k}{4c}\right)\left\|x^k - \tilde{x}^k\right\|^2\right). \tag{3.2}$$

In the next theorem we show that α_k^* and $\Psi(\alpha_k^*)$ are lower bounded away from zero, whenever $x^k \neq \tilde{x}^k$ and it is one of the keys to prove the global convergence results.

Theorem 3.1 *For given $x^k \in R^n_{++}$, let $\tilde{x}^k$ satisfy the condition* (2.2) *and β_k satisfy*

$$0 < \beta_l \le \inf_{k=0}^{\infty} \beta_k \le \sup_{k=0}^{\infty} \beta_k \le \beta_u < 4c(1-\mu),$$

then we have the following:

$$\alpha_k^* \ge \frac{1-\eta}{5-4\eta+\mu} > 0 \tag{3.3}$$

and

$$\Psi\left(\alpha_k^*\right) \ge \frac{(1-\eta)^2}{(5-4\eta+\mu)(1+\mu)} \left\|x^k - \tilde{x}^k\right\|^2. \tag{3.4}$$

Proof It follows from (2.9) and (2.10) that

$$\begin{aligned} \alpha_k^* &= \frac{(x^k - \tilde{x}^k)^T D(x^k, \beta_k) + \frac{1}{1+\mu}(1-\mu-\frac{\beta_k}{4c})\|x^k - \tilde{x}^k\|^2}{\|D(x^k, \beta_k) + x^k - \tilde{x}^k\|^2} \\ &\ge \frac{(x^k - \tilde{x}^k)^T D(x^k, \beta_k)}{\|D(x^k, \beta_k) + x^k - \tilde{x}^k\|^2} \\ &= \frac{(x^k - \tilde{x}^k)^T D(x^k, \beta_k)}{\|D(x^k, \beta_k)\|^2 + 2D(x^k, \beta_k)^T (x^k - \tilde{x}^k) + \|x^k - \tilde{x}^k\|^2} \\ &\ge \frac{(x^k - \tilde{x}^k)^T D(x^k, \beta_k)}{(4 + \frac{1+\mu}{1-\eta}) D(x^k, \beta_k)^T (x^k - \tilde{x}^k)} \\ &= \frac{1-\eta}{5-4\eta+\mu} > 0. \end{aligned}$$

Using (3.2), (3.3), and (2.9), we have

$$\begin{aligned} \Psi\left(\alpha_k^*\right) &\ge \left(\frac{1-\eta}{5-4\eta+\mu}\right)\left(\frac{1-\eta}{1+\mu} + \frac{1}{1+\mu}\left(1-\mu-\frac{\beta_k}{4c}\right)\right)\left\|x^k - \tilde{x}^k\right\|^2 \\ &\ge \frac{(1-\eta)^2}{(5-4\eta+\mu)(1+\mu)}\left\|x^k - \tilde{x}^k\right\|^2. \end{aligned}$$

□

Remark 3.1 Note that $\alpha_{k_2}^* = \min\{(1-\mu-\frac{\beta_k}{4c})/(1+\mu), \frac{(x^k-\tilde{x}^k)^T D(x^k,\beta_k)}{\|D(x^k,\beta_k)\|^2 + 2D(x^k,\beta_k)^T (x^k-\tilde{x}^k)}\}$ is the optimal step size used in [18]. Since α_k^* is to maximize the profit function $\Psi(\alpha_k)$, we have

$$\Psi\left(\alpha_k^*\right) \ge \Psi\left(\alpha_{k_2}^*\right). \tag{3.5}$$

Inequality (3.5) shows theoretically that the proposed method is expected to make more progress than that in [18] at each iteration, and so it explains theoretically that the proposed method outperforms the method in [18].

