Lee and Jung *Journal of Inequalities and Applications* (2015) 2015:66
DOI 10.1186/s13660-015-0586-7

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

A general uniqueness theorem concerning the stability of monomial functional equations in fuzzy spaces

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Abstract

In this paper, we prove a general uniqueness theorem that can easily be applied to the (generalized) Hyers-Ulam stability of a large class of functional equations, which includes monomial functional equations (*e.g.* the Cauchy additive functional equation, the quadratic functional equation, and the cubic functional equation, *etc.*). This uniqueness theorem can save us much trouble in proving the uniqueness of relevant solutions repeatedly appearing in the stability problems for functional equations in fuzzy spaces.

MSC: Primary 39B82; 46S40; secondary 26E50; 03E72; 39B52

Keywords: uniqueness; stability; Hyers-Ulam stability; generalized Hyers-Ulam stability; monomial functional equation; monomial mapping

1 Introduction

Let X and Y be real vector spaces and let n be a positive integer. For a given mapping $f : X \to Y$, we define a mapping $D_nf : X \times X \to Y$ by

$$D_nf(x,y) := \sum_{i=0}^{n} {}_nC_i(-1)^{n-i}f(ix+y) - n!f(x)$$

for all $x, y \in X$, where ${}_nC_i = \frac{n!}{i!(n-i)!}$. A mapping $f : X \to Y$ is called a monomial mapping of degree n if f satisfies the monomial functional equation $D_nf(x,y) = 0$ for all $x, y \in X$. The mapping $f(x) := ax^n$ satisfies the functional equation $D_nf(x,y) = 0$ for all $x, y \in \mathbf{R}$. In particular, a mapping $f : X \to Y$ is called an additive mapping, a quadratic mapping, a cubic mapping, a quadratic mapping, respectively, if f satisfies the functional equation $D_1f(x,y) = 0$, $D_2f(x,y) = 0$, $D_3f(x,y) = 0$, $D_4f(x,y) = 0$, respectively. We notice that if a mapping $f : X \to Y$ is a monomial mapping of degree n, then $f(rx) = r^nf(x)$ for all $x \in X$ and all rational numbers r (see [1, 2]).

In the study of Hyers-Ulam stability problems of monomial functional equations, we have been frequently requested to prove the uniqueness of the monomial mappings under various conditions. We can find in the books [3–6] a lot of references concerning the Hyers-Ulam stability of functional equations (see also [7–11]).

In this paper, we prove a general uniqueness theorem that can easily be applied to the (generalized) Hyers-Ulam stability of a large class of functional equations, which includes monomial functional equations. Indeed, this uniqueness theorem can save us much trouble in proving the uniqueness of relevant solutions repeatedly appearing in the stability problems for various functional equations in fuzzy spaces (see [12–14]).

2 Preliminaries

We first introduce the definition of fuzzy normed spaces (see [15–17]).

Definition 2.1 Let X be a real vector space. A mapping $N : X \times \mathbf{R} \to [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $c, s, t \in \mathbf{R}$,

(N_1) $N(x,t) = 0$ for all $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, t/|c|)$ for all $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing mapping on $\mathbf{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

The pair (X,N) is called a *fuzzy normed space*.

Example 2.2 ([17]) Let $(X, \|\cdot\|)$ be a normed space and let $k > 0$ be an arbitrary real number. If we define a mapping $N_k : X \times \mathbf{R} \to [0,1]$ by

$$N_k(x,t) := \begin{cases} \frac{t}{t+k\|x\|} & \text{if } t > 0, \\ 0 & \text{otherwise,} \end{cases}$$

then N_k is a fuzzy norm on X.

Let (X,N) be a fuzzy normed space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we write $N\text{-}\lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists a positive integer n_0 such that for all $n \geq n_0$ and all integers $p > 0$ we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$. It is well known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed space is called a *fuzzy Banach space*.

3 Main result

Throughout this section, let (Y, N_Y) and (Z, N_Z) be fuzzy normed spaces and let X be a real vector space. To the best of our knowledge, Mirmostafaee *et al.* seem to be the first authors who investigated the (generalized) Hyers-Ulam stability of functional equations in fuzzy spaces [15–17].

