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Some criteria for concave conformal mappings

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Abstract
The main purpose of this paper is to derive some criteria for concave conformal mappings.

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1 Introduction

A conformal, meromorphic function f on the *punctured* unit disk

$$\mathbb{U}^* := \{z \in \mathbb{C} : 0 < |z| < 1\} =: \mathbb{U} \backslash \{0\}$$

is said to be a concave mapping if $f(\mathbb{U}^*)$ is the complement of a compact, convex set.

Let Σ denote the class of analytic functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k \quad (z \in \mathbb{U}^*), \tag{1.1}$$

then the necessary and sufficient condition for f to be a concave mapping is

$$1 + \Re\left(\frac{zf''(z)}{f'(z)}\right) < 0 \quad (z \in \mathbb{U}), \tag{1.2}$$

where

$$\frac{zf''(z)}{f'(z)} = -2 - 2b_1 z^2 - 6b_2 z^3 - (12b_3 + 2b_1^2)z^4 - \cdots.$$

Recently, Bhowmik *et al.* [1], Chuaqui *et al.* [2], Ibrahim and Sokół [3] derived some interesting properties of concave conformal mappings. In this paper, we aim at proving several criteria for the function $f \in \Sigma$ to be a concave mapping.

To prove our main results, we need the following two lemmas.

Lemma 1.1 (Jack's lemma [4]) *Let $h(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots$ be a non-constant analytic function in $\mathbb{U}$. If $|h(z)|$ attains its maximum value on the circle $|z| = r < 1$, then*

$$z_0 h'(z_0) = kh(z_0),$$

where k is a real number with $k \geq n$.

Lemma 1.2 (See [5]) *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that Φ is a mapping from $\mathbb{C}^2 \times \mathbb{U}$ to $\mathbb{C}$ which satisfies $\Phi(ix, y; z) \notin \Omega$ for $z \in \mathbb{U}$ and for all real x, y such that $y \leq -\frac{1+x^2}{2}$. If the function $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathbb{U}$ and $\Phi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{U}$, then $\Re(p(z)) > 0$.*

2 Main results

We first give the following result.

Theorem 2.1 *Suppose that $f \in \Sigma$ with $(zf'(z))' \neq 0$. If f satisfies the condition*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{z(2f''(z) + zf'''(z))}{f'(z) + zf''(z)}\right| < \lambda \quad \left(0 < \lambda \leq \frac{1}{2}\right), \tag{2.1}$$

then f is concave in $\mathbb{U}^*$.

Proof Assume that

$$\phi(z) := \frac{(1-\lambda)\frac{f'(z)}{f'(z)+zf''(z)} + 1}{\lambda} - 1 \quad \left(0 < \lambda \leq \frac{1}{2}; z \in \mathbb{U}\right). \tag{2.2}$$

Then the function ϕ is analytic in $\mathbb{U}$ with $\phi(0) = 0$. From (2.2), we know that

$$\frac{f'(z)}{f'(z) + zf''(z)} = \frac{\lambda\phi(z) + \lambda - 1}{1 - \lambda}. \tag{2.3}$$

By differentiating both sides of (2.3) with respect to z logarithmically, we get

$$\frac{zf''(z)}{f'(z)} - \frac{z(2f''(z) + zf'''(z))}{f'(z) + zf''(z)} = \frac{\lambda z\phi'(z)}{\lambda\phi(z) + \lambda - 1}. \tag{2.4}$$

From (2.1) and (2.4), we find that

$$\left|\frac{zf''(z)}{f'(z)} - \frac{z(2f''(z) + zf'''(z))}{f'(z) + zf''(z)}\right| = \lambda\left|\frac{\lambda z\phi'(z)}{\lambda\phi(z) + \lambda - 1}\right| < \lambda. \tag{2.5}$$

Now, we can claim that $|\phi(z)| < 1$. If not, there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} \left|\phi(z)\right| = \left|\phi(z_0)\right| = 1.$$

By Lemma 1.1, we know that

$$z_0\phi'(z_0) = k\phi(z_0) = ke^{i\theta} \quad (0 \leq \theta < 2\pi; k \geq 1). \tag{2.6}$$

For $z = z_0$, we find from (2.4) and (2.6) that

$$\left|\frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0(2f''(z_0) + z_0 f'''(z_0))}{f'(z_0) + z_0 f''(z_0)}\right| = \lambda\left|\frac{k}{\lambda + (\lambda - 1)e^{-i\theta}}\right| \geq \lambda. \tag{2.7}$$