For fast convergence, we take a relaxation factor $\gamma \in [1,2)$ and set the step size α_k in (2.3) by $\alpha_k = \gamma\alpha_k^*$, it follows from (2.11), (2.12), and Theorem 3.1 that

$$\begin{aligned} \Theta(\alpha_k) &\ge \gamma(2-\gamma)(1-\rho)\Psi\left(\alpha_k^*\right) \\ &\ge \gamma(2-\gamma)(1-\rho)\frac{(1-\eta)^2}{(5-4\eta+\mu)(1+\mu)}\left\|x^k - \tilde{x}^k\right\|^2. \end{aligned} \tag{3.6}$$

Then from definition of $\Theta(\alpha_k)$ and (3.6) there is a constant

$$\tau := \gamma(2-\gamma)(1-\rho)\frac{(1-\eta)^2}{(5-4\eta+\mu)(1+\mu)} > 0$$

such that

$$\left\|x^{k+1}(\alpha_k) - x^*\right\|^2 \le \left\|x^k - x^*\right\|^2 - \tau\left\|x^k - \tilde{x}^k\right\|^2, \quad \forall x^* \in \Omega^*. \tag{3.7}$$

The following result can be proved by similar arguments to those in [12, 14, 18, 21]. Hence the proof will be omitted.

Theorem 3.2 [12, 14, 18, 21] *If* $\inf_{k=0}^{\infty} \beta_k = \beta_l > 0$, *then the sequence* $\{x^k\}$ *generated by the proposed method converges to some* x^∞ *which is a solution of NCP.*

The detailed algorithm is as follows.

Step 0. Let $\beta_0 = 1$, $\eta(:= 0.9) < 1$, $0 < \rho < 1$, $0 < \mu < 1$, $\gamma = 1.9$, $\epsilon = 10^{-7}$, $k = 0$, and $x^0 > 0$.

Step 1. If $\|\min(x, F(x))\|_\infty \le \epsilon$, then stop. Otherwise, go to Step 2.

Step 2. (Prediction step)

$$s := (1-\mu)x^k - \beta_k F(x^k), \qquad \tilde{x}_i^k := \left(s_i + \sqrt{(s_i)^2 + 4\mu\left(x_i^k\right)^2}\right)/2,$$
$$\xi^k := \beta_k\left(F(\tilde{x}^k) - F(x^k)\right), \qquad r := \left\|\xi^k\right\|/\left\|x^k - \tilde{x}^k\right\|$$

while $(r > \eta)$

$$\beta_k := \beta_k * 0.8/r,$$
$$s := (1-\mu)x^k - \beta_k F(x^k), \qquad \tilde{x}_i^k := \left(s_i + \sqrt{(s_i)^2 + 4\mu\left(x_i^k\right)^2}\right)/2,$$
$$\xi^k := \beta_k\left(F(\tilde{x}^k) - F(x^k)\right), \qquad r := \left\|\xi^k\right\|/\left\|x^k - \tilde{x}^k\right\|.$$

end while

Step 3. (Correction step)

$$D(x^k, \beta_k) := (x^k - \tilde{x}^k) + \frac{1}{1+\mu}\xi^k, \qquad d(x^k, \beta_k) := (x^k - \tilde{x}^k) + \frac{\beta_k}{1+\mu}F(\tilde{x}^k),$$
$$\alpha_k^* = \frac{(x^k - \tilde{x}^k)^T D(x^k, \beta_k) + \frac{1}{1+\mu}(1 - \mu - \frac{\beta_k}{4c})\|x^k - \tilde{x}^k\|^2}{\|D(x^k, \beta_k) + x^k - \tilde{x}^k\|^2}, \qquad \alpha_k = \gamma\alpha_k^*,$$
$$x^{k+1} = \rho x^k + (1-\rho)P_{R_+^n}\left[x^k - \alpha_k d(x^k, \beta_k)\right].$$

Step 4.

$$\beta_{k+1} = \begin{cases} \frac{\beta_k * 0.7}{r}, & \text{if } r \le 0.3; \\ \beta_k, & \text{otherwise.} \end{cases}$$

Step 5. $k := k + 1$; go to Step 1.

4 Preliminary computational results

In this section, we consider two examples to illustrate the efficiency and the performance of the proposed algorithm.

4.1 Numerical experiments I

We consider the nonlinear complementarity problems

$$x \geq 0, \quad F(x) \geq 0, \qquad x^T F(x) = 0, \tag{4.1}$$

where

$$F(x) = D(x) + Mx + q,$$

$D(x)$ and $Mx + q$ are the nonlinear part and linear part of $F(x)$, respectively.