In the following theorem, we prove that if, for any given mapping f, there exists a mapping F (near f) with some properties which are certainly satisfied by monomial mappings, then the mapping F is uniquely determined.

Theorem 3.1 *Let $a \neq 1$ be a positive real constant, let b be a real constant, let $\Phi : X\backslash\{0\} \to (Z, N_Z)$ be a mapping satisfying either*

$$\lim_{n\to\infty} N_Z\left(\frac{1}{a^{bn}}\Phi\left(a^n x\right), t\right) = 1 \tag{1}$$

for all $x \in X \backslash \{0\}$ *and all* $t > 0$, *or*

$$\lim_{n\to\infty} N_Z\left(a^{bn}\Phi\left(\frac{x}{a^n}\right), t\right) = 1 \tag{2}$$

for all $x \in X\backslash\{0\}$ *and all* $t > 0$, *and let* $f : X \to (Y, N_Y)$ *be an arbitrarily given mapping. If there exists a mapping* $F : X \to (Y, N_Y)$ *such that*

$$N_Y\big(f(x) - F(x), t\big) \geq N_Z\big(\Phi(x), t\big) \tag{3}$$

for all $x \in X\backslash\{0\}$ *and all* $t > 0$, *and if* F *satisfies*

$$F(ax) = a^b F(x) \tag{4}$$

for all $x \in X$, *then* F *is determined by*

$$F(x) = \begin{cases} N_Y\text{-}\lim_{n\to\infty} \frac{1}{a^{bn}} f(a^n x) & \text{if } \Phi \text{ satisfies (1)}, \\ N_Y\text{-}\lim_{n\to\infty} a^{bn} f(\frac{x}{a^n}) & \text{if } \Phi \text{ satisfies (2)} \end{cases}$$

for all $x \in X\backslash\{0\}$. *In other words,* F *is the unique mapping satisfying* (3) *and* (4).

Proof Assume that F is a mapping satisfying (3) and (4) for a given mapping $f : X \to (Y, N_Y)$.

First, we consider the case when Φ satisfies the condition (1) for all $x \in X\backslash\{0\}$ and all $t > 0$. It then follows from (N_3), (1), (3), and (4) that

$$\begin{aligned} N_Y\left(F(x) - \frac{1}{a^{bn}} f\left(a^n x\right), t\right) &= N_Y\left(\frac{1}{a^{bn}}\left(F\left(a^n x\right) - f\left(a^n x\right)\right), t\right) \\ &= N_Y\left(f\left(a^n x\right) - F\left(a^n x\right), a^{bn} t\right) \\ &\geq N_Z\left(\Phi\left(a^n x\right), a^{bn} t\right) \\ &= N_Z\left(\frac{1}{a^{bn}}\Phi\left(a^n x\right), t\right) \\ &\to 1, \quad \text{as } n \to \infty \end{aligned}$$

for all $x \in X\backslash\{0\}$ and all $t > 0$. Thus, we have $F(x) = N_Y\text{-}\lim_{n\to\infty} \frac{1}{a^{bn}} f(a^n x)$ for all $x \in X\backslash\{0\}$.

On the other hand, if Φ satisfies the condition (2) for all $x \in X\backslash\{0\}$ and all $t > 0$, it then follows from (N_3), (2), (3), and (4) that

$$\begin{aligned} N_Y\left(F(x) - a^{bn} f\left(\frac{x}{a^n}\right), t\right) &= N_Y\left(a^{bn} F\left(\frac{x}{a^n}\right) - a^{bn} f\left(\frac{x}{a^n}\right), t\right) \\ &= N_Y\left(f\left(\frac{x}{a^n}\right) - F\left(\frac{x}{a^n}\right), \frac{t}{a^{bn}}\right) \\ &\geq N_Z\left(\Phi\left(\frac{x}{a^n}\right), \frac{t}{a^{bn}}\right) \\ &= N_Z\left(a^{bn}\Phi\left(\frac{x}{a^n}\right), t\right) \\ &\to 1, \quad \text{as } n \to \infty \end{aligned}$$

for all $x \in X\backslash\{0\}$ and all $t > 0$. Therefore, we have $F(x) = N_Y\text{-}\lim_{n\to\infty} a^{bn} f(\frac{x}{a^n})$ for all $x \in X\backslash\{0\}$. $\square$

