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Limit properties of exceedance point processes of strongly dependent normal sequences

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Abstract
In this paper, we define an in plane Cox process and prove the time-normalized point process of exceedances by a dependent normal sequence converging to the Cox process in distribution under some mild conditions. As some applications of the convergence result, two important joint asymptotic distributions for the order statistics are derived.
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1 Introduction

Let $\{\xi_i, i \geq 1\}$ be a standardized normal sequence with correlation coefficients $r_{ij} = \mathrm{Cov}(\xi_i, \xi_j)$ and $M_n^{(k)}$ be the kth largest maxima of $\{\xi_i, 1 \leq i \leq n\}$. A conventional assumption is that $r_{ij} \to 0$ as $j - i \to +\infty$ at different rates according to which dependent normal sequences are classified into two different types: 'weakly dependent' and 'strongly dependent', respectively. Leadbetter *et al.* [1] considered the case: $r_{ij} = r_{|j-i|}$ and $r_n \log n \to 0$ as $n \to +\infty$, *i.e.* $\{\xi_i, i \geq 1\}$ is a weakly dependent stationary normal sequence. By using asymptotic independence, they focused on $M_n^{(k)}$ and its location (which is written as $L_n^{(k)}$) and obtained the asymptotic behavior of the probabilities $P(a_n(M_n^{(2)} - b_n) \leq x, L_n^{(2)}/n \leq t)$, and $P(a_n(M_n^{(1)} - b_n) \leq x_1, a_n(M_n^{(2)} - b_n) \leq x_2)$; here and in the sequel the standardized constants a_n and b_n are defined as

$$a_n = (2 \log n)^{1/2}, \qquad b_n = a_n - (2a_n)^{-1}(\log\log n + \log 4\pi). \tag{1.1}$$

Mittal and Ylvisaker [2] showed that if $r_{ij} = r_{|j-i|}$ and $r_n \log n \to \gamma > 0$ as $n \to +\infty$ (the strongly dependent stationary case) then $a_n(M_n^{(1)} - b_n)$ tends in distribution to a convolution of $\exp(-e^{-x})$ and a normal distribution function, and further if $r_n \log n \to \infty$ then by a different normalization, the limiting distribution is normal. Recently, several important results for extremes of dependent normal sequences were established. Ho and Hsing [3] and Tan and Peng [4] investigated the joint asymptotic distributions of the maximum of $\{\xi_i, 1 \leq i \leq n\}$ and $\sum_{i=1}^n \xi_i$ for dependent Gaussian sequences. Hashorva *et al.* [5] considered the joint limit distributions of maxima of complete and incomplete samples, respectively, *i.e.* the Piterbarg theorem under some conditions on convergence rate of the correlations. Leadbetter *et al.* [1] developed an important tool: the weak convergence of exceedance point processes which is crucial to study the joint asymptotic distributions of some extremes. Many authors further studied the asymptotic behavior of exceedance point processes under different conditions. We refer to Piterbarg [6], Hu *et al.* [7], Falk *et al.* [8], Peng *et al.* [9] and Hashorva *et al.* [10] for point processes of exceedances by weakly dependent stationary sequences including Gaussian ones and Wiśniewski [11], Lin *et al.* [12] for point processes of exceedances by strongly dependent Gaussian vector sequences.

Throughout this paper, let $\{\xi_i, i \geq 1\}$ be a standardized strongly dependent stationary normal sequence with correlation coefficients $r_{ij} = \operatorname{Cov}(\xi_i, \xi_j)$. C stands for a constant which may vary from line to line and '$\to$' for the convergence as $n \to \infty$. The remainder of the paper is organized as follows. In Section 2, we define an in plane Cox process and prove that the time-normalized point process N_n of exceedances of levels $u_n^{(1)}, u_n^{(2)}, \ldots, u_n^{(r)}$ by $\{\xi_i, 1 \leq i \leq n\}$ converges in distribution to the in plane Cox process. In Section 3, as the applications of our main result, the asymptotic results of the probabilities $P(a_n(M_n^{(2)} - b_n) \leq x, L_n^{(2)}/n \leq t)$, and $P(a_n(M_n^{(1)} - b_n) \leq x_1, a_n(M_n^{(2)} - b_n) \leq x_2)$ are established.