We form the linear part in the test problems similarly to Harker and Pang [4]. The matrix $M = A^T A + B$, where A is an $n \times n$ matrix whose entries are randomly generated in the interval $(-5, +5)$ and a skew-symmetric matrix B is generated in the same way. The vector q is generated from a uniform distribution in the interval $(-500, 500)$ or in $(-500, 0)$. In $D(x)$, the nonlinear part of $F(x)$, the components are chosen to be $D_j(x) = d_j * \arctan(x_j)$, where d_j is a random variable in $(0, 1)$.

In all tests we take the logarithmic proximal parameter $\mu = 0.01$, $\rho = 0.01$, and $c = 0.9$. All iterations start with $x^0 = (1, \ldots, 1)^T$ and $\beta_0 = 1$, and we have the stopping criterion whenever

$$\left\| \min\bigl(x^k, F\bigl(x^k\bigr)\bigr) \right\|_\infty \leq 10^{-7}.$$

All codes were written in Matlab, and we compare the proposed method with that in [18]. The test results for problem (4.1) are reported in Tables 1 and 2. k is the number of iteration and l denotes the number of evaluations of mapping F.

Tables 1 and 2 show that the proposed method is more efficient. Numerical results indicate that the proposed method can be save about $49 \sim 65$ percent of the number of iterations and about $48 \sim 63$ of the amount of computing the value of function F.

4.2 Numerical experiments II

In this subsection, we apply the proposed method to the traffic equilibrium problems and present corresponding numerical results.

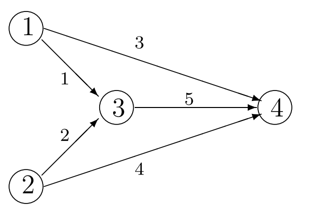
Consider a network $[N, L]$ of nodes N and directed links L, which consists of a finite sequence of connecting links with a certain orientation. Let a, b, *etc.* denote the links, and

Table 1 Numerical results for problem (4.1) with $q \in (-500, 500)$

n	**The method in [18]**			**The proposed method**		
	k	***l***	**CPU (Sec.)**	***k***	***l***	**CPU (Sec.)**
200	297	651	0.068	117	279	0.016
300	329	708	0.094	129	310	0.029
400	333	721	0.13	169	367	0.08
500	368	801	0.22	171	381	0.12
700	364	751	0.41	142	334	0.12
1000	339	743	1.74	139	328	0.66

Table 2 Numerical results for problem (4.1) with $q \in (-500, 0)$

n	The method in [18]			The proposed method		
	k	*l*	CPU (Sec.)	*k*	*l*	CPU (Sec.)
200	578	1246	0.09	217	495	0.04
300	584	1257	0.14	212	497	0.06
400	769	1586	0.26	284	633	0.11
500	821	1762	0.38	282	645	0.15
700	699	1524	0.61	245	571	0.27
1000	813	1709	3.37	294	679	1.45



O/D Pairs	Path No. & the link on the path
ω_1 : ①→ ④	$p_1 = \{3\}$ $p_2 = \{1,5\}$
ω_2 : ②→ ④	$p_3 = \{4\}$ $p_4 = \{2,5\}$

Figure 1 An illustrative example of given directed network and the O/D pairs.

let p, q, *etc.* denote the paths. We let ω denote an origin/destination (O/D) pair of nodes of the network and P_ω denotes the set of all paths connecting O/D pair ω. Note that the path-arc incidence matrix and the path-O/D pair incidence matrix, denoted by A and B, respectively, are determined by the given network and O/D pairs. To see how to convert a traffic equilibrium problem into a variational inequality, we take into account a simple example as depicted in Figure 1.

