Jakšetić et al. *Journal of Inequalities and Applications* (2015) 2015:162
DOI 10.1186/s13660-015-0682-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

A monotonic refinement of Levinson's inequality

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Abstract
In this paper we give a monotonic refinement of the probabilistic version of Levinson's inequality based on the monotonic refinement of Jensen's inequality obtained by Cho *et al.* (Panam. Math. J. 12:43-50, 2002).

MSC: 26D15

Keywords: Levinson's inequality; Jensen's inequality; 3-convexity at a point

1 Introduction

Levinson's inequality and its converse are summarized in the following result taken from Bullen [1].

Theorem 1.1 (a) *If $f:[a,b]\to\mathbb{R}$ is 3-convex and p_i, x_i, y_i, $i=1,2,\dots,n$, are such that $p_i>0$, $\sum_{i=1}^n p_i=1$, $a\le x_i, y_i\le b$,*

$$\max(x_1,\dots,x_n)\le\min(y_1,\dots,y_n) \tag{1}$$

and

$$x_1+y_1=x_2+y_2=\cdots=x_n+y_n=2c \tag{2}$$

for some $c\in[a,b]$, then

$$\sum_{i=1}^n p_i f(x_i)-f(\overline{x})\le\sum_{i=1}^n p_i f(y_i)-f(\overline{y}), \tag{3}$$

where $\overline{x}=\sum_{i=1}^n p_i x_i$ and $\overline{y}=\sum_{i=1}^n p_i y_i$ denote the weighted arithmetic means.

(b) *If for a continuous function f inequality* (3) *holds for all n, all $c\in[a,b]$, all $2n$ distinct points $x_i, y_i\in[a,b]$ satisfying* (1) *and* (2) *and all weights $p_i>0$ such that $\sum_{i=1}^n p_i=1$, then f is 3-convex.*

Levinson [2] originally proved the inequality for functions $f:(0,2c)\to\mathbb{R}$ such that $f'''\ge 0$. Popoviciu [3] showed that the assumption of nonnegativity of the third derivative can be weakened to 3-convexity of f. Bullen [1] gave another proof of the inequality (rescaled to a general interval $[a,b]$) as well as its converse given in part (b) of Theorem 1.1. Pečarić and Raşa [4] extended the inequality by using the method of index set functions; in the process they weakened assumption (1) and obtained a monotonic refinement of the inequality.

The above version of the inequality assumes that the sequences x_i's and y_i's are symmetrically distributed around the point c. Mercer [5] made a significant improvement by replacing this condition of symmetric distribution with the weaker one that the variances of the two sequences are equal.

Theorem 1.2 *If $f : [a,b] \to \mathbb{R}$ satisfies $f''' \geq 0$ and p_i, x_i, y_i, $i = 1,2,\dots,n$, are such that $p_i > 0$, $\sum_{i=1}^n p_i = 1$, $a \leq x_i, y_i \leq b$,* (1) *holds and*

$$\sum_{i=1}^{n} p_i(x_i - \overline{x})^2 = \sum_{i=1}^{n} p_i(y_i - \overline{y})^2,$$

then (3) *holds.*

Witkowski [6] extended Mercer's result to 3-convex functions and a more general probabilistic setting. Baloch *et al.* [7] showed that inequality (3) holds for a larger class of functions they introduced and called 3-convex functions at a point.

Definition 1.3 Let I be an interval in $\mathbb{R}$ and $c \in I$. A function $f : I \to \mathbb{R}$ is said to be 3-convex at point c if there exists a constant A such that the function $F(s) = f(s) - \frac{A}{2}s^2$ is concave on $I \cap (-\infty, c]$ and convex on $I \cap [c, \infty)$.

Baloch *et al.* [7] also proved the converse of the inequality, *i.e.*, 3-convex functions at a point are the largest class of functions for which Levinson's inequality holds under the equal variances assumption. Probabilistic version of Levinson's inequality and its converse are summarized in the following result taken from Pečarić *et al.* [8].