But (2.7) contradicts (2.5). Thus, we deduce that $|\phi(z)| < 1$, which implies that

$$\left|\frac{(1-\lambda)\frac{f'(z)}{f'(z)+zf''(z)}+1}{\lambda}-1\right| < 1, \tag{2.8}$$

or equivalently,

$$\left|\frac{f'(z)}{f'(z)+zf''(z)}+1\right| < \frac{\lambda}{1-\lambda}. \tag{2.9}$$

From (2.9), we get

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) < \lambda - 1 < 0 \quad \left(0 < \lambda \leq \frac{1}{2}\right),$$

which shows that the function f is concave in $\mathbb{U}^*$. □

Theorem 2.2 *Suppose that $f \in \Sigma$ with $f'(z) \neq 0$. If f satisfies the inequality*

$$\Re\left(\frac{z[(f''(z)+zf'''(z))f'(z)-z(f''(z))^2]}{f'(z)(zf''(z)+3f'(z))}\right) < 1, \tag{2.10}$$

then f is concave in $\mathbb{U}^*$.

Proof Define the function $\varphi(z)$ by

$$\varphi(z) := \frac{zf''(z)}{f'(z)} + 2 \quad (z \in \mathbb{U}). \tag{2.11}$$

It is easy to see that

$$\varphi(z) = -2b_1z^2 - 6b_2z^3 - \left(12b_3 + 2b_1^2\right)z^4 - \cdots$$

is analytic in $\mathbb{U}$ with $\varphi(0) = \varphi'(0) = 0$. From (2.11), we obtain

$$\frac{zf''(z)}{f'(z)} + 3 = 1 + \varphi(z) \quad (z \in \mathbb{U}). \tag{2.12}$$

Taking logarithmical derivatives of both sides of (2.12) with respect to z, we get

$$\frac{z[(f''(z)+zf'''(z))f'(z)-z(f''(z))^2]}{f'(z)(zf''(z)+3f'(z))} = \frac{z\varphi'(z)}{1+\varphi(z)}. \tag{2.13}$$

We now show that $|\varphi(z)| < 1$. If not, there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z|\leq|z_0|}\left|\varphi(z)\right| = \left|\varphi(z_0)\right| = 1.$$

By Jack's lemma, we know that

$$z_0\varphi'(z_0) = k\varphi(z_0) = ke^{i\theta} \quad (0 \leq \theta < 2\pi; k \geq 2). \tag{2.14}$$

For $z = z_0$, we have

$$\begin{aligned} &\Re\left(\frac{z_0[(f''(z_0) + z_0f'''(z_0))f'(z_0) - z_0(f''(z_0))^2]}{f'(z_0)(z_0f''(z_0) + 3f'(z_0))}\right) \\ &\quad = \Re\left(\frac{z_0\varphi'(z_0)}{1 + \varphi(z_0)}\right) = \Re\left(\frac{ke^{i\theta}}{1 + e^{i\theta}}\right) \geq \frac{k}{2} \geq 1. \end{aligned} \tag{2.15}$$

But (2.15) is a contradiction to condition (2.10), which implies that $|\varphi(z)| < 1$. Consequently, we deduce from (2.11) that

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) = \Re\big(\varphi(z)\big) - 1 \leq \big|\varphi(z)\big| - 1 < 0,$$

which implies that f is concave in $\mathbb{U}^*$. □

Theorem 2.3 *Suppose that $f \in \Sigma$ with $f'(z) \neq 0$. If f satisfies the condition*

$$\Re\left(\frac{zf'(z)}{(zf'(z))'}\left(\frac{(zf'(z))'}{f'(z)}\right)'\right) > \begin{cases} \frac{\delta}{2(\delta-1)} & (0 \leqq \delta \leqq \frac{1}{2}), \\ \frac{\delta-1}{2\delta} & (\frac{1}{2} \leqq \delta < 1), \end{cases} \tag{2.16}$$

then f is concave in $\mathbb{U}^*$.