2 Convergence of point processes of exceedances

Let $\{\xi_i, i \geq 1\}$ be a standardized normal sequence with correlation coefficients $r_{ij} = \operatorname{Cov}(\xi_i, \xi_j)$ satisfying the following assumptions:

$$r_{ij} = r_{|j-i|} \quad \text{and} \quad r_n \log n \to \gamma \in (0, \infty) \quad \text{as } n \to +\infty. \tag{2.1}$$

We concentrate on deriving the convergence of the time-normalized exceedance point process N_n of the levels $u_n^{(1)}, u_n^{(2)}, \ldots, u_n^{(r)}$ by $\{\xi_i, 1 \leq i \leq n\}$ where $u_n^{(k)} = x_k/a_n + b_n$, $k = 1, 2, \ldots, r$. In order to prove the main result in this section, we shall use the famous Berman's inequality which is based on the early work of Slepian [13], Berman [14] and is polished up in Li and Shao [15]. The latest results related to Berman's inequality are Hashorva and Weng [16] and Lu and Wang [17]. The former gave a detailed introduction to Berman's inequality and derived the inequality for some general scaling random variable and thus obtained a Berman inequality for non-normal random vector. The upper bound of Berman's inequality gives an estimate of the difference between two standardized n-dimensional distribution functions by a convenient function of their covariances. According to Hashorva and Weng [16], some results for normal sequences may be extended to non-normal cases. The following lemmas are also needed in the proof of our result.

Lemma 2.1 *Let $d > 0$ and $\gamma \geq 0$ be constants, put $\rho_n = \gamma / \log n$ and suppose that $r_n \log n \to \gamma$ as $n \to \infty$. Then, for any sequence $\{u_n\}$ such that $n(1 - \Phi(u_n))$ is bounded, we have*

$$nd \sum_{k=1}^{[nd]} |r_k - \rho_n| \exp\left(-\frac{u_n^2}{1 + w_k}\right) \to 0 \quad \text{as } n \to +\infty,$$

where $w_k = \max\{|r_k|, \rho_n\}$.

Proof The proof can be found on p.134 in Leadbetter *et al.* [1]. □

Leadbetter *et al.* [1] defined a Cox process N with intensity $\exp(-x-\gamma+\sqrt{2\gamma}\zeta)$, where ζ is a standard normal random variable, *i.e.* the process has the distribution determined by the following probability:

$$P\left(\bigcap_{i=1}^{k}\{N(B_i)=k_i\}\right)=\int_{-\infty}^{\infty}\prod_{i=1}^{k}\left(\frac{(m(B_i)\exp(-x-\gamma+\sqrt{2\gamma}z))^{k_i}}{k_i!}\cdot\exp\left(-m(B_i)e^{-x-\gamma+\sqrt{2\gamma}z}\right)\right)\phi(z)\,dz, \tag{2.2}$$

where $m(\cdot)$ is the Lebesgue measure and proved that a point process N_n of time-normalized exceedances converges in distribution to N on $(0,+\infty)$ which is summarized as Lemma 2.2 below.

Lemma 2.2 *Suppose* $\{\xi_i, i\geq 1\}$ *is a standard stationary normal sequence with covariances satisfying* (2.1). *Then the point process* N_n *of time-normalized exceedances of the level* $u_n(u_n=x/a_n+b_n)$ *converges in distribution to* N *on* $(0,+\infty)$, *where* N *is the Cox process defined by* (2.2).

Proof The proof can be found on p.136 in Leadbetter *et al.* [1]. □

In Theorem 2.1 below, we extend Lemma 2.2 to the case of exceedances of several levels and study a vector of point processes $N_n=(N_n^{(1)},N_n^{(2)},\ldots,N_n^{(r)})$ which arises when $\{\xi_i, 1\leq i\leq n\}$ exceeds the levels $u_n^{(1)},u_n^{(2)},\ldots,u_n^{(r)}$, where $u_n^{(k)}=x_k/a_n+b_n$, $1\leq k\leq r$. For clarity, we record the locations of $u_n^{(1)},u_n^{(2)},\ldots,u_n^{(r)}$ along fixed horizontal lines $L_1,L_2,\ldots,L_r$ in the plane. The structure of the process vector is the same as that of the exceedances process on pp.111-112 in Leadbetter *et al.* [1] where the authors presented a detailed and visualized introduction. According to Lemma 2.2, each one-dimensional point process, on a given L_k, *i.e.* $N_n^{(k)}$ converges to a Cox process in distribution under appropriate conditions. Before presenting Theorem 2.1, we first give two definitions, one of which concerns a two-dimension Cox process, *i.e.* an in plane Cox process.