Theorem 1.4 (a) *Let $f : [a,b] \to \mathbb{R}$ be* 3*-convex at point c and $X : \Omega \to [a,c]$ and $Y : \Omega \to [c,b]$ be two random variables such that* $\operatorname{Var}(X) = \operatorname{Var}(Y)$. *Then*

$$\mathbb{E}\big(f(X)\big) - f\big(\mathbb{E}(X)\big) \leq \mathbb{E}\big(f(Y)\big) - f\big(\mathbb{E}(Y)\big). \tag{4}$$

(b) *Let $f : [a,b] \to \mathbb{R}$ be continuous and $c \in (a,b)$ fixed. Suppose that inequality* (4) *holds for all discrete random variables X and Y taking two values $x_1, x_2 \in [a,c]$ and $y_1, y_2 \in [c,b]$, respectively, each with probability $\frac{1}{2}$ and such that* $\operatorname{Var}(X) = \operatorname{Var}(Y)$ (*i.e.* $|x_2 - x_1| = |y_2 - y_1|$). *Then f is* 3*-convex at c.*

Remark 1.5 Results in [8] were stated for f defined on an arbitrary interval I. In that case, the finiteness of $\operatorname{Var}(X) = \operatorname{Var}(Y)$, $\mathbb{E}[f(X)]$ and $\mathbb{E}[f(Y)]$ needs to be assumed. For simplicity, in this paper we will work with the closed interval $[a,b]$ since in this case the function f and all random variables are bounded and the aforementioned finiteness assumptions are satisfied.

If X and Y are discrete random variables taking values x_i and y_i, respectively, with probabilities p_i, then Theorem 1.4(a) gives Theorem 1.2. In [8] it was proven that a function defined on an interval is 3-convex if and only if it is 3-convex at every point of the interval. Therefore, the converse stated in Theorem 1.4(b) strengthens the converse stated in Theorem 1.1(b).

Theorem 1.4 shows that 3-convex functions at a point are characterized by Levinson's inequality in a similar way that convex functions are characterized by Jensen's inequality. Cho *et al.* [9] constructed two mappings connected to Jensen's inequality and proved their monotonicity and convexity properties. Throughout the rest of the paper Ω denotes a measurable space with a finite measure μ, and we assume all mappings to be measurable. Further, $\mathbb{E}[\cdot]$ and $\operatorname{Var}(\cdot)$ denote the expectation and variance operators with respect to the probability measure $\frac{1}{\mu(\Omega)}\mu$, *i.e.*, for $z:\Omega\to\mathbb{R}$,

$$\mathbb{E}[z] = \frac{1}{\mu(\Omega)}\int_\Omega z(s)\,d\mu(s),$$
$$\operatorname{Var}[z] = \frac{1}{\mu(\Omega)}\int_\Omega \bigl(z(s)-\mathbb{E}[z]\bigr)^2\,d\mu(s) = \mathbb{E}\bigl[z^2\bigr]-\mathbb{E}^2[z].$$

The following is a result from [9].

Theorem 1.6 *Let* $f:[a,b]\to\mathbb{R}$ *be convex*, $x:\Omega\to[a,b]$ *and* $H,V:[0,1]\to\mathbb{R}$ *the mappings*

$$H(t) = \frac{1}{\mu(\Omega)}\int_\Omega f\bigl(tx(s)+(1-t)\mathbb{E}[x]\bigr)\,d\mu(s)$$

and

$$V(t) = \frac{1}{\mu(\Omega)^2}\int_\Omega\int_\Omega f\bigl(tx(s)+(1-t)x(u)\bigr)\,d\mu(s)\,d\mu(u).$$

Then:

(a) *the mappings* H *and* V *are convex on* $[0,1]$,

(b) *the mapping* H *is nondecreasing on* $[0,1]$, *while the mapping* V *is nonincreasing on* $[0,\frac{1}{2}]$ *and nondecreasing on* $[\frac{1}{2},1]$,

