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A reverse Hölder inequality for α, β-symmetric integral and some related results

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Abstract

In this paper, we establish a reversed Hölder inequality via an α, β-symmetric integral, which is defined as a linear combination of the α-forward and the β-backward integrals, and then we give some generalizations of the α, β-symmetric integral Hölder inequality which is due to Brito da Cruz *et al.*; some related inequalities are also given.

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1 Introduction

Let $a_k \geq 0$, $b_k \geq 0$ $(k = 1, 2, \dots, n)$, $p > 1$, $1/p + 1/q = 1$. The classical Hölder inequality [1] is stated as follows:

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p \right)^{1/p} \left(\sum_{k=1}^{n} b_k^q \right)^{1/q}. \tag{1.1}$$

Similarly, the integral version of the Hölder inequality [1] is

$$\int_a^b f(x)g(x)\,dx \leq \left(\int_a^b f^p(x)\,dx \right)^{1/p} \left(\int_a^b g^q(x)\,dx \right)^{1/q}, \tag{1.2}$$

where $f(x) > 0$, $g(x) > 0$, $p > 1$, $1/p + 1/q = 1$, and $f(x)$ and $g(x)$ are continuous real-valued functions on $[a, b]$.

If $p = q = 2$, then inequalities (1.1) and (1.2) reduce to the well-known Cauchy inequalities [2] of the discrete form and the continuous form, respectively.

The Hölder inequality and Cauchy inequality play an important role in many areas of pure and applied mathematics. A large number of generalizations, refinements, variations, and applications of these inequalities have been investigated in [3–18] and references therein. Recently, Brito da Cruz *et al.* in [19] gave a α, β-symmetric integral Hölder inequality as follows.

Let $f,g:\mathrm{R}\to\mathrm{R}$ and $a,b\in\mathrm{R}$ with $a<b$. If $|f|$ and $|g|$ are α,β-symmetric integrable on $[a,b]$, $p>1$ with $q=p/(p-1)$. Then

$$\int_a^b \bigl|f(t)g(t)\bigr|\,d_{\alpha,\beta}t \le \left(\int_a^b \bigl|f(t)\bigr|^p\,d_{\alpha,\beta}t\right)^{1/p}\left(\int_a^b \bigl|g(t)\bigr|^q\,d_{\alpha,\beta}t\right)^{1/q}, \tag{1.3}$$

with equality if and only if the functions $|f|$ and $|g|$ are proportional.

The aim of this work is to establish a reversed version of the α,β-symmetric integral Hölder inequality and some generalizations of the α,β-symmetric integral Hölder inequality. Moreover, the obtained results will be applied to establish the α,β-symmetric integral reverse Minkowski inequality, Dresher inequality, and their corresponding reverse versions. This paper is organized as follows. In Section 2, we recall some basic definitions and properties of α,β-symmetric integral, which can also be found in [19, 20]; in Section 3, we establish a α,β-symmetric integral reverse Hölder inequality and give some generalizations of the α,β-symmetric integral Hölder inequality, we apply the obtained results to establish the reverse Minkowski inequality, Dresher inequality, and its reverse form involving α,β-symmetric integral, some extensions of the Minkowski and Dresher inequalities are also given; in Section 4, we establish some further generalizations and refinements of the α,β-symmetric integral Hölder inequality; in Section 5, we present a subdividing of the α,β-symmetric integral Hölder inequality.

2 Preliminaries

In the section, we recall some basic definitions and properties of α,β-symmetric integral.

The α-forward and β-backward differences are defined as follows (see [19]):

$$\Delta_\alpha[f](t):=\frac{f(t+\alpha)-f(t)}{\alpha},\qquad \nabla_\beta[f](t):=\frac{f(t)-f(t-\beta)}{\beta},$$

where $\alpha>0$, $\beta>0$.

Definition 2.1 (see [19]) Assume that $I\subseteq\mathrm{R}$ with $\sup I=+\infty$ and let $a,b\in I$ with $a<b$. If $f:I\to\mathrm{R}$ and $\alpha>0$, then the α-forward integral of f is defined by

$$\int_a^b f(t)\Delta_\alpha t=\int_a^\infty f(t)\Delta_\alpha t-\int_b^\infty f(t)\Delta_\alpha t,$$

where $\int_x^{+\infty} f(t)\Delta_\alpha t=\alpha\sum_{k=0}^{+\infty} f(x+k\alpha)$, provided the series converges at $x=a$ and $x=b$.

The α-forward integral has the following properties.

Proposition 2.1 (see [19]) *Assume that* $f,g:I\to\mathrm{R}$ *are* α*-forward integrable on* $[a,b]$, $c\in[a,b]$, $k\in\mathrm{R}$, *then*:

1. $\int_a^a f(t)\Delta_\alpha t=0$;
2. $\int_a^b f(t)\Delta_\alpha t=\int_a^c f(t)\Delta_\alpha t+\int_c^b f(t)\Delta_\alpha t$, *when the integrals exist*;
3. $\int_a^b f(t)\Delta_\alpha t=-\int_b^a f(t)\Delta_\alpha t$;
4. kf *is* α*-forward integrable on* $[a,b]$ *and*

$$\int_a^b kf(t)\Delta_\alpha t=k\int_a^b f(t)\Delta_\alpha t;$$

5. *$f+g$ is α-forward integrable on $[a,b]$ and*

$$\int_a^b (f+g)(t)\Delta_\alpha t = \int_a^b f(t)\Delta_\alpha t + \int_a^b g(t)\Delta_\alpha t;$$

6. *if $f \equiv 0$, then $\int_a^b f(t)\Delta_\alpha t = 0$.*

Proposition 2.2 (see [19]) *Assume that $f : I \to \mathrm{R}$ is α-forward integrable on $[a,b]$. Let $g : I \to \mathrm{R}$ be a nonnegative α-forward integrable function on $[a,b]$. Then fg is α-forward integrable on $[a,b]$.*

Proposition 2.3 (see [19]) *Assume that $f : I \to \mathrm{R}$ and let $|f|$ be α-forward integrable on $[a,b]$. If $p > 1$, then $|f|^p$ is also α-forward integrable on $[a,b]$.*

Proposition 2.4 (see [19]) *Assume that $f, g : I \to \mathrm{R}$ are α-forward integrable on $[a,b]$ with $b = a + k\alpha$ for some $k \in \mathrm{N}_0$. We have*:
1. *If $f(t) \geq 0$ for all $t \in \{a + k\alpha : k \in \mathrm{N}_0\}$, then $\int_a^b f(t)\Delta_\alpha t \geq 0$.*
2. *If $g(t) \geq f(t)$ for all $t \in \{a + k\alpha : k \in \mathrm{N}_0\}$, then $\int_a^b g(t)\Delta_\alpha t \geq \int_a^b f(t)\Delta_\alpha t$.*

Proposition 2.5 (Fundamental theorem of the α-forward integral, see [19]) *Assume that $f : I \to \mathrm{R}$ is α-forward integrable over I. Let $x \in I$ and define $F(x) = \int_a^x f(t)\Delta_\alpha t$. Then $\Delta_\alpha[F](x) = f(x)$. Conversely, $\int_a^b \Delta_\alpha[f](t)\Delta_\alpha t = f(b) - f(a)$.*

Proposition 2.6 (α-Forward integration by parts, see [19]) *Assume that $f, g : I \to \mathrm{R}$. Let fg and $f\Delta_\alpha[g]$ be α-forward integrable on $[a,b]$. Then*

$$\int_a^b f(t)\Delta_\alpha[g](t)\Delta_\alpha t = f(t)g(t)|_a^b - \int_a^b \Delta_\alpha[f](t)g(t+\alpha)\Delta_\alpha[g](t)\Delta_\alpha t.$$

Similarly, the β-backward integral is defined by Definition 2.2.