For the given example in Figure 1, the path-arc incidence matrix A and the path-O/D pair incidence matrix B have the following forms:

$$\text{No. link} \quad \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \qquad\qquad \text{No. O/Dpair} \quad \begin{matrix} \omega_1 & \omega_2 \end{matrix}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let x_p represent the traffic flow on path p and f_a denote the link load on link a, then the arc-flow vector f is given by

$$f = A^T x.$$

Let d_ω denote the traffic amount between O/D pair ω, which must satisfy

$$d_\omega = \sum_{p \in P_\omega} x_p.$$

Thus, the O/D pair-traffic amount vector d is given by

$$d = B^T x.$$

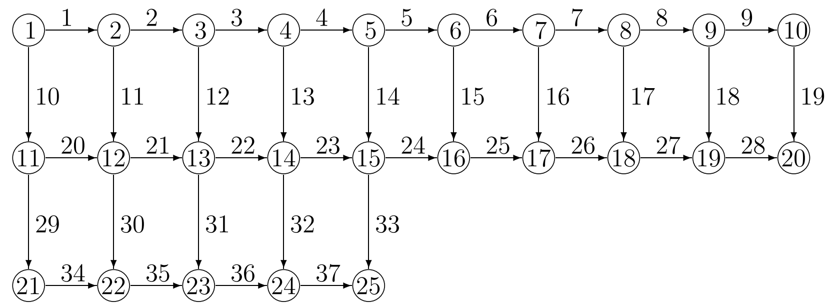


Figure 2 A directed network with 25 nodes and 37 links.

Let $t(f) = \{t_a, a \in L\}$ be the vector of link travel costs, which is a function of the link flow. A user traveling on path p incurs a (path) travel cost θ_p. For given link travel cost vector t, the path travel cost vector θ is given by

$$\theta = At(f) \quad \text{and thus} \quad \theta(x) = At(A^T x).$$

Associated with every O/D pair ω, there is a travel disutility $\lambda_\omega(d)$. Since both the path costs and the travel disutilities are functions of the flow pattern x, the traffic network equilibrium problem is to seek the path flow pattern x^* such that

$$x^* \geq 0, \quad (x - x^*)^T F(x^*) \geq 0, \quad \forall x \geq 0, \tag{4.2}$$

where

$$F_p(x) = \theta_p(x) - \lambda_\omega(d(x)), \quad \forall \omega, p \in P_\omega,$$

and thus

$$F(x) = At(A^T x) - B\lambda(B^T x).$$

We apply the proposed method to the example taken from [22] (Example 7.5 in [22]), which consisted of 25 nodes, 37 links and six O/D pairs. The network is depicted in Figure 2.

For this example, there are together 55 paths for the six given O/D pairs and hence the dimension of the variable x is 55. Therefore, the path-arc incidence matrix A is a 55×37 matrix and the path-O/D pair incidence matrix B is a 55×6 matrix. The user cost of traversing link a is given in Table 3. The disutility function is given by

$$\lambda_\omega(d) = -m_\omega d_\omega + q_\omega \tag{4.3}$$

and the coefficients m_ω and q_ω in the disutility function of different O/D pairs for this example are given in Table 4.

The test results for problems (4.2) for different ε are reported in Table 5, k is the number of iterations and l denotes the number of evaluations of mapping F. The stopping criterion