(c) *the following equalities hold*:

$$\inf_{t\in[0,1]} H(t) = H(0) = f\bigl(\mathbb{E}[x]\bigr),$$
$$\sup_{t\in[0,1]} H(t) = H(1) = \mathbb{E}\bigl[f(x)\bigr],$$
$$\inf_{t\in[0,1]} V(t) = V\left(\frac{1}{2}\right) = \frac{1}{\mu(\Omega)^2}\int_\Omega\int_\Omega f\left(\frac{x(s)+x(u)}{2}\right)d\mu(s)\,d\mu(u),$$
$$\sup_{t\in[0,1]} V(t) = V(0) = V(1) = \mathbb{E}\bigl[f(x)\bigr],$$

(d) *the following inequality holds for all* $t\in[0,1]$:

$$V(t) \geq \max\bigl\{H(t),H(1-t)\bigr\}.$$

Remark 1.7 Theorem 1.6 was proven in [9] for the case when Ω is an interval in $\mathbb{R}$ and μ is a measure with density, *i.e.*, $d\mu(s) = p(s)\,ds$. But, from the proofs given there, it is obvious that the statements hold under the more general setting given here.

If we denote $x_{(t)}(s) = tx(s) + (1-t)\mathbb{E}[x]$, then $H(t) = \mathbb{E}[f(x_{(t)})]$. As t ranges from 0 to 1, the function (*i.e.*, random variable) $x_{(t)}$ ranges from the constant $\mathbb{E}[x]$ to the function x itself. In the process the expectation $\mathbb{E}[f(x_{(t)})]$ increases by the monotonicity property from Theorem 1.6(b). Therefore, for $0 \leq s \leq t \leq 1$, the following monotonic refinement of Jensen's inequality holds:

$$f\bigl(\mathbb{E}[x]\bigr) = H(0) \leq \mathbb{E}\bigl[f(x_{(s)})\bigr] \leq \mathbb{E}\bigl[f(x_{(t)})\bigr] \leq H(1) = \mathbb{E}\bigl[f(x)\bigr].$$

Furthermore, if x' and x'' are two independent identically distributed 'copies' of x on the product space $\Omega \times \Omega$, then $V(t) = \mathbb{E}[f(\tilde{x}_{(t)})]$, where $\tilde{x}_{(t)} = tx' + (1-t)x''$, and Theorem 1.6(d) can be interpreted as $\mathbb{E}[f(x_{(t)})] \leq \mathbb{E}[f(\tilde{x}_{(t)})]$.

In this paper we will construct the corresponding two mappings in connection with Levinson's inequality and show their monotonicity and convexity properties.

2 Main results

The following is our main result.

Theorem 2.1 *Let $f : [a,b] \to \mathbb{R}$ be 3-convex at point c, $x : \Omega \to [a,c]$ and $y : \Omega \to [c,b]$ such that* $\mathrm{Var}(x) = \mathrm{Var}(y)$ *and $H, V : [0,1] \to \mathbb{R}$ the mappings*

$$H(t) = \frac{1}{\mu(\Omega)} \int_\Omega \bigl[f\bigl(ty(s) + (1-t)\mathbb{E}[y]\bigr) - f\bigl(tx(s) + (1-t)\mathbb{E}[x]\bigr)\bigr]\,d\mu(s)$$

and

$$V(t) = \frac{1}{\mu(\Omega)^2} \int_\Omega \int_\Omega \bigl[f\bigl(ty(s) + (1-t)y(u)\bigr) - f\bigl(tx(s) + (1-t)x(u)\bigr)\bigr]\,d\mu(s)\,d\mu(u).$$

Then:

(a) *the mappings H and V are convex on* $[0,1]$,

(b) *the mapping H is nondecreasing on* $[0,1]$, *while the mapping V is nonincreasing on* $[0,\frac{1}{2}]$ *and nondecreasing on* $[\frac{1}{2},1]$,