Definition 2.2 (see [19]) Assume that $I \subseteq \mathrm{R}$ with $\inf I = -\infty$ and let $a, b \in I$ with $a < b$. For $f : I \to \mathrm{R}$ and $\beta > 0$, the β-backward integral of f is defined by

$$\int_a^b f(t)\nabla_\beta t = \int_{-\infty}^b f(t)\nabla_\beta t - \int_{-\infty}^a f(t)\nabla_\beta t,$$

where $\int_{-\infty}^x f(t)\nabla_\beta t = \beta \sum_{k=0}^{+\infty} f(x - k\beta)$, provided the series converges at $x = a$ and $x = b$.

The β-backward integral has similar results and properties to the α-forward integral. In particular, the β-backward integral is the inverse operator of ∇_β.

We recall the α,β-symmetric integral which is defined as a linear combination of the α-forward and the β-backward integrals.

Definition 2.3 (see [19]) Assume that $f : \mathrm{R} \to \mathrm{R}$ and $a, b \in \mathrm{R}$ with $a < b$. If f is α-forward and β-backward integrable on $[a,b]$, $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$, then we define the α,β-symmetric integral of f from a to b by

$$\int_a^b f(t)\, d_{\alpha,\beta} t = \frac{\alpha}{\alpha+\beta}\int_a^b f(t)\Delta_\alpha t + \frac{\beta}{\alpha+\beta}\int_a^b f(t)\nabla_\beta t.$$

The function f is α,β-symmetric integrable if it is α,β-symmetric integrable for all $a,b \in \mathrm{R}$.

The α,β-symmetric integral has the following properties.

Proposition 2.7 (see [19]) *Suppose that $f,g : \mathrm{R} \to \mathrm{R}$ are α,β-symmetric integrable on $[a,b]$. Let $c \in [a,b]$ and $k \in \mathrm{R}$. Then*:

1. $\int_a^a f(t)\,d_{\alpha,\beta}t = 0$;
2. $\int_a^b f(t)\,d_{\alpha,\beta}t = \int_a^c f(t)\,d_{\alpha,\beta}t + \int_c^b f(t)\,d_{\alpha,\beta}t$, *when the integrals exist*;
3. $\int_a^b f(t)\,d_{\alpha,\beta}t = -\int_b^a f(t)\,d_{\alpha,\beta}t$;
4. *kf is α,β-symmetric integrable on $[a,b]$ and*

$$\int_a^b kf(t)\,d_{\alpha,\beta}t = k\int_a^b f(t)\,d_{\alpha,\beta}t;$$

5. *$f+g$ is α,β-symmetric integrable on $[a,b]$ and*

$$\int_a^b (f+g)(t)\,d_{\alpha,\beta}t = \int_a^b f(t)\,d_{\alpha,\beta}t + \int_a^b g(t)\,d_{\alpha,\beta}t;$$

6. *fg is α,β-symmetric integrable on $[a,b]$ provided g is a nonnegative function.*

Proposition 2.8 (see [19]) *Suppose that $f : \mathrm{R} \to \mathrm{R}$ and $p > 1$. Let $|f|$ be symmetric α,β-integrable on $[a,b]$, then $|f|^p$ is also α,β-symmetric integrable on $[a,b]$.*

Proposition 2.9 (see [19]) *Assume that $f,g : \mathrm{R} \to \mathrm{R}$ are α,β-symmetric integrable functions on $[a,b]$, $A = \{a + k\alpha : k \in \mathrm{N}_0\}$ and $B = \{b - k\beta : k \in \mathrm{N}_0\}$. For $b \in A$ and $a \in B$, we have*:

1. *if $|f(t)| \leq g(t)$ for all $t \in A \cup B$, then*

$$\left|\int_a^b f(t)\,d_{\alpha,\beta}t\right| \leq \int_a^b g(t)\,d_{\alpha,\beta}t;$$

2. *if $f(t) \geq 0$ for all $t \in A \cup B$, then*

$$\int_a^b f(t)\,d_{\alpha,\beta}t \geq 0;$$

3. *if $g(t) \geq f(t)$ for all $t \in A \cup B$, then*

$$\int_a^b g(t)\,d_{\alpha,\beta}t \geq \int_a^b f(t)\,d_{\alpha,\beta}t.$$

In Proposition 2.9 we assume that $a,b \in \mathrm{R}$ with $b \in A = \{a + k\alpha : k \in \mathrm{N}_0\}$ and $a \in B = \{b - k\beta : k \in \mathrm{N}_0\}$, where $\alpha,\beta \in \mathrm{R}_0^+$, $\alpha + \beta \neq 0$.

Proposition 2.10 (Mean value theorem, see [19]) *If $f,g : \mathrm{R} \to \mathrm{R}$ are bounded and α,β-symmetric integrable on $[a,b]$ with g nonnegative. Let m and M be the infimum and the* *supremum, respectively, of the function f. Then there exists a real number K satisfying the inequalities $m \leq K \leq M$ such that*

$$\int_a^b f(t)g(t)\,d_{\alpha,\beta}t = K\int_a^b g(t)\,d_{\alpha,\beta}t.$$

3 Main results

As before, let $a, b \in \mathrm{R}$ with $b \in A = \{a + k\alpha : k \in \mathrm{N}_0\}$ and $a \in B = \{b - k\beta : k \in \mathrm{N}_0\}$, where $\alpha, \beta \in \mathrm{R}_0^+$, $\alpha + \beta \neq 0$.

Theorem 3.1 (Reverse Hölder inequality) *Let $f, g : \mathrm{R} \to \mathrm{R}$ and $a, b \in \mathrm{R}$ with $a < b$. If $|f|$ and $|g|$ are α,β-symmetric integrable on $[a, b]$, $0 < p < 1$ (or $p < 0$) with $q = p/(p-1)$, then*

$$\int_a^b \bigl|f(t)g(t)\bigr|\,d_{\alpha,\beta}t \geq \left(\int_a^b \bigl|f(t)\bigr|^p\,d_{\alpha,\beta}t\right)^{1/p}\left(\int_a^b \bigl|g(t)\bigr|^q\,d_{\alpha,\beta}t\right)^{1/q}, \tag{3.1}$$

with equality if and only if the functions $|f|$ and $|g|$ are proportional.

Proof We assume that

$$\left(\int_a^b \bigl|f(t)\bigr|^p\,d_{\alpha,\beta}t\right)^{1/p}\left(\int_a^b \bigl|g(t)\bigr|^q\,d_{\alpha,\beta}t\right)^{1/q} \neq 0,$$

and let

$$\xi(t) = \bigl|f(t)\bigr|^p \Big/ \int_a^b \bigl|f(\tau)\bigr|^p\,d_{\alpha,\beta}\tau$$

and

$$\gamma(t) = \bigl|g(t)\bigr|^q \Big/ \int_a^b \bigl|f(\tau)\bigr|^q\,d_{\alpha,\beta}\tau.$$

Since the two functions $\xi(t)$ and $\gamma(t)$ are symmetric α,β-integrable on $[a, b]$, applying the following reverse Young inequality (see [21]):

$$xy \geq \frac{1}{p}x^p + \frac{1}{q}y^q, \quad x, y > 0, 0 < p < 1, 1/p + 1/q = 1,$$

with equality holding iff $x = y$, we have

$$\begin{aligned}
&\int_a^b \frac{|f(t)|}{(\int_a^b |f(\tau)|^p\,d_{\alpha,\beta}\tau)^{1/p}}\frac{|g(t)|}{(\int_a^b |g(\tau)|^q\,d_{\alpha,\beta}\tau)^{1/q}}\,d_{\alpha,\beta}t \\
&\quad = \int_a^b \xi^{1/p}(t)\gamma^{1/q}(t)\,d_{\alpha,\beta}t \geq \int_a^b \left(\frac{\xi(t)}{p} + \frac{\gamma(t)}{p}\right)d_{\alpha,\beta}t \\
&\quad = \frac{1}{p}\int_a^b \left(\frac{|f(t)|^p}{\int_a^b |f(\tau)|^p\,d_{\alpha,\beta}\tau}\right)d_{\alpha,\beta}t + \frac{1}{q}\int_a^b \left(\frac{|g(t)|^q}{\int_a^b |g(\tau)|^q\,d_{\alpha,\beta}\tau}\right)d_{\alpha,\beta}t = 1.
\end{aligned}$$

Therefore, we obtain the desired inequality. □

Combining (1.3) and (3.1), we have Corollary 3.1.