In general, it is not easy to apply Theorem 3.1 in practical applications. Hence, we introduce some corollaries which are easily applicable to investigating the uniqueness problems in the stability of various functional equations.

Corollary 3.2 *Let $a \neq 1$ be a positive real constant, let b be a real constant, let $\phi : X\backslash\{0\} \to (Z, N_Z)$ be a mapping satisfying either*

$$\Phi(x) := N_Z\text{-}\lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{a^{bi}} \phi(a^i x) \in Z \tag{5}$$

for all $x \in X\backslash\{0\}$, or

$$\Phi(x) := N_Z\text{-}\lim_{n\to\infty} \sum_{i=0}^{n} a^{bi} \phi\left(\frac{x}{a^i}\right) \in Z \tag{6}$$

for all $x \in X\backslash\{0\}$, and let $f : X \to (Y, N_Y)$ be an arbitrarily given mapping. If there exists a mapping $F : X \to (Y, N_Y)$ satisfying (3) *for all $x \in X\backslash\{0\}$ and $t > 0$ and* (4) *for all $x \in X$, then F is a unique mapping satisfying* (3) *and* (4).

Proof If ϕ satisfies (5) for all $x \in X\backslash\{0\}$, then we have

$$\Phi(a^m x) = N_Z\text{-}\lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{a^{bi}} \phi(a^{i+m} x)$$

or equivalently

$$\lim_{n\to\infty} N_Z\left(\Phi(a^m x) - \sum_{i=0}^{n} \frac{1}{a^{bi}} \phi(a^{i+m} x), \frac{a^{bm}}{2} t\right) = 1 \tag{7}$$

for all $x \in X\backslash\{0\}$ and $m \in \mathbf{N}$, where we write '$\frac{a^{bm}}{2} t$' instead of 't' in (7) (it is not bad because $\frac{a^{bm}}{2}$ is a positive real constant). It now follows from (N_3) and (7) that

$$\begin{aligned} &\lim_{n\to\infty} N_Z\left(\frac{1}{a^{bm}} \Phi(a^m x) - \frac{1}{a^{bm}} \sum_{i=0}^{n} \frac{1}{a^{bi}} \phi(a^{i+m} x), \frac{t}{2}\right) \\ &\quad = \lim_{n\to\infty} N_Z\left(\Phi(a^m x) - \sum_{i=0}^{n} \frac{1}{a^{bi}} \phi(a^{i+m} x), \frac{a^{bm}}{2} t\right) = 1 \end{aligned} \tag{8}$$

for any $x \in X\backslash\{0\}$ and $m \in \mathbf{N}$. Hence, by (N_4) and (8), we get

$$\begin{aligned} &N_Z\left(\frac{1}{a^{bm}} \Phi(a^m x), t\right) \\ &\quad = \lim_{n\to\infty} N_Z\left(\frac{1}{a^{bm}} \Phi(a^m x), t\right) \end{aligned}$$

$$\begin{aligned}
&\geq \lim_{n\to\infty} \min\Biggl\{N_Z\Biggl(\frac{1}{a^{bm}}\Phi\bigl(a^m x\bigr) - \frac{1}{a^{bm}}\sum_{i=0}^{n}\frac{1}{a^{bi}}\phi\bigl(a^{i+m}x\bigr), \frac{t}{2}\Biggr),\\
&\quad N_Z\Biggl(\sum_{i=0}^{n}\frac{1}{a^{b(i+m)}}\phi\bigl(a^{i+m}x\bigr), \frac{t}{2}\Biggr)\Biggr\}\\
&= \lim_{n\to\infty} N_Z\Biggl(\sum_{i=m}^{m+n}\frac{1}{a^{bi}}\phi\bigl(a^i x\bigr), \frac{t}{2}\Biggr)
\end{aligned}$$

for all $x \in X\backslash\{0\}$ and $t > 0$.