Definition 2.1 The locations of order statistics are the places where order statistics appear in the index set, for example the location of the maxima of $\{\xi_i, 1\leq i\leq n\}$ varying among $1,\ldots,n$.

Definition 2.2 Let $\{\sigma_{1j}, j=1,2,\ldots\}$ be the points of a Cox process $N^{(k)}$ on L_r with (stochastic) intensity $\exp(-x_r-\gamma+\sqrt{2\gamma}\zeta)$, where ζ is a standard normal random variable, *i.e.* $N^{(k)}$ has the distribution characterized in (2.2). Let $\beta_j, j=1,2,\ldots$ be independent and identically distributed (i.i.d.) random variables, independent also of the Cox process on L_r, taking values $1,2,\ldots,r$ with conditional probabilities

$$P(\beta_j=s|\zeta=z)=\begin{cases}(\tau_{r-s+1}-\tau_{r-s})/\tau_r, & \text{for } s=1,2,\ldots,r-1,\\ \tau_1/\tau_r, & \text{for } s=r,\end{cases}$$

i.e. $P(\beta_j\geq s|\zeta=z)=\tau_{r-s+1}/\tau_r$ for $s=1,2,\ldots,r$ where $\tau_i=e^{-x_i-\gamma+\sqrt{2\gamma}z}$, $i=1,2,\ldots,r$. For each j, place points $\sigma_{2j},\sigma_{3j},\ldots,\sigma_{\beta_j j}$ on β_j-1 lines $L_{r-1},L_{r-2},\ldots,L_{r-\beta_j+1}$, vertically above σ_{1j}, we can obtain an in plane Cox process N. Obviously the probability that a point appears L_{r-1}

above σ_{1j} is $P(\beta_j \geq 2|\zeta = z) = \tau_{r-1}/\tau_r$ and the deletions are conditionally independent, so that $N^{(r-1)}$ is obtained as a conditionally independent thinning of the Cox process $N^{(r)}$.

The structure of the in plane Cox process N is very similar to that of the Poisson process on p.112 in Leadbetter *et al.* [1], but the independent thinning is replaced with the conditionally independent thinning here.

Theorem 2.1 *Suppose* $\{\xi_i, i \geq 1\}$ *is a standardized normal sequence satisfying the conditions in Lemma* 2.2. *Let* $u_n^{(k)} = x_k/a_n + b_n$ *satisfy* $u_n^{(1)} \geq u_n^{(2)} \geq \cdots \geq u_n^{(r)}$ $(1 \leq k \leq r)$ *where* a_n *and* b_n *are defined in* (1.1). *Then the time-normalized point process* N_n *of exceedances of levels* $u_n^{(1)}, u_n^{(2)}, \ldots, u_n^{(r)}$ *by* $\{\xi_i, 1 \leq i \leq n\}$ *converges in distribution to the above-mentioned in plane Cox process.*

Proof It is sufficient to show that when n goes to ∞:

(a) $E(N_n(B)) \to E(N(B))$ for all sets B of the form $(c,d] \times (r,\delta]$, $r < \delta$, $0 < c < d$, where $d \leq 1$ and $E(\cdot)$ is the expectation,

(b) $P(N_n(B) = 0) \to P(N(B) = 0)$ for all sets B which are finite unions of disjoint sets of this form.

Focus on (a) firstly. If $B = (c,d] \times (r,\delta]$ intersects any of the lines, suppose these are $L_s, L_{s+1}, \ldots, L_t$ $(1 \leq s \leq t \leq r)$. Then

$$N_n(B) = \sum_{k=s}^{t} N_n^{(k)}\big((c,d]\big), \qquad N(B) = \sum_{k=s}^{t} N^{(k)}\big((c,d]\big)$$

and the number of points j/n in $(c,d]$ is $[nd] - [nc]$. As in the proof of Theorem 5.5.1 on p.113 in Leadbetter *et al.* [1], we have $E(N_n(B)) = ([nd] - [nc]) \sum_{k=s}^{t}(1 - F(u_n^{(k)}))$, where

$$1 - F\big(u_n^{(k)}\big) = 1 - \Phi\big(u_n^{(k)}\big), \quad 1 \leq j \leq n.$$

Obviously

$$n\big(1 - \Phi\big(u_n^{(k)}\big)\big) = n\big(1 - \Phi(x_k/a_n + b_n)\big) \sim e^