Corollary 3.1 *Let $f_j : \mathrm{R} \to \mathrm{R}$, $p_j \in \mathrm{R}$, $j = 1, 2, \ldots, m$, $\sum_{j=1}^{m} 1/p_j = 1$. If $|f_j|$ is α, β-symmetric integrable on $[a, b]$, then*:

(1) *For $p_j > 1$, we have*

$$\int_a^b \prod_{j=1}^{m} |f_j(t)| \, d_{\alpha,\beta} t \le \prod_{j=1}^{m} \left(\int_a^b |f_j(t)|^{p_j} \, d_{\alpha,\beta} t \right)^{1/p_j}. \tag{3.2}$$

(2) *For $0 < p_1 < 1$, $p_j < 0$, $j = 2, \ldots, m$, we have*

$$\int_a^b \prod_{j=1}^{m} |f_j(t)| \, d_{\alpha,\beta} t \ge \prod_{j=1}^{m} \left(\int_a^b |f_j(t)|^{p_j} \, d_{\alpha,\beta} t \right)^{1/p_j}. \tag{3.3}$$

Theorem 3.2 (Reverse Minkowski inequality) *Let $f, g : \mathrm{R} \to \mathrm{R}$ and $a, b, p \in \mathrm{R}$ with $a < b$ and $0 < p < 1$ (or $p < 0$). If $|f|$ and $|g|$ are α, β-symmetric integrable on $[a, b]$, then*

$$\begin{aligned} &\left(\int_a^b |f(t) + g(t)|^p \, d_{\alpha,\beta} t \right)^{1/p} \\ &\quad \ge \left(\int_a^b |f(t)|^p \, d_{\alpha,\beta} t \right)^{1/p} + \left(\int_a^b |g(t)|^p \, d_{\alpha,\beta} t \right)^{1/p}, \end{aligned} \tag{3.4}$$

with equality if and only if the functions $|f|$ and $|g|$ are proportional.

Proof Let

$$\begin{aligned} &M = \int_a^b |f(t)|^p \, d_{\alpha,\beta} t, \qquad N = \int_a^b |g(t)|^p \, d_{\alpha,\beta} t, \\ &W = \left(\int_a^b |f(t)|^p \, d_{\alpha,\beta} t \right)^{1/p} + \left(\int_a^b |g(t)|^p \, d_{\alpha,\beta} t \right)^{1/p}. \end{aligned}$$

By the α, β-symmetric integral Hölder inequality [12], in view of $0 < p < 1$, we have

$$\begin{aligned} W &= \int_a^b \left(|f(t)|^p M^{1/p-1} + |g(t)|^p N^{1/p-1} \right) d_{\alpha,\beta} t \\ &\le \int_a^b |f(t) + g(t)|^p \left(M^{1/p} + N^{1/p} \right)^{1-p} d_{\alpha,\beta} t \\ &= W^{1-p} \int_a^b |f(t) + g(t)|^p \, d_{\alpha,\beta} t. \end{aligned} \tag{3.5}$$

By using (3.5), we immediately arrive at the Minkowski inequality and the theorem is completely proved. □

An improvement of inequality (3.4) and its corresponding reverse form are obtained in Theorem 3.3.

Theorem 3.3 *Assume that $f, g : \mathrm{R} \to \mathrm{R}$ and $a, b \in \mathrm{R}$ with $a < b$. Assume that $|f|$ and $|g|$ are α, β-symmetric integrable on $[a, b]$, $p > 0$, $s, t \in \mathrm{R} \backslash \{0\}$, and $s \ne t$.*

(1) *Suppose that* $p, s, t \in \mathrm{R}$ *are different, such that* $s, t > 1$ *and* $(s-t)/(p-t) > 1$, *then*

$$\begin{aligned}
&\int_a^b |f(x)+g(x)|^p \, d_{\alpha,\beta}x \\
&\quad \leq \left[\left(\int_a^b |f(x)|^s \, d_{\alpha,\beta}x\right)^{\frac{1}{s}} + \left(\int_a^b |g(x)|^s \, d_{\alpha,\beta}x\right)^{\frac{1}{s}}\right]^{s(p-t)/(s-t)} \\
&\qquad \times \left[\left(\int_a^b |f(x)|^t \, d_{\alpha,\beta}x\right)^{\frac{1}{t}} + \left(\int_a^b |g(x)|^t \, d_{\alpha,\beta}x\right)^{\frac{1}{t}}\right]^{t(s-p)/(s-t)}.
\end{aligned} \tag{3.6}$$

(2) *Suppose that* $p, s, t \in \mathrm{R}$ *are different, such that* $0 < s, t < 1$ *and* $(s-t)/(p-t) < 1$, *then*

$$\begin{aligned}
&\int_a^b |f(x)+g(x)|^p \, d_{\alpha,\beta}x \\
&\quad \geq \left[\left(\int_a^b |f(x)|^s \, d_{\alpha,\beta}x\right)^{\frac{1}{s}} + \left(\int_a^b |g(x)|^s \, d_{\alpha,\beta}x\right)^{\frac{1}{s}}\right]^{s(p-t)/(s-t)} \\
&\qquad \times \left[\left(\int_a^b |f(x)|^t \, d_{\alpha,\beta}x\right)^{\frac{1}{t}} + \left(\int_a^b |g(x)|^t \, d_{\alpha,\beta}x\right)^{\frac{1}{t}}\right]^{t(s-p)/(s-t)}.
\end{aligned} \tag{3.7}$$

Proof (1) From the assumption, we have $(s-t)/(p-t) > 1$, and it is obvious that

$$\begin{aligned}
&\int_a^b |f(x)+g(x)|^p \, d_{\alpha,\beta}x \\
&\quad = \int_a^b \left(|f(x)+g(x)|^s\right)^{(p-t)/(s-t)} \left(|f(x)+g(x)|^t\right)^{(s-p)/(s-t)} \, d_{\alpha,\beta}x.
\end{aligned}$$

From the Hölder inequality (see [19]) with indices $(s-t)/(p-t)$ and $(s-t)/(s-p)$, it follows that

$$\begin{aligned}
&\int_a^b |f(x)+g(x)|^p \, d_{\alpha,\beta}x \\
&\quad \leq \left(\int_a^b |f(x)+g(x)|^s \, d_{\alpha,\beta}x\right)^{(p-t)/(s-t)} \left(\int_a^b |f(x)+g(x)|^t \, d_{\alpha,\beta}x\right)^{(s-p)/(s-t)}.
\end{aligned} \tag{3.8}$$

On the other hand, from the Minkowski inequality (see [19]) for $s > 1$ and $t > 1$, respectively, we obtain

$$\begin{aligned}
&\left(\int_a^b |f(x)+g(x)|^s \, d_{\alpha,\beta}x\right)^{\frac{1}{s}} \\
&\quad \leq \left(\int_a^b |f(x)|^s \, d_{\alpha,\beta}x\right)^{\frac{1}{s}} + \left(\int_a^b |g(x)|^s \, d_{\alpha,\beta}x\right)^{\frac{1}{s}}
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&\left(\int_a^b |f(x)+g(x)|^t \, d_{\alpha,\beta}x\right)^{\frac{1}{t}} \\
&\quad \leq \left(\int_a^b |f(x)|^t \, d_{\alpha,\beta}x\right)^{\frac{1}{t}} + \left(\int_a^b |g(x)|^t \, d_{\alpha,\beta}x\right)^{\frac{1}{t}}.
\end{aligned} \tag{3.10}$$

From (3.8), (3.9), and (3.10), the desired result is obtained.