Hence, by using (N_4) and (5), we have

$$\begin{aligned}
&\lim_{m\to\infty} N_Z\biggl(\frac{1}{a^{bm}}\Phi\bigl(a^m x\bigr), t\biggr)\\
&\quad\geq \lim_{m\to\infty}\lim_{n\to\infty} N_Z\Biggl(\sum_{i=m}^{m+n}\frac{1}{a^{bi}}\phi\bigl(a^i x\bigr), \frac{t}{2}\Biggr)\\
&\quad\geq \lim_{m\to\infty}\lim_{n\to\infty} \min\Biggl\{N_Z\Biggl(\sum_{i=0}^{m+n}\frac{1}{a^{bi}}\phi\bigl(a^i x\bigr) - \Phi(x), \frac{t}{4}\Biggr),\\
&\qquad N_Z\Biggl(\Phi(x) - \sum_{i=0}^{m-1}\frac{1}{a^{bi}}\phi\bigl(a^i x\bigr), \frac{t}{4}\Biggr)\Biggr\}\\
&\quad= 1
\end{aligned}$$

for each $x \in X\backslash\{0\}$ and $t > 0$, *i.e.*, Φ satisfies the condition (1).

If ϕ satisfies (6) for any $x \in X\backslash\{0\}$, then we get

$$\Phi\biggl(\frac{x}{a^m}\biggr) = N_Z\text{-}\lim_{n\to\infty}\sum_{i=0}^{n} a^{bi}\phi\biggl(\frac{x}{a^{i+m}}\biggr)$$

or equivalently

$$\lim_{n\to\infty} N_Z\Biggl(\Phi\biggl(\frac{x}{a^m}\biggr) - \sum_{i=0}^{n} a^{bi}\phi\biggl(\frac{x}{a^{i+m}}\biggr), \frac{t}{2a^{bm}}\Biggr) = 1 \tag{9}$$

for every $x \in X\backslash\{0\}$ and $m \in \mathbf{N}$, where we write '$\frac{t}{2a^{bm}}$' instead of 't' in (9) (it is not bad because $\frac{1}{2a^{bm}}$ is a positive real constant). In view of (N_3) and (9), we have

$$\begin{aligned}
&\lim_{n\to\infty} N_Z\Biggl(a^{bm}\Phi\biggl(\frac{x}{a^m}\biggr) - a^{bm}\sum_{i=0}^{n} a^{bi}\phi\biggl(\frac{x}{a^{i+m}}\biggr), \frac{t}{2}\Biggr)\\
&\quad= \lim_{n\to\infty} N_Z\Biggl(\Phi\biggl(\frac{x}{a^m}\biggr) - \sum_{i=0}^{n} a^{bi}\phi\biggl(\frac{x}{a^{i+m}}\biggr), \frac{t}{2a^{bm}}\Biggr) = 1
\end{aligned} \tag{10}$$

for any $x \in X\backslash\{0\}$ and $m \in \mathbf{N}$. Thus, it follows from (N_4) and (10) that

$$\begin{aligned}
&N_Z\biggl(a^{bm}\Phi\biggl(\frac{x}{a^m}\biggr), t\biggr)\\
&\quad= \lim_{n\to\infty} N_Z\biggl(a^{bm}\Phi\biggl(\frac{x}{a^m}\biggr), t\biggr)
\end{aligned}$$

$$
\begin{aligned}
&\geq \lim_{n\to\infty} \min\left\{ N_Z\left(a^{bm}\Phi\left(\frac{x}{a^m}\right) - a^{bm}\sum_{i=0}^{n} a^{bi}\phi\left(\frac{x}{a^{i+m}}\right), \frac{t}{2}\right), \right.\\
&\quad \left. N_Z\left(\sum_{i=0}^{n} a^{b(i+m)}\phi\left(\frac{x}{a^{i+m}}\right), \frac{t}{2}\right)\right\}\\
&= \lim_{n\to\infty} N_Z\left(\sum_{i=m}^{m+n} a^{bi}\phi\left(\frac{x}{a^{i}}\right), \frac{t}{2}\right)
\end{aligned}
$$

for any $x \in X\backslash\{0\}$ and $t > 0$.