(c) *the following equalities hold*:

$$\inf_{t\in[0,1]} H(t) = H(0) = f\bigl(\mathbb{E}[y]\bigr) - f\bigl(\mathbb{E}[x]\bigr),$$

$$\sup_{t\in[0,1]} H(t) = H(1) = \mathbb{E}\bigl[f(y)\bigr] - \mathbb{E}\bigl[f(x)\bigr],$$

$$\begin{aligned}\inf_{t\in[0,1]} V(t) = V\left(\frac{1}{2}\right) = \frac{1}{\mu(\Omega)^2} \int_\Omega \int_\Omega \biggl[& f\left(\frac{y(s)+y(u)}{2}\right) \\ & - f\left(\frac{x(s)+x(u)}{2}\right)\biggr]\,d\mu(s)\,d\mu(u),\end{aligned}$$

$$\sup_{t\in[0,1]} V(t) = V(0) = V(1) = \mathbb{E}\bigl[f(y)\bigr] - \mathbb{E}\bigl[f(x)\bigr],$$

(d) *the following inequality holds for all* $t \in [0,1]$:

$$V(t) \geq \max\{H(t), H(1-t)\}.$$

Proof Let the constant A be as in Definition 1.3, *i.e.*, such that the function $F(s) = f(s) - \frac{A}{2}s^2$ is concave on $[a,c]$ and convex on $[c,b]$.

Since the function y takes values in $[c,b]$, so does the function $y_{(t)} = ty + (1-t)\mathbb{E}[y]$ for every $t \in [0,1]$. Furthermore, since the function F is convex on $[c,b]$, by Theorem 1.6 the mapping

$$H_1(t) = \frac{1}{\mu(\Omega)} \int_\Omega F\bigl(ty(s) + (1-t)\mathbb{E}[y]\bigr)\, d\mu(s)$$

is convex and nondecreasing on $[0,1]$. We have

$$\begin{aligned} H_1(t) &= \frac{1}{\mu(\Omega)} \int_\Omega f\bigl(ty(s) + (1-t)\mathbb{E}[y]\bigr)\, d\mu(s) \\ &\quad - \frac{A}{2\mu(\Omega)} \int_\Omega \bigl(ty(s) + (1-t)\mathbb{E}[y]\bigr)^2\, d\mu(s) \\ &= \mathbb{E}\bigl[f(y_{(t)})\bigr] - \frac{A}{2}t^2\mathbb{E}\bigl[y^2\bigr] - At(1-t)\mathbb{E}[y]\mathbb{E}[y] - \frac{A}{2}(1-t)^2\mathbb{E}^2[y] \\ &= \mathbb{E}\bigl[f(y_{(t)})\bigr] - \frac{A}{2}t^2\bigl(\mathbb{E}\bigl[y^2\bigr] - \mathbb{E}^2[y]\bigr) - \frac{A}{2}\mathbb{E}^2[y] \\ &= \mathbb{E}\bigl[f(y_{(t)})\bigr] - \frac{A}{2}t^2 \operatorname{Var}(y) - \frac{A}{2}\mathbb{E}^2[y]. \end{aligned}$$

Similarly, the function $x_{(t)} = tx + (1-t)\mathbb{E}[x]$ takes values in $[a,c]$ for every $t \in [0,1]$ and $-F$ is convex on $[a,c]$, so by Theorem 1.6 the mapping

$$H_2(t) = -\frac{1}{\mu(\Omega)} \int_\Omega F\bigl(tx(s) + (1-t)\mathbb{E}[x]\bigr)\, d\mu(s)$$

is convex and nondecreasing on $[0,1]$, and we have

$$\begin{aligned} H_2(t) &= -\frac{1}{\mu(\Omega)} \int_\Omega f\bigl(tx(s) + (1-t)\mathbb{E}[x]\bigr)\, d\mu(s) \\ &\quad + \frac{A}{2\mu(\Omega)} \int_\Omega \bigl(tx(s) + (1-t)\mathbb{E}[x]\bigr)^2\, d\mu(s) \\ &= -\mathbb{E}\bigl[f(x_{(t)})\bigr] + \frac{A}{2}t^2 \operatorname{Var}(x) + \frac{A}{2}\mathbb{E}^2[x]. \end{aligned}$$