(2) Based on the assumption, we have $(s-t)/(p-t) < 1$ and in view of

$$\begin{aligned}&\int_a^b \left|f(x)+g(x)\right|^p d_{\alpha,\beta}x \\ &\quad= \int_a^b \left(\left|f(x)+g(x)\right|^s\right)^{(p-t)/(s-t)} \left(\left|f(x)+g(x)\right|^t\right)^{(s-p)/(s-t)} d_{\alpha,\beta}x,\end{aligned}$$

by using inequality (3.1) with indices $(s-t)/(p-t)$ and $(s-t)/(s-p)$, we have

$$\begin{aligned}&\int_a^b \left|f(x)+g(x)\right|^p d_{\alpha,\beta}x \\ &\quad\geq \left(\int_a^b \left|f(x)+g(x)\right|^s d_{\alpha,\beta}x\right)^{(p-t)/(s-t)} \left(\int_a^b \left|f(x)+g(x)\right|^t d_{\alpha,\beta}x\right)^{(s-p)/(s-t)}. \qquad (3.11)\end{aligned}$$

On the other hand, thanks to the Minkowski inequality (3.4) for the cases of $0 < s < 1$ and $0 < t < 1$,

$$\begin{aligned}&\left(\int_a^b \left|f(x)+g(x)\right|^s d_{\alpha,\beta}x\right)^{\frac{1}{s}} \\ &\quad\geq \left(\int_a^b \left|f(x)\right|^s d_{\alpha,\beta}x\right)^{\frac{1}{s}} + \left(\int_a^b \left|g(x)\right|^s d_{\alpha,\beta}x\right)^{\frac{1}{s}} \qquad (3.12)\end{aligned}$$

and

$$\begin{aligned}&\left(\int_a^b \left|f(x)+g(x)\right|^t d_{\alpha,\beta}x\right)^{\frac{1}{t}} \\ &\quad\geq \left(\int_a^b \left|f(x)\right|^t (dx)^{\alpha}\right)^{\frac{1}{t}} + \left(\int_a^b \left|g(x)\right|^t d_{\alpha,\beta}x\right)^{\frac{1}{t}}. \qquad (3.13)\end{aligned}$$

It follows from (3.11), (3.12), and (3.13) that the desired result is obtained. □

Remark 3.1

(1) For Theorem 3.3, for $p > 1$, letting $s = p + \varepsilon$, $t = p - \varepsilon$, when p, s, t are different, $s, t > 1$, and $(s-t)/(p-t)/2 > 1$, and letting $\varepsilon \to 0$, we obtain the result of [19].
(2) For Theorem 3.3, for $0 < p < 1$, letting $s = p + \varepsilon$, $t = p - \varepsilon$, when p, s, t are different, $0 < s, t < 1$, and $0 < (s-t)/(p-t)/2 < 1$, and letting $\varepsilon \to 0$, we obtain (3.4).

From the Minkowski inequality [19] and the reverse Minkowski inequality involving α, β-symmetric integral, we can deduce the following generalization.

Corollary 3.2 *Let $f_j : \mathrm{R} \to \mathrm{R}$, $j = 1, 2, \ldots, m$. If $|f_j|$ is α, β-symmetric integrable on $[a, b]$, then*:

(1) *For $p > 1$, we have*

$$\left(\int_a^b \left|\sum_{j=1}^m f_j(x)\right|^p d_{\alpha,\beta}x\right)^{1/p} \leq \sum_{j=1}^m \left(\int_a^b \left|f_j(x)\right|^p d_{\alpha,\beta}x\right)^{1/p}. \qquad (3.14)$$

(2) *For* $0 < p < 1$, *we have*

$$\left(\int_a^b \left|\sum_{j=1}^m f_j(x)\right|^p d_{\alpha,\beta}x\right)^{1/p} \geq \sum_{j=1}^m \left(\int_a^b |f_j(x)|^p d_{\alpha,\beta}x\right)^{1/p}. \tag{3.15}$$

Corollary 3.3 is an analog of Corollary 3.2.

Corollary 3.3 *Let* $f_j : \mathrm{R} \to \mathrm{R}$, $j = 1, 2, \ldots, m$. *If* $|f_j|$ *is* α,β-*symmetric integrable on* $[a,b]$, *then*:

(1) *For* $p > 1$, *we have*

$$\int_a^b \left(\sum_{j=1}^m |f_j(x)|\right)^p d_{\alpha,\beta}x \geq \sum_{j=1}^m \int_a^b |f_j(x)|^p d_{\alpha,\beta}x. \tag{3.16}$$

(2) *For* $0 < p < 1$, *we have*

$$\int_a^b \left(\sum_{j=1}^m |f_j(x)|\right)^p d_{\alpha,\beta}x \leq \sum_{j=1}^m \int_a^b |f_j(x)|^p d_{\alpha,\beta}x. \tag{3.17}$$

Proof (1) For $p > 1$, let $s = p$, $r = 1$, by the Jensen inequality [3], it follows that

$$|f_1(x)| + |f_2(x)| + \cdots + |f_m(x)| \geq \left(|f_1(x)|^p + |f_2(x)|^p + \cdots + |f_m(x)|^p\right)^{1/p},$$

from the above inequality, we obtain

$$\left(|f_1(x)| + |f_2(x)| + \cdots + |f_m(x)|\right)^p \geq |f_1(x)|^p + |f_2(x)|^p + \cdots + |f_m(x)|^p,$$

by integrating the above inequality with respect to x, we obtain the desired result.