Therefore, by (N_4) and (6), we obtain

$$
\begin{aligned}
&\lim_{m\to\infty} N_Z\left(a^{bm}\Phi\left(\frac{x}{a^m}\right), t\right)\\
&\quad \geq \lim_{m\to\infty}\lim_{n\to\infty} N_Z\left(\sum_{i=m}^{m+n} a^{bi}\phi\left(\frac{x}{a^{i}}\right), \frac{t}{2}\right)\\
&\quad \geq \lim_{m\to\infty}\lim_{n\to\infty} \min\left\{ N_Z\left(\sum_{i=0}^{m+n} a^{bi}\phi\left(\frac{x}{a^{i}}\right) - \Phi(x), \frac{t}{4}\right), \right.\\
&\qquad \left. N_Z\left(\Phi(x) - \sum_{i=0}^{m-1} a^{bi}\phi\left(\frac{x}{a^{i}}\right), \frac{t}{4}\right)\right\}\\
&\quad = 1
\end{aligned}
$$

for each $x \in X\backslash\{0\}$ and $t > 0$, *i.e.*, Φ satisfies the condition (2). Hence, our assertion is true in view of Theorem 3.1. □

Corollary 3.3 *Let $a \neq 1$ be a positive real constant, let b be a real constant, let Y and Z be real normed spaces, let $\phi : X\backslash\{0\} \to Z$ be a mapping satisfying either*

$$
\Phi(x) := \sum_{i=0}^{\infty} \frac{1}{a^{bi}}\phi\left(a^i x\right) \in Z \tag{11}
$$

for all $x \in X\backslash\{0\}$, or

$$
\Phi(x) := \sum_{i=0}^{\infty} a^{bi}\phi\left(\frac{x}{a^i}\right) \in Z \tag{12}
$$

for all $x \in X\backslash\{0\}$, and let $f : X \to Y$ be an arbitrarily given mapping. If there exists a mapping $F : X \to Y$ satisfying

$$
\left\|f(x) - F(x)\right\| \leq \left\|\Phi(x)\right\| \tag{13}
$$

for all $x \in X\backslash\{0\}$ and if F satisfies (4) *for all $x \in X$, then F is a unique mapping satisfying* (4) *and* (13).

Proof If we define the fuzzy norms $N_Y : Y \times \mathbf{R} \to [0,1]$ on Y and $N_Z : Z \times \mathbf{R} \to [0,1]$ on Z by

$$N_Y(y,t) := \begin{cases} \frac{t}{t+\|y\|} & \text{if } t > 0, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad N_Z(z,t) := \begin{cases} \frac{t}{t+\|z\|} & \text{if } t > 0, \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in Y$ and $z \in Z$, then (Y, N_Y) and (Z, N_Z) are fuzzy normed spaces. We then know that (11) and (12) imply (1) and (2), respectively.

In view of (13), we have

$$\frac{t}{t + \|f(x) - F(x)\|} \geq \frac{t}{t + \|\Phi(x)\|}$$

for all $x \in X\backslash\{0\}$ and $t > 0$, *i.e.*, $N_Y(f(x) - F(x), t) \geq N_Z(\Phi(x), t)$ for all $x \in X\backslash\{0\}$ and $t > 0$. Therefore, by Theorem 3.1, we conclude that our assertion is true. □