Let us also denote the (constant) mapping $H_3(t) = \frac{A}{2}(\mathbb{E}^2[y] - \mathbb{E}^2[x])$. All three of the mappings H_i, $i = 1,2,3$, are convex and nondecreasing and, therefore, so is their sum. Since $\operatorname{Var}(x) = \operatorname{Var}(y)$, we have $H = H_1 + H_2 + H_3$, and this proves the convexity and monotonicity properties of H from parts (a) and (b), while the first two equalities in (c) follow by simple calculation.

As for the mapping V, first of all, it is easy to see that $V(t) = V(1-t)$ for all $t \in [0,1]$, that is, V is symmetric with respect to $t = \frac{1}{2}$. Next, since y takes values in $[c,b]$ and F is convex on that interval, by Theorem 1.6 the mapping

$$V_1(t) = \frac{1}{\mu(\Omega)^2}\int_\Omega\int_\Omega F\bigl(ty(s)+(1-t)y(u)\bigr)\,d\mu(s)\,d\mu(u)$$

is convex on $[0,1]$ and nondecreasing on $[\frac{1}{2},1]$. We have

$$\begin{aligned}
V_1(t) &= \frac{1}{\mu(\Omega)^2}\int_\Omega\int_\Omega f\bigl(ty(s)+(1-t)y(u)\bigr)\,d\mu(s)\,d\mu(u) \\
&\quad - \frac{A}{2\mu(\Omega)^2}\int_\Omega\int_\Omega \bigl(ty(s)+(1-t)y(u)\bigr)^2\,d\mu(s)\,d\mu(u)\,d\mu(u) \\
&= \frac{1}{\mu(\Omega)^2}\int_\Omega\int_\Omega f\bigl(ty(s)+(1-t)y(u)\bigr)\,d\mu(s)\,d\mu(u) \\
&\quad - \frac{A}{2}t^2\mathbb{E}\bigl[y^2\bigr] - At(1-t)\mathbb{E}[y]\mathbb{E}[y] - \frac{A}{2}(1-t)^2\mathbb{E}\bigl[y^2\bigr] \\
&= \frac{1}{\mu(\Omega)^2}\int_\Omega\int_\Omega f\bigl(ty(s)+(1-t)y(u)\bigr)\,d\mu(s)\,d\mu(u) \\
&\quad + At(1-t)\bigl(\mathbb{E}\bigl[y^2\bigr] - \mathbb{E}^2[y]\bigr) - \frac{A}{2}\mathbb{E}\bigl[y^2\bigr] \\
&= \frac{1}{\mu(\Omega)^2}\int_\Omega\int_\Omega f\bigl(ty(s)+(1-t)y(u)\bigr)\,d\mu(s)\,d\mu(u) \\
&\quad + At(1-t)\operatorname{Var}(y) - \frac{A}{2}\mathbb{E}\bigl[y^2\bigr].
\end{aligned}$$

Similarly, since x takes values in $[a,c]$ and $-F$ is convex on that interval, by Theorem 1.6 the mapping

$$V_2(t) = -\frac{1}{\mu(\Omega)^2}\int_\Omega\int_\Omega F\bigl(tx(s)+(1-t)x(u)\bigr)\,d\mu(s)\,d\mu(u)$$

is convex on $[0,1]$ and nondecreasing on $[\frac{1}{2},1]$ and we have

$$\begin{aligned}
V_2(t) &= -\frac{1}{\mu(\Omega)^2}\int_\Omega\int_\Omega f\bigl(tx(s)+(1-t)x(u)\bigr)\,d\mu(s)\,d\mu(u) \\
&\quad - At(1-t)\operatorname{Var}(x) + \frac{A}{2}\mathbb{E}\bigl[x^2\bigr].
\end{aligned}$$