(2) For $0 < p < 1$, let $s = 1$, $r = p$, by the Jensen inequality [3], we have

$$|f_1(x)| + |f_2(x)| + \cdots + |f_m(x)| \leq \left(|f_1(x)|^p + |f_2(x)|^p + \cdots + |f_m(x)|^p\right)^{1/p},$$

it follows from the above inequality that

$$\left(|f_1(x)| + |f_2(x)| + \cdots + |f_m(x)|\right)^p \leq |f_1(x)|^p + |f_2(x)|^p + \cdots + |f_m(x)|^p,$$

by integrating the above inequality with respect to x, the desired result is obtained. □

Theorem 3.4 (Dresher inequality) *Let* $f, g : \mathrm{R} \to \mathrm{R}$ *and* $a, b \in \mathrm{R}$ *with* $a < b$. *If* $|f|$ *and* $|g|$ *are* α,β-*symmetric integrable on* $[a,b]$, $0 < r < 1 < p$, *then*

$$\begin{aligned}
&\left(\frac{\int_a^b |f(t)+g(t)|^p d_{\alpha,\beta}t}{\int_a^b |f(t)+g(t)|^r d_{\alpha,\beta}t}\right)^{1/(p-r)} \\
&\quad\leq \left(\frac{\int_a^b |f(t)|^p d_{\alpha,\beta}t}{\int_a^b |f(t)|^r d_{\alpha,\beta}t}\right)^{1/(p-r)} + \left(\frac{\int_a^b |g(t)|^p d_{\alpha,\beta}t}{\int_a^b |g(t)|^r d_{\alpha,\beta}t}\right)^{1/(p-r)},
\end{aligned} \tag{3.18}$$

with equality if and only if the functions $|f|$ *and* $|g|$ *are proportional.*

Proof Based on the α,β-symmetric integral Hölder and Minkowski inequalities [19], we have

$$\begin{aligned}
&\left(\int_a^b \left|f(t)+g(t)\right|^p d_{\alpha,\beta}t\right)^{1/(p-r)} \\
&\quad\le \left(\left(\int_a^b \left|f(t)\right|^p d_{\alpha,\beta}t\right)^{1/p} + \left(\int_a^b \left|g(t)\right|^p d_{\alpha,\beta}t\right)^{1/p}\right)^{p/(p-r)} \\
&\quad= \left(\left(\frac{\int_a^b |f(t)|^p d_{\alpha,\beta}t}{\int_a^b |f(t)|^r d_{\alpha,\beta}t}\right)^{1/p}\left(\int_a^b \left|f(t)\right|^r d_{\alpha,\beta}t\right)^{1/p}\right. \\
&\qquad\left. + \left(\frac{\int_a^b |g(t)|^p d_{\alpha,\beta}x}{\int_a^b |g(t)|^r d_{\alpha,\beta}t}\right)^{1/p}\left(\int_a^b \left|g(t)\right|^r d_{\alpha,\beta}t\right)^{1/p}\right)^{p/(p-r)} \\
&\quad\le \left(\left(\frac{\int_a^b |f(t)|^p d_{\alpha,\beta}t}{\int_a^b |f(t)|^r d_{\alpha,\beta}t}\right)^{1/(p-r)} + \left(\frac{\int_a^b |g(t)|^p d_{\alpha,\beta}t}{\int_a^b |g(t)|^r d_{\alpha,\beta}t}\right)^{1/(p-r)}\right) \\
&\qquad\times \left(\left(\int_a^b \left|f(t)\right|^r d_{\alpha,\beta}t\right)^{1/r} + \left(\int_a^b \left|g(t)\right|^r d_{\alpha,\beta}t\right)^{1/r}\right)^{r/(p-r)}.
\end{aligned} \tag{3.19}$$

From Theorem 3.2, we get

$$\begin{aligned}
&\left(\left(\int_a^b \left|f(t)\right|^r d_{\alpha,\beta}t\right)^{1/r} + \left(\int_a^b \left|g(t)\right|^r d_{\alpha,\beta}t\right)^{1/r}\right)^r \\
&\quad\le \int_a^b \left|f(t)+g(t)\right|^r d_{\alpha,\beta}t.
\end{aligned} \tag{3.20}$$

From (3.5) and (3.20), we get (3.18). Hence, the theorem is completely proved. □

Corollary 3.4 *Let $f_j : \mathrm{R} \to \mathrm{R}$, $0 < r < 1 < p$, $j = 1, 2, \ldots, m$. If $|f_j|$ is α,β-symmetric integrable on $[a,b]$, then*

$$\left(\frac{\int_a^b |\sum_{j=1}^m f_j(t)|^p d_{\alpha,\beta}t}{\int_a^b |\sum_{j=1}^m f_j(t)|^r d_{\alpha,\beta}t}\right)^{1/(p-r)} \le \sum_{j=1}^m \left(\frac{\int_a^b |f_j(t)|^p d_{\alpha,\beta}t}{\int_a^b |f_j(t)|^r d_{\alpha,\beta}t}\right)^{1/(p-r)}. \tag{3.21}$$

Theorem 3.5 (Reverse Dresher inequality) *Let $f, g : \mathrm{R} \to \mathrm{R}$ and $a, b \in \mathrm{R}$ with $a < b$. If $|f|$ and $|g|$ are α,β-symmetric integrable on $[a,b]$, $p \le 0 \le r \le 1$, then*

$$\begin{aligned}
&\left(\frac{\int_a^b |f(t)+g(t)|^p d_{\alpha,\beta}t}{\int_a^b |f(t)+g(t)|^r d_{\alpha,\beta}t}\right)^{1/(p-r)} \\
&\quad\ge \left(\frac{\int_a^b |f(t)|^p d_{\alpha,\beta}t}{\int_a^b |f(t)|^r d_{\alpha,\beta}t}\right)^{1/(p-r)} + \left(\frac{\int_a^b |g(t)|^p d_{\alpha,\beta}t}{\int_a^b |g(t)|^r d_{\alpha,\beta}t}\right)^{1/(p-r)},
\end{aligned} \tag{3.22}$$

with equality if and only if the functions $|f|$ and $|g|$ are proportional.

Proof Let $\alpha_1 \ge 0$, $\alpha_2 \ge 0$, $\beta_1 > 0$, and $\beta_2 > 0$, and $-1 < \lambda < 0$, using the following Radon inequality (see [3]):

$$\sum_{k=1}^n \frac{a_k^p}{b_k^{p-1}} < \frac{(\sum_{k=1}^n a_k)^p}{(\sum_{k=1}^n b_k)^{p-1}}, \quad x_k \ge 0, a_k > 0, 0 < p < 1,$$

we have

$$\frac{\alpha_1^{\lambda+1}}{\beta_1^{\lambda}} + \frac{\alpha_2^{\lambda+1}}{\beta_2^{\lambda}} \le \frac{(\alpha_1+\alpha_2)^{\lambda+1}}{(\beta_1+\beta_2)^{\lambda}}, \tag{3.23}$$

with equality if and only if (α) and (β) are proportional. Let

$$\alpha_1 = \left(\int_a^b |f|^p \, d_{\alpha,\beta}t\right)^{1/p}, \qquad \beta_1 = \left(\int_a^b |f|^r \, d_{\alpha,\beta}t\right)^{1/r}, \tag{3.24}$$

$$\alpha_2 = \left(\int_a^b |g|^p \, d_{\alpha,\beta}t\right)^{1/p}, \qquad \beta_2 = \left(\int_a^b |g|^r \, d_{\alpha,\beta}t\right)^{1/r}, \tag{3.25}$$

and set $\lambda = \frac{r}{p-r}$. From (3.23)-(3.25), we have

$$\begin{aligned}\frac{\alpha_1^{\lambda+1}}{\beta_1^{\lambda}} + \frac{\alpha_2^{\lambda+1}}{\beta_2^{\lambda}} &= \frac{(\int_a^b |f|^p \, d_{\alpha,\beta}t)^{(\lambda+1)/p}}{(\int_a^b |f|^r \, d_{\alpha,\beta}t)^{\lambda/r}} + \frac{(\int_a^b |g|^p \, d_{\alpha,\beta}t)^{(\lambda+1)/p}}{(\int_a^b |g|^r \, d_{\alpha,\beta}t)^{\lambda/r}} \\ &= \left(\frac{\int_a^b |f|^p \, d_{\alpha,\beta}t}{\int_a^b |f|^r \, d_{\alpha,\beta}t}\right)^{1/(p-r)} + \left(\frac{\int_a^b |g|^p \, d_{\alpha,\beta}t}{\int_a^b |g|^r \, d_{\alpha,\beta}t}\right)^{1/(p-r)} \le \frac{(\alpha_1+\alpha_2)^{\lambda+1}}{(\beta_1+\beta_2)^{\lambda}} \\ &= \frac{[(\int_a^b |f|^p \, d_{\alpha,\beta}t)^{1/p} + (\int_a^b |g|^p \, d_{\alpha,\beta}t)^{1/p}]^{p/(p-r)}}{[(\int_a^b |f|^r \, d_{\alpha,\beta}t)^{1/r} + (\int_a^b |g|^r \, d_{\alpha,\beta}t)^{1/r}]^{r/(p-r)}}.\end{aligned} \tag{3.26}$$