Corollary 3.4 *Let $a \neq 1$ be a positive real constant, let b be a real constant, let $\phi : X\backslash\{0\} \to [0,\infty)$ be a mapping satisfying either*

$$\Phi(x) := \sum_{i=0}^{\infty} \frac{1}{a^{bi}} \phi\left(a^i x\right) < \infty \tag{14}$$

for all $x \in X\backslash\{0\}$, or

$$\Phi(x) := \sum_{i=0}^{\infty} a^{bi} \phi\left(\frac{x}{a^i}\right) < \infty \tag{15}$$

for all $x \in X\backslash\{0\}$, and let $f : X \to (Y, N_Y)$ be an arbitrarily given mapping. Let $N_{\mathbf{R}} : \mathbf{R} \times \mathbf{R} \to [0,1]$ be the fuzzy norm on $\mathbf{R}$ defined by

$$N_{\mathbf{R}}(r,t) := \begin{cases} \frac{t}{t+|r|} & \textit{if } t > 0, \\ 0 & \textit{otherwise.} \end{cases}$$

If there exists a mapping $F : X \to (Y, N_Y)$ satisfying

$$N_Y\big(f(x) - F(x), t\big) \geq N_{\mathbf{R}}\big(\Phi(x), t\big) \tag{16}$$

for all $x \in X\backslash\{0\}$ and all $t > 0$ and, moreover, if F satisfies (4) *for all $x \in X$, then F is a unique mapping satisfying* (4) *and* (16).

We now prove a general uniqueness theorem that can easily be applied to the (generalized) Hyers-Ulam stability of the monomial functional equations.

Corollary 3.5 *Let $a \neq 1$ be a positive real constant, let Y be a normed space, let $\phi : X\backslash\{0\} \to [0,\infty)$ be a mapping satisfying either* (14) *for all $x \in X\backslash\{0\}$ or* (15) *for all $x \in X\backslash\{0\}$, and let $f : X \to Y$ be an arbitrary mapping. If there exists a mapping $F : X \to Y$ satisfying*

$$\big\|f(x) - F(x)\big\| \leq \Phi(x) \tag{17}$$

for all $x \in X\backslash\{0\}$, *and, moreover, if* F *satisfies* (4) *for all* $x \in X$, *then* F *is a unique mapping satisfying* (4) *and* (17).

Proof If we set $Z := \mathbf{R}$,

$$N_Y(y,t) := \begin{cases} \frac{t}{t+\|y\|} & \text{if } t > 0, \\ 0 & \text{otherwise} \end{cases}$$

and

$$N_Z(z,t) = N_{\mathbf{R}}(z,t) := \begin{cases} \frac{t}{t+|z|} & \text{if } t > 0, \\ 0 & \text{otherwise} \end{cases}$$

it then follows from (17) that

$$\frac{t}{t+\|f(x)-F(x)\|} \geq \frac{t}{t+|\Phi(x)|} \quad \text{or} \quad N_Y\bigl(f(x)-F(x),t\bigr) \geq N_Z\bigl(\Phi(x),t\bigr)$$

for all $x \in X\backslash\{0\}$ and $t > 0$.

Finally, we use Corollaries 3.3 and 3.4 to show that our assertion of this corollary is true. □

Corollary 3.6 *Let* $a \neq 1$ *and* θ *be positive real numbers, let* b *and* p *be real numbers with* $p \neq b$, *and let* $f : \mathbf{R} \to Y$ *be a mapping from* $\mathbf{R}$ *into a normed space* Y. *If there is a mapping* $F : \mathbf{R} \to Y$ *satisfying the inequality*

$$\bigl\|f(x) - F(x)\bigr\| \leq \theta|x|^p \tag{18}$$

for any $x \in \mathbf{R}\backslash\{0\}$, *and if* F, *moreover, satisfies* (4) *for all* $x \in \mathbf{R}$, *then* F *is a unique mapping satisfying* (4) *and* (18).

Proof We define fuzzy norms $N_{\mathbf{R}}$ and N_Y by

$$N_{\mathbf{R}}(x,t) := \begin{cases} \frac{t}{t+|x|} & \text{if } t > 0, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad N_Y(y,t) := \begin{cases} \frac{t}{t+\|y\|} & \text{if } t > 0, \\ 0 & \text{otherwise} \end{cases}$$

for any $x \in \mathbf{R}$ and $y \in Y$. Moreover, we define a mapping $\Phi : \mathbf{R}\backslash\{0\} \to (\mathbf{R}, N_{\mathbf{R}})$ by $\Phi(x) := \theta x^p$.