Table 3 The link traversing cost functions $t_a(f)$ in the example

$t_1(f) = 5 \cdot 10^{-5} f_1^4 + 5f_1 + 2f_2 + 500$	$t_{20}(f) = 3 \cdot 10^{-5} f_{20}^4 + 6f_{20} + f_{21} + 300$
$t_2(f) = 3 \cdot 10^{-5} f_2^4 + 4f_2 + 4f_1 + 200$	$t_{21}(f) = 4 \cdot 10^{-5} f_{21}^4 + 4f_{21} + f_{22} + 400$
$t_3(f) = 5 \cdot 10^{-5} f_3^4 + 3f_3 + f_4 + 350$	$t_{22}(f) = 2 \cdot 10^{-5} f_{22}^4 + 6f_{22} + f_{23} + 500$
$t_4(f) = 3 \cdot 10^{-5} f_4^4 + 6f_4 + 3f_5 + 400$	$t_{23}(f) = 3 \cdot 10^{-5} f_{23}^4 + 9f_{23} + 2f_{24} + 350$
$t_5(f) = 6 \cdot 10^{-5} f_5^4 + 6f_5 + 4f_6 + 600$	$t_{24}(f) = 2 \cdot 10^{-5} f_{24}^4 + 8f_{24} + f_{25} + 400$
$t_6(f) = 7f_6 + 3f_7 + 500$	$t_{25}(f) = 3 \cdot 10^{-5} f_{25}^4 + 9f_{25} + 3f_{26} + 450$
$t_7(f) = 8 \cdot 10^{-5} f_7^4 + 8f_7 + 2f_8 + 400$	$t_{26}(f) = 6 \cdot 10^{-5} f_{26}^4 + 7f_{26} + 8f_{27} + 300$
$t_8(f) = 4 \cdot 10^{-5} f_8^4 + 5f_8 + 2f_9 + 650$	$t_{27}(f) = 3 \cdot 10^{-5} f_{27}^4 + 8f_{27} + 3f_{28} + 500$
$t_9(f) = 10^{-5} f_9^4 + 6f_9 + 2f_10 + 700$	$t_{28}(f) = 3 \cdot 10^{-5} f_{28}^4 + 7f_{28} + 650$
$t_{10}(f) = 4f_{10} + f_{12} + 800$	$t_{29}(f) = 3 \cdot 10^{-5} f_{29}^4 + 3f_{29} + f_{30} + 450$
$t_{11}(f) = 7 \cdot 10^{-5} f_{11}^4 + 7f_{11} + 4f_{12} + 650$	$t_{30}(f) = 4 \cdot 10^{-5} f_{30}^4 + 7f_{30} + 2f_{31} + 600$
$t_{12}(f) = 8f_{12} + 2f_{13} + 700$	$t_{31}(f) = 3 \cdot 10^{-5} f_{31}^4 + 8f_{31} + f_{32} + 750$
$t_{13}(f) = 10^{-5} f_{13}^4 + 7f_{13} + 3f_{18} + 600$	$t_{32}(f) = 6 \cdot 10^{-5} f_{32}^4 + 8f_{32} + 3f_{33} + 650$
$t_{14}(f) = 8f_{14} + 3f_{15} + 500$	$t_{33}(f) = 4 \cdot 10^{-5} f_{33}^4 + 9f_{33} + 2f_{31} + 750$
$t_{15}(f) = 3 \cdot 10^{-5} f_{15}^4 + 9f_{15} + 2f_{14} + 200$	$t_{34}(f) = 6 \cdot 10^{-5} f_{34}^4 + 7f_{34} + 3f_{30} + 550$
$t_{16}(f) = 8f_{16} + 5f_{12} + 300$	$t_{35}(f) = 3 \cdot 10^{-5} f_{35}^4 + 8f_{35} + 3f_{32} + 600$
$t_{17}(f) = 3 \cdot 10^{-5} f_{17}^4 + 7f_{17} + 2f_{15} + 450$	$t_{36}(f) = 2 \cdot 10^{-5} f_{36}^4 + 8f_{36} + 4f_{31} + 750$
$t_{18}(f) = 5f_{18} + f_{16} + 300$	$t_{37}(f) = 6 \cdot 10^{-5} f_{37}^4 + 5f_{37} + f_{36} + 350$
$t_{19}(f) = 8f_{19} + 3f_{17} + 600$	

Table 4 The O/D pairs and the parameters in (4.3) of the example

(O,D) Pair ω	(1, 20)	(1, 25)	(2, 20)	(3, 25)	(1, 24)	(11, 25)
m_ω	1	6	10	5	7	9
q_ω	1,000	800	2,000	6,000	8,000	7,000
$\|P_\omega\|$	10	15	9	6	10	5

Table 5 Numerical results for different ε

Different ε	The method in [18]			The proposed method		
	k	***l***	**CPU (Sec.)**	***k***	***l***	**CPU (Sec.)**
10^{-5}	201	445	0.04	90	216	0.11
10^{-6}	263	580	0.034	115	276	0.01
10^{-7}	321	708	0.054	150	352	0.019
10^{-8}	380	837	0.058	183	426	0.018
10^{-9}	438	963	0.061	211	491	0.02

is

$$\frac{\| \min\{x, F(x)\}\|_\infty}{\| \min\{x^0, F(x^0)\}\|_\infty} \leq \varepsilon.$$

Table 5 shows that the new method is more flexible and efficient to solve a traffic equilibrium problem. Moreover, it demonstrates computationally that the new method is more effective than the method presented in [18] in the sense that the new method needs fewer iteration and less evaluation numbers of F, which clearly illustrates its efficiency.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have made equal contributions. All authors read and approved the final manuscript.

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