Since $-1 < \lambda = \frac{r}{p-r} < 0$, we may assume $p < 0 < r$, and by Theorem 3.2 and $0 < r \le 1$, we obtain, respectively,

$$\left[\left(\int_a^b |f|^p \, d_{\alpha,\beta}t\right)^{1/p} + \left(\int_a^b |g|^p \, d_{\alpha,\beta}t\right)^{1/p}\right]^p \ge \int_a^b |f+g|^p \, d_{\alpha,\beta}t, \tag{3.27}$$

with equality if and only if f and g are proportional, and

$$\left[\left(\int_a^b |f|^r \, d_{\alpha,\beta}t\right)^{1/r} + \left(\int_a^b |g|^r \, d_{\alpha,\beta}t\right)^{1/r}\right]^r \le \int_a^b |f+g|^r \, d_{\alpha,\beta}t, \tag{3.28}$$

with equality if and only if $|f|$ and $|g|$ are proportional.

From the equality conditions for (3.23), (3.27), and (3.28), it follows that the sign of equality in (3.22) holds if and only if $|f|$ and $|g|$ are proportional.

From (3.26)-(3.28), we obtain the reverse Dresher inequality and the theorem is completely proved. □

Corollary 3.5 *Let $f_j : \mathrm{R} \to \mathrm{R}$, $p \le 0 \le r < 1$, $j = 1, 2, \ldots, m$. If $|f_j|$ is α, β-symmetric integrable on $[a, b]$, then*

$$\left(\frac{\int_a^b |\sum_{j=1}^m f_j(t)|^p \, d_{\alpha,\beta}t}{\int_a^b |\sum_{j=1}^m f_j(t)|^r \, d_{\alpha,\beta}t}\right)^{1/(p-r)} \ge \sum_{j=1}^m \left(\frac{\int_a^b |f_j(t)|^p \, d_{\alpha,\beta}t}{\int_a^b |f_j(t)|^r \, d_{\alpha,\beta}t}\right)^{1/(p-r)}. \tag{3.29}$$

4 Some further generalizations of the Hölder inequality

Theorem 4.1 *Suppose that $p_k > 0$, $\alpha_{kj} \in \mathrm{R}$ ($j = 1, 2, \ldots, m$, $k = 1, 2, \ldots, s$), $\sum_k^s \frac{1}{p_k} = 1$, $\sum_{k=1}^s \alpha_{kj} = 0$, $f_j : \mathrm{R} \to \mathrm{R}$. If $|f_j|$ is α, β-symmetric integrable on $[a, b]$, then*:

(1) *For* $p_k > 1$, *we have*

$$\int_a^b \left|\prod_{j=1}^m f_j(t)\right| d_{\alpha,\beta}t \le \prod_{k=1}^s \left(\int_a^b \prod_{j=1}^m \left|f_j(t)\right|^{1+p_k\alpha_{kj}} d_{\alpha,\beta}t\right)^{1/p_k}. \tag{4.1}$$

(2) *For* $0 < p_s < 1$, $p_k < 0$ $(k = 1, 2, \ldots, s-1)$, *we have*

$$\int_a^b \left|\prod_{j=1}^m f_j(t)\right| d_{\alpha,\beta}t \ge \prod_{k=1}^s \left(\int_a^b \prod_{j=1}^m \left|f_j(t)\right|^{1+p_k\alpha_{kj}} d_{\alpha,\beta}t\right)^{1/p_k}. \tag{4.2}$$

Proof (1) Let

$$g_k(t) = \left(\prod_{j=1}^m f_j^{1+p_k\alpha_{kj}}(t)\right)^{1/p_k}. \tag{4.3}$$

Applying the assumptions $\sum_k^s \frac{1}{p_k} = 1$ and $\sum_{k=1}^s \alpha_{kj} = 0$, it follows from a direct computation that

$$\begin{aligned}
\prod_{k=1}^s g_k(t) &= g_1(t)g_2(t)\cdots g_s(t) \\
&= \left(\prod_{j=1}^m f_j^{1+p_1\alpha_{1j}}(t)\right)^{1/p_1}\left(\prod_{j=1}^m f_j^{1+p_2\alpha_{2j}}(t)\right)^{1/p_2}\cdots\left(\prod_{j=1}^m f_j^{1+p_s\alpha_{sj}}(t)\right)^{1/p_s} \\
&= \prod_{j=1}^m f_j^{1/p_1+\alpha_{1j}}(t)\prod_{j=1}^m f_j^{1/p_2+\alpha_{2j}}(t)\cdots\prod_{j=1}^m f_j^{1/p_s+\alpha_{sj}}(t) \\
&= \prod_{j=1}^m f_j^{1/p_1+1/p_2+\cdots 1/p_s+\alpha_{1j}+\alpha_{2j}+\cdots+\alpha_{sj}}(t) = \prod_{j=1}^m f_j(t).
\end{aligned}$$

That is,

$$\prod_{k=1}^s g_k(t) = \prod_{j=1}^m f_j(t).$$

It is obvious that

$$\int_a^b \left|\prod_{j=1}^m f_j(t)\right| d_{\alpha,\beta}t = \int_a^b \left|\prod_{k=1}^s g_k(t)\right| d_{\alpha,\beta}t. \tag{4.4}$$

From the Hölder inequality (3.2), it follows that

$$\int_a^b \left|\prod_{k=1}^s g_k(t)\right| d_{\alpha,\beta}t \le \prod_{k=1}^s \left(\int_a^b \left|g_k(t)\right|^{p_k} d_{\alpha,\beta}t\right)^{1/p_k}. \tag{4.5}$$

Substituting $g_k(t)$ in (4.5), we have inequality (4.1) immediately.

(2) The proof of inequality (4.2) is similar to the proof of inequality (4.1); by (4.3), (4.4), and (3.3), we obtain

$$\int_a^b \left|\prod_{k=1}^s g_k(t)\right| d_{\alpha,\beta}t \geq \prod_{k=1}^s \left(\int_a^b \left|g_k(t)\right|^{p_k} d_{\alpha,\beta}t\right)^{1/p_k}. \tag{4.6}$$

Substitution of $g_k(t)$ in Eq. (4.6) gives inequality (4.2) immediately. □

Remark 4.1 Let $s = m$, $\alpha_{kj} = -1/p_k$ for $j \neq k$, and $\alpha_{kk} = 1 - 1/p_k$. Then the inequalities (4.1) and (4.2) are respectively reduced to (3.2) and (3.3).

Many existing inequalities concerned with the Hölder inequality are special cases of the inequalities (4.1) and (4.2). For example, we have the following.