If $a^b > a^p$, then Φ satisfies (1) for all $x \in \mathbf{R}\backslash\{0\}$ and all $t > 0$. On the other hand, if $a^b < a^p$, then Φ satisfies (2) for all $x \in \mathbf{R}\backslash\{0\}$ and all $t > 0$. It, moreover, follows from (18) that

$$N_Y\bigl(f(x)-F(x),t\bigr) \geq N_{\mathbf{R}}\bigl(\Phi(x),t\bigr)$$

for all $x \in \mathbf{R}\backslash\{0\}$ and $t > 0$. Therefore, in view of Theorem 3.1, we conclude that our corollary is true. □

In the following corollaries, we prove that if there exists an additive mapping F near a given mapping f, then the mapping F is uniquely determined. These corollaries are immediate consequences of Corollaries 3.2, 3.3, 3.4, 3.5, and 3.6, because every additive mapping satisfies the condition (4) for each positive rational number $a \neq 1$ provided $b = 1$.

Corollary 3.7 *Let $a \neq 1$ and $b = 1$ be positive rational numbers, let $\phi : X\backslash\{0\} \to (Z, N_Z)$ be a mapping satisfying either* (5) *for all $x \in X\backslash\{0\}$ or* (6) *for all $x \in X\backslash\{0\}$, and let $f : X \to (Y, N_Y)$ be an arbitrarily given mapping. If there exists an additive mapping $F : X \to (Y, N_Y)$ satisfying the inequality* (3) *for all $x \in X\backslash\{0\}$ and $t > 0$, then F is uniquely determined.*

Corollary 3.8 *Let $a \neq 1$ and $b = 1$ be positive rational numbers, let Y and Z be real normed spaces, let $\phi : X\backslash\{0\} \to Z$ be a mapping satisfying either* (11) *for all $x \in X\backslash\{0\}$, or* (12) *for all $x \in X\backslash\{0\}$, and let $f : X \to Y$ be an arbitrarily given mapping. If there exists an additive mapping $F : X \to Y$ satisfying* (13) *for all $x \in X\backslash\{0\}$, then F is a unique additive mapping satisfying* (13).

Corollary 3.9 *Let $a \neq 1$ and $b = 1$ be positive rational numbers, let $\phi : X\backslash\{0\} \to [0, \infty)$ be a mapping satisfying either* (14) *for all $x \in X\backslash\{0\}$ or* (15) *for all $x \in X\backslash\{0\}$, and let $f : X \to (Y, N_Y)$ be an arbitrarily given mapping. Let $N_{\mathbf{R}}$ be the fuzzy norm on $\mathbf{R}$ defined as in Corollary* 3.4. *If there exists an additive mapping $F : X \to (Y, N_Y)$ satisfying* (16) *for all $x \in X\backslash\{0\}$ and all $t > 0$, then F is a unique additive mapping satisfying* (16).

Corollary 3.10 *Let $a \neq 1$ and $b = 1$ be positive rational numbers, let Y be a normed space, let $\phi : X\backslash\{0\} \to [0, \infty)$ be a mapping satisfying either* (14) *for all $x \in X\backslash\{0\}$ or* (15) *for all $x \in X\backslash\{0\}$, and let $f : X \to Y$ be an arbitrary mapping. If there exists an additive mapping $F : X \to Y$ satisfying* (17) *for all $x \in X\backslash\{0\}$, then F is a unique additive mapping satisfying* (17).

Corollary 3.11 *Let $a \neq 1$, $b = 1$, and θ be positive real numbers, let $p \neq 1$ be a real number, and let f be a mapping from $\mathbf{R}$ into a normed space Y. If there is an additive mapping $F : \mathbf{R} \to Y$ satisfying the inequality* (18) *for each $x \in \mathbf{R}\backslash\{0\}$, then F is a unique additive mapping satisfying* (18).