Proof Suppose that

$$\psi(z) = \frac{-\frac{zf''(z)}{f'(z)} - 1 - \delta}{1 - \delta} \quad (0 \leq \delta < 1; z \in \mathbb{U}). \tag{2.17}$$

Then ψ is analytic in $\mathbb{U}$. From (2.17), we find that

$$\frac{zf'(z)}{(zf'(z))'}\left(\frac{(zf'(z))'}{f'(z)}\right)' = \frac{(1 - \delta)z\psi'(z)}{\delta + (1 - \delta)\psi(z)} = \Phi\big(\psi(z), z\psi'(z); z\big), \tag{2.18}$$

where

$$\Phi(r, s; t) = \frac{(1 - \delta)s}{\delta + (1 - \delta)r}.$$

For the real numbers x and y satisfying the condition $y \leq -\frac{1+x^2}{2}$, we know that

$$\begin{aligned} \Re\big(\Phi(ix, y; z)\big) &= \frac{(1 - \delta)\delta y}{\delta^2 + (1 - \delta)^2x^2} \\ &\leq -\frac{(1 - \delta)\delta}{2} \cdot \frac{1 + x^2}{\delta^2 + (1 - \delta)^2x^2} \\ &\leq \begin{cases} \frac{\delta}{2(\delta-1)} & (0 \leqq \delta \leqq \frac{1}{2}), \\ \frac{\delta-1}{2\delta} & (\frac{1}{2} \leqq \delta < 1). \end{cases} \end{aligned} \tag{2.19}$$

Now, we take

$$\Omega = \left\{\xi : \Re(\xi) > \begin{cases} \frac{\delta}{2(\delta-1)} & (0 \leqq \delta \leqq \frac{1}{2}) \\ \frac{\delta-1}{2\delta} & (\frac{1}{2} \leqq \delta < 1) \end{cases}\right\},$$

then $\Phi(ix, y; z) \notin \Omega$ for all real x, y such that $y \leq -\frac{1+x^2}{2}$. Furthermore, by virtue of (2.16), we know that $\Phi(\psi(z), z\psi'(z); z) \in \Omega$. Thus, by Lemma 1.2, we get $\Re(\psi(z)) > 0$, which shows that f is concave in $\mathbb{U}^*$. □

Finally, we correct an error of Theorem 2.1 in [3], the condition

$$\Re\left(\frac{zf'''(z)}{f''(z)}\right) < 0 \quad (z \in \mathbb{U})$$

in it should be changed into

$$\Re\left(\frac{zf'''(z)}{f''(z)}\right) > -3 \quad (z \in \mathbb{U}).$$

Theorem 2.4 *Suppose that $f \in \Sigma$ with $f'(z) \neq 0$. If f satisfies the inequality*

$$\Re\left(\frac{zf'''(z)}{f''(z)}\right) > -3 \quad (z \in \mathbb{U}), \tag{2.20}$$

then f is concave in $\mathbb{U}^*$.

Proof Define the function $\omega(z)$ by

$$-1 - \frac{zf''(z)}{f'(z)} = \frac{1 + \omega(z)}{1 - \omega(z)}. \tag{2.21}$$

Then ω is analytic in $\mathbb{U}$ with $\omega(0) = \omega'(0) = 0$. From (2.21), we get

$$\frac{zf''(z)}{f'(z)} = \frac{-2}{1 - \omega(z)}. \tag{2.22}$$

Differentiating both sides of (2.22) logarithmically, we get

$$\frac{zf'''(z)}{f''(z)} = \frac{z\omega'(z)}{1 - \omega(z)} + \frac{zf''(z)}{f'(z)} - 1. \tag{2.23}$$

Now, we show that $|\omega(z)| < 1$. If not, there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} \left|\omega(z)\right| = \left|\omega(z_0)\right| = 1.$$

By Jack's lemma, we know that

$$\begin{aligned}
\Re\left(\frac{z_0 f'''(z_0)}{f''(z_0)}\right) &= \Re\left(\frac{(k+1)\omega(z_0) - 3}{1 - \omega(z_0)}\right) \\
&= \Re\left(\frac{(k+1)(\cos\theta + i\sin\theta) - 3}{1 - \cos\theta - i\sin\theta}\right) \\
&= \frac{(k+4)\cos\theta - (k+4)}{(1-\cos\theta)^2 + \sin^2\theta} \\
&= \frac{(k+4)(\cos\theta - 1)}{(1-\cos\theta)^2 + \sin^2\theta}
\end{aligned}$$

$$= -\frac{k+4}{2}$$
$$\leq -3,$$

where $k \geq 2$, but this contradicts (2.20), which implies that $|\omega(z)| < 1$. Thus, f is concave in $\mathbb{U}^*$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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