Corollary 4.1 *Under the assumptions of Theorem* 4.1, *assume that* $s = m$, $\alpha_{kj} = -t/p_k$ *for* $j \neq k$, *and* $\alpha_{kk} = t(1 - 1/p_k)$ *with* $t \in \mathrm{R}$, *then*:

(1) *For* $p_k > 1$, *one obtains*

$$\int_a^b \left|\prod_{j=1}^m f_j(t)\right| d_{\alpha,\beta}t \leq \prod_{k=1}^m \left(\int_a^b \left(\prod_{j=1}^m \left|f_j(t)\right|\right)^{1-t} \left(\left|f_k(t)\right|^{p_k}\right)^t d_{\alpha,\beta}t\right)^{1/p_k}. \tag{4.7}$$

(2) *For* $0 < p_m < 1$, $p_k < 0$ $(k = 1, 2, \dots, m-1)$, *one obtains*

$$\int_a^b \left|\prod_{j=1}^m f_j(t)\right| d_{\alpha,\beta}t \geq \prod_{k=1}^m \left(\int_a^b \left(\prod_{j=1}^m \left|f_j(t)\right|\right)^{1-t} \left(\left|f_k(t)\right|^{p_k}\right)^t d_{\alpha,\beta}t\right)^{1/p_k}. \tag{4.8}$$

Theorem 4.2 *Suppose that* $p_k > 0$, $r \in \mathrm{R}$, $\alpha_{kj} \in \mathrm{R}$ $(j = 1, 2, \dots, m, k = 1, 2, \dots, s)$, $\sum_k^s \frac{1}{p_k} = r$, $\sum_{k=1}^s \alpha_{kj} = 0$, $f_j : \mathrm{R} \to \mathrm{R}$. *If* $|f_j|$ *is* α, β*-symmetric integrable on* $[a, b]$, *then*:

(1) *For* $rp_k > 1$, *we have*

$$\int_a^b \left|\prod_{j=1}^m f_j(t)\right| d_{\alpha,\beta}t \leq \prod_{k=1}^s \left(\int_a^b \prod_{j=1}^m \left|f_j(t)\right|^{1+rp_k\alpha_{kj}} d_{\alpha,\beta}t\right)^{1/rp_k}. \tag{4.9}$$

(2) *For* $0 < rp_s < 1$, $rp_k < 0$ $(k = 1, 2, \dots, s-1)$, *we have*

$$\int_a^b \left|\prod_{j=1}^m f_j(t)\right| d_{\alpha,\beta}t \geq \prod_{k=1}^s \left(\int_a^b \prod_{j=1}^m \left|f_j(t)\right|^{1+rp_k\alpha_{kj}} d_{\alpha,\beta}t\right)^{1/rp_k}. \tag{4.10}$$

Proof (1) Since $rp_k > 1$ and $\sum_k^s \frac{1}{p_k} = r$, it follows that $\sum_k^s \frac{1}{rp_k} = 1$. Then by (4.1), we immediately find inequality (4.9).

(2) From $0 < rp_s < 1$, $rp_k < 0$, and $\sum_k^s \frac{1}{p_k} = r$, it follows that $\sum_k^s \frac{1}{rp_k} = 1$. By (4.2), we immediately get inequality (4.10). This completes the proof. □

From Theorem 4.2, we establish Corollary 4.2, which is a generalization of Theorem 4.2.

Corollary 4.2 *Under the assumptions of Theorem* 4.2, *assume that* $s = 2$, $p_1 = p$, $p_2 = q$, $\alpha_{1j} = -\alpha_{2j} = \alpha_j$, *then*:

(1) *For* $rp > 1$, *we get*

$$\int_a^b \left|\prod_{j=1}^m f_j(t)\right| d_{\alpha,\beta}t \\ \leq \left(\int_a^b \prod_{j=1}^m \left|f_j(t)\right|^{1+rp\alpha_j} d_{\alpha,\beta}t\right)^{1/rp} \left(\int_a^b \prod_{j=1}^m \left|f_j(t)\right|^{1-rq\alpha_j} d_{\alpha,\beta}t\right)^{1/rq}. \tag{4.11}$$

(2) *For* $0 < rp < 1$, *we get*

$$\int_a^b \left|\prod_{j=1}^m f_j(t)\right| d_{\alpha,\beta}t \\ \geq \left(\int_a^b \prod_{j=1}^m \left|f_j(t)\right|^{1+rp\alpha_j} d_{\alpha,\beta}t\right)^{1/rp} \left(\int_a^b \prod_{j=1}^m \left|f_j(t)\right|^{1-rq\alpha_j} d_{\alpha,\beta}t\right)^{1/rq}. \tag{4.12}$$

Now we present a refinement of inequalities (4.9) and (4.10), respectively.

Theorem 4.3 *Under the assumptions of Theorem* 4.2:

(1) *For* $rp_k > 1$, *one has*

$$\int_a^b \left|\prod_{j=1}^m f_j(t)\right| d_{\alpha,\beta}t \leq \varphi(c) \leq \prod_{k=1}^s \left(\int_a^b \prod_{j=1}^m \left|f_j(t)\right|^{1+rp_k\alpha_{kj}} d_{\alpha,\beta}t\right)^{1/rp_k}, \tag{4.13}$$

where

$$\varphi(c) \equiv \int_a^c \prod_{j=1}^m \left|f_j(t)\right| d_{\alpha,\beta}t + \prod_{k=1}^s \left(\int_c^b \prod_{j=1}^m \left|f_j(t)\right|^{1+rp_k\alpha_{kj}} d_{\alpha,\beta}t\right)^{1/rp_k}$$

is a nonincreasing function with $a \leq c \leq b$.

(2) *For* $0 < rp_s < 1$, $rp_k < 0$ $(k = 1, 2, \ldots, s-1)$, *one has*

$$\int_a^b \left|\prod_{j=1}^m f_j(t)\right| d_{\alpha,\beta}t \geq \varphi(c) \geq \prod_{k=1}^s \left(\int_a^b \prod_{j=1}^m \left|f_j(t)\right|^{1+rp_k\alpha_{kj}} d_{\alpha,\beta}t\right)^{1/rp_k}, \tag{4.14}$$

where

$$\varphi(c) \equiv \int_a^c \prod_{j=1}^m \left|f_j(t)\right| d_{\alpha,\beta}t + \prod_{k=1}^s \left(\int_c^b \prod_{j=1}^m \left|f_j(t)\right|^{1+rp_k\alpha_{kj}} d_{\alpha,\beta}t\right)^{1/rp_k}$$

is a nondecreasing function with $a \leq c \leq b$.

Proof (1) Let

$$g_k(t) = \left(\prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(t)\right)^{1/rp_k}.$$

By rearrangement and the assumptions of Theorem 4.2, it follows that

$$\prod_{j=1}^{m} f_j(t) = \prod_{k=1}^{s} g_k(t).$$

Then thanks to the Hölder inequality (3.2), we have

$$\begin{aligned}
\int_a^b \left|\prod_{j=1}^{m} f_j(t)\right| d_{\alpha,\beta}t &= \int_a^b \left|\prod_{k=1}^{s} g_k(t)\right| d_{\alpha,\beta}t \\
&= \int_a^c \left|\prod_{k=1}^{s} g_k(t)\right| d_{\alpha,\beta}t + \int_c^b \left|\prod_{k=1}^{s} g_k(t)\right| d_{\alpha,\beta}t \\
&\le \int_a^c \left|\prod_{k=1}^{s} g_k(t)\right| d_{\alpha,\beta}t + \prod_{k=1}^{s} \left(\int_c^b \left|g_k(t)\right|^{rp_k} d_{\alpha,\beta}t\right)^{1/rp_k} \\
&\le \prod_{k=1}^{s} \left(\int_a^c \left|g_k(t)\right|^{rp_k} d_{\alpha,\beta}t + \int_c^b \left|g_k(t)\right|^{rp_k} d_{\alpha,\beta}t\right)^{1/rp_k} \\
&= \prod_{k=1}^{s} \left(\int_a^b \left|g_k(t)\right|^{rp_k} d_{\alpha,\beta}t\right)^{1/rp_k} \\
&= \prod_{k=1}^{s} \left(\int_a^b \prod_{j=1}^{m} \left|f_j(t)\right|^{1+rp_k\alpha_{kj}} d_{\alpha,\beta}t\right)^{1/rp_k}.
\end{aligned}$$

Therefore, we obtain the desired result.