Because each quadratic mapping satisfies the condition (4) for any given positive rational number $a \neq 1$ provided $b = 2$, we can replace the words 'additive mapping' with 'quadratic mapping' in Corollaries 3.7, 3.8, 3.9, 3.10, and 3.11.

Corollary 3.12 *Let $a \neq 1$ and $b = 2$ be positive rational numbers, let $\phi : X\backslash\{0\} \to (Z, N_Z)$ be a mapping satisfying either* (5) *for all $x \in X\backslash\{0\}$ or* (6) *for all $x \in X\backslash\{0\}$, and let $f : X \to (Y, N_Y)$ be an arbitrarily given mapping. If there exists a quadratic mapping $F : X \to (Y, N_Y)$ satisfying the inequality* (3) *for all $x \in X\backslash\{0\}$ and $t > 0$, then F is uniquely determined.*

Corollary 3.13 *Let $a \neq 1$ and $b = 2$ be positive rational numbers, let Y and Z be real normed spaces, let $\phi : X\backslash\{0\} \to Z$ be a mapping satisfying either* (11) *for all $x \in X\backslash\{0\}$ or* (12) *for all $x \in X\backslash\{0\}$, and let $f : X \to Y$ be an arbitrarily given mapping. If there exists a quadratic mapping $F : X \to Y$ satisfying* (13) *for all $x \in X\backslash\{0\}$, then F is a unique quadratic mapping satisfying* (13).

Corollary 3.14 *Let $a \neq 1$ and $b = 2$ be positive rational numbers, let $\phi : X\backslash\{0\} \to [0, \infty)$ be a mapping satisfying either* (14) *for all $x \in X\backslash\{0\}$ or* (15) *for all $x \in X\backslash\{0\}$, and let $f : X \to (Y, N_Y)$ be an arbitrarily given mapping. Let $N_{\mathbf{R}}$ be the fuzzy norm on $\mathbf{R}$ defined as* *in Corollary* 3.4. *If there exists a quadratic mapping* $F : X \to (Y, N_Y)$ *satisfying* (16) *for all* $x \in X\backslash\{0\}$ *and all* $t > 0$, *then* F *is a unique quadratic mapping satisfying* (16).

Corollary 3.15 *Let* $a \neq 1$ *and* $b = 2$ *be positive rational numbers, let* Y *be a normed space, let* $\phi : X\backslash\{0\} \to [0, \infty)$ *be a mapping satisfying either* (14) *for all* $x \in X\backslash\{0\}$ *or* (15) *for all* $x \in X\backslash\{0\}$, *and let* $f : X \to Y$ *be an arbitrary mapping. If there exists a quadratic mapping* $F : X \to Y$ *satisfying* (17) *for all* $x \in X\backslash\{0\}$, *then* F *is a unique quadratic mapping satisfying* (17).

Corollary 3.16 *Let* $a \neq 1$, $b = 2$, *and* θ *be positive real numbers, let* $p \neq 2$ *be a real number, and let* f *be a mapping from* $\mathbf{R}$ *into a normed space* Y. *If there is a quadratic mapping* $F : \mathbf{R} \to Y$ *satisfying* (18) *for each* $x \in X\backslash\{0\}$, *then* F *is a unique quadratic mapping satisfying* (18).

Remark 3.17 (1) Because every cubic mapping satisfies the condition (4) for any given positive rational number $a \neq 1$ provided $b = 3$, we can replace the words 'additive mapping' with 'cubic mapping' in Corollaries 3.7, 3.8, 3.9, 3.10, and 3.11 when $b = 3$.

(2) Because each quartic mapping satisfies the condition (4) for any given positive rational number $a \neq 1$ provided $b = 4$, we can replace the words 'additive mapping' with 'quartic mapping' in Corollaries 3.7, 3.8, 3.9, 3.10, and 3.11 when $b = 4$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Acknowledgements
Soon-Mo Jung was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2013R1A1A2005557).

Received: 2 December 2014 Accepted: 3 February 2015 Published online: 24 February 2015

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