Let us also denote the (constant) mapping $V_3(t) = \frac{A}{2}(\mathbb{E}[y^2] - \mathbb{E}[x^2])$. All three of the mappings V_i, $i = 1,2,3$, are convex and nondecreasing on $[\frac{1}{2},1]$ and, therefore, so is their sum. Since $\operatorname{Var}(x) = \operatorname{Var}(y)$, we have $V = V_1 + V_2 + V_3$. Furthermore, since V is symmetric around $t = \frac{1}{2}$, it follows that it is nonincreasing on $[0,\frac{1}{2}]$, its minimum is attained at $t = \frac{1}{2}$ and its maximum is attained at $t = 0$ and $t = 1$. This proves the convexity and monotonicity properties of V.

Finally, as for part (d), since V is symmetric around $t = \frac{1}{2}$ and H is nondecreasing, it is enough to prove that $V(t) \geq H(t)$ for $t \in [\frac{1}{2},1]$. This inequality holds since $V_1(t) \geq H_1(t)$ and $V_2(t) \geq H_2(t)$ by Theorem 1.6(d) and $V_3(t) = H_3(t)$ since $\operatorname{Var}(x) = \operatorname{Var}(y)$ and this finishes the proof. □

A monotonic refinement of Levinson's inequality (4) based on Theorem 2.1 is the following: if $x_{(t)}$ and $y_{(t)}$ for $t \in [0,1]$ are as in the proof of Theorem 2.1, then $H(t) = \mathbb{E}[f(y_{(t)})] - \mathbb{E}[f(x_{(t)})]$ and for $0 \leq s \leq t \leq 1$ it holds

$$f\bigl(\mathbb{E}[y]\bigr) - f\bigl(\mathbb{E}[x]\bigr) = H(0) \leq \mathbb{E}\bigl[f(y_{(s)})\bigr] - \mathbb{E}\bigl[f(x_{(s)})\bigr] \\ \leq \mathbb{E}\bigl[f(y_{(t)})\bigr] - \mathbb{E}\bigl[f(x_{(t)})\bigr] \leq H(1) = \mathbb{E}\bigl[f(y)\bigr] - \mathbb{E}\bigl[f(x)\bigr].$$

Remark 2.2 The convexity and monotonicity property of the mapping H in the case when x and y are two discrete random variables taking values x_i and y_i, respectively, with probabilities p_i, $i = 1, \ldots, n$, was proven in [7].

Remark 2.3 The assumption of equal variances in Theorem 2.1 can be weakened. If we denote $B = A(\operatorname{Var}(y) - \operatorname{Var}(x))$, then the assumption $\operatorname{Var}(x) = \operatorname{Var}(y)$ can be relaxed to $B \geq 0$. Indeed, what we have shown in the proof of Theorem 2.1 is that

$$H = \sum_{i=1}^{4} H_i \quad \text{and} \quad V = \sum_{i=1}^{4} V_i,$$

where $H_4(t) = \frac{1}{2}Bt^2$ and $V_4(t) = Bt(t-1)$. For $B \geq 0$ the mapping H_4 is convex and nondecreasing, while the mapping V_4 is convex, symmetric around $t = \frac{1}{2}$ and nondecreasing on $[\frac{1}{2}, 1]$. Therefore, the convexity and monotonicity properties of H and V are preserved.

Furthermore, $V_3(t) - H_3(t) = \frac{1}{2}B$, so from $V_1(t) \geq H_1(t)$, $V_2(t) \geq H_2(t)$ and $V_3(t) + V_4(t) - H_3(t) - H_4(t) = \frac{1}{2}B(1-t)^2 \geq 0$ it follows that $V(t) \geq H(t)$, *i.e.*, part (d) also holds.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results, and they read and approved the final manuscript.

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Acknowledgements
This work has been fully supported by Croatian Science Foundation under the project 5435.

Received: 2 November 2014 Accepted: 30 April 2015 Published online: 16 May 2015

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