(2) The proof of inequality (4.14) is similar to the proof of inequality (4.13). □

5 A subdividing of the Hölder inequality

Theorem 5.1 *Let $f,g : \mathrm{R} \to \mathrm{R}$ and $a,b \in \mathrm{R}$ with $a < b$. Assume that $|f|$ and $|g|$ are α,β-symmetric integrable on $[a,b]$, and $s,t \in \mathrm{R}$, and let $p = (s-t)/1-t$, $p = (s-t)/(s-1)$.*

(1) *If $s < 1 < t$ or $s > 1 > t$, then*

$$\int_a^b \left|f(x)g(x)\right| d_{\alpha,\beta}x \le \left(\int_a^b \left|f(x)\right|^{sp} d_{\alpha,\beta}x\right)^{1/p^2} \left(\int_a^b \left|g(x)\right|^{tq} d_{\alpha,\beta}x\right)^{1/q^2} \times \left(\int_a^b \left|f(x)\right|^{tp} d_{\alpha,\beta}x \int_a^b \left|g(x)\right|^{sq} d_{\alpha,\beta}x\right)^{1/pq} \tag{5.1}$$

with equality if and only if $f(x)$ and $g(x)$ are proportional.

(2) *If $s > t > 1$ or $s < t < 1$; $t > s > 1$ or $t < s < 1$, then*

$$\int_a^b \left|f(x)g(x)\right| d_{\alpha,\beta}x \ge \left(\int_a^b \left|f(x)\right|^{sp} d_{\alpha,\beta}x\right)^{1/p^2} \left(\int_a^b \left|g(x)\right|^{tq} d_{\alpha,\beta}x\right)^{1/q^2} \times \left(\int_a^b \left|f(x)\right|^{tp} d_{\alpha,\beta}x \int_a^b \left|g(x)\right|^{sq} d_{\alpha,\beta}x\right)^{1/pq} \tag{5.2}$$

with equality if and only if $f(x)$ and $g(x)$ are proportional.

Proof (1) Set $p = \frac{s-t}{1-t}$, and it follows from $s < 1 < t$ or $s > 1 > t$ that

$$p = \frac{s-t}{1-t} > 1.$$

From inequality (1.3) with indices $\frac{s-t}{1-t}$ and $\frac{s-t}{s-1}$, it follows that

$$\begin{aligned}
&\int_a^b \left|f(x)g(x)\right| d_{\alpha,\beta}x \\
&\quad = \int_a^b \left|f(x)g(x)\right|^{s(1-t)/(s-t)} \left|f(x)g(x)\right|^{t(s-1)/(s-t)} d_{\alpha,\beta}x \\
&\quad \le \left(\int_a^b \left|f(x)g(x)\right|^s d_{\alpha,\beta}x\right)^{(1-t)/(s-t)} \left(\int_a^b \left|f(x)g(x)\right|^t d_{\alpha,\beta}x\right)^{(s-1)/(s-t)},
\end{aligned} \tag{5.3}$$

with equality if and only if $(fg)^s$ and $(fg)^t$ are proportional.

On the other hand, by the Hölder inequality again, for $p = \frac{s-t}{1-t} > 1$, the following two inequalities are obtained:

$$\begin{aligned}
&\int_a^b \left|f(x)g(x)\right|^s d_{\alpha,\beta}x \\
&\quad \le \left(\int_a^b \left|f(x)\right|^{s(s-t)/(1-t)} d_{\alpha,\beta}x\right)^{(1-t)/(s-t)} \left(\int_a^b \left|g(x)\right|^{s(s-t)/(s-1)} d_{\alpha,\beta}x\right)^{(s-1)/(s-t)},
\end{aligned} \tag{5.4}$$

with equality if and only if $f^{s(s-t)/(1-t)}$ and $g^{s(s-t)/(s-1)}$ are proportional, and

$$\begin{aligned}
&\int_a^b \left|f(x)g(x)\right|^t d_{\alpha,\beta}x \\
&\quad \le \left(\int_a^b \left|f(x)\right|^{t(s-t)/(1-t)} d_{\alpha,\beta}x\right)^{(1-t)/(s-t)} \left(\int_a^b \left|g(x)\right|^{t(s-t)/(s-1)} d_{\alpha,\beta}x\right)^{(s-1)/(s-t)},
\end{aligned} \tag{5.5}$$

with equality if and only if $f^{t(s-t)/(1-t)}$ and $g^{t(s-t)/(s-1)}$ are proportional.

It follows from (5.3), (5.4), and (5.5) that the case (1) of Theorem 5.1 is proved.

(2) Let $p = \frac{s-t}{1-t}$, and in view of $s > t > 1$ or $s < t < 1$, we have

$$p = \frac{s-t}{1-t} < 0,$$

and $t > s > 1$ or $t < s < 1$, we have $0 < \frac{s-t}{1-t} < 1$, by inequality (3.1) with indices $\frac{s-t}{1-t}$ and $\frac{s-t}{s-1}$, we have

$$\begin{aligned}
&\int_a^b \left|f(x)g(x)\right| d_{\alpha,\beta}x \\
&\quad = \int_a^b \left|f(x)g(x)\right|^{s(1-t)/(s-t)} \left|f(x)g(x)\right|^{t(s-1)/(s-t)} d_{\alpha,\beta}x \\
&\quad \ge \left(\int_a^b \left|f(x)g(x)\right|^s d_{\alpha,\beta}x\right)^{(1-t)/(s-t)} \left(\int_a^b \left|f(x)g(x)\right|^t d_{\alpha,\beta}x\right)^{(s-1)/(s-t)},
\end{aligned} \tag{5.6}$$

with equality if and only if $(fg)^s$ and $(fg)^t$ are proportional.

On the other hand, from the reverse Hölder inequality again for $0 < p = \frac{s-t}{1-t} < 1$ or $p = \frac{s-t}{1-t} < 0$, we obtain the following two inequalities:

$$\begin{aligned}&\int_a^b \left|f(x)g(x)\right|^s d_{\alpha,\beta}x\\&\quad\geq \left(\int_a^b \left|f(x)\right|^{s(s-t)/(1-t)} d_{\alpha,\beta}x\right)^{(1-t)/(s-t)} \left(\int_a^b \left|g(x)\right|^{s(s-t)/(s-1)} d_{\alpha,\beta}x\right)^{(s-1)/(s-t)},\end{aligned} \tag{5.7}$$

with equality if and only if $f^{s(s-t)/(1-t)}$ and $g^{s(s-t)/(s-1)}$ are proportional, and

$$\begin{aligned}&\int_a^b \left|f(x)g(x)\right|^t d_{\alpha,\beta}x\\&\quad\geq \left(\int_a^b \left|f(x)\right|^{t(s-t)/(1-t)} d_{\alpha,\beta}x\right)^{(1-t)/(s-t)} \left(\int_a^b \left|g(x)\right|^{t(s-t)/(s-1)} d_{\alpha,\beta}x\right)^{(s-1)/(s-t)},\end{aligned} \tag{5.8}$$

with equality if and only if $f^{t(s-t)/(1-t)}$ and $g^{t(s-t)/(s-1)}$ are proportional.

From (5.6), (5.7), and (5.8), the proof of the case (2) of Theorem 5.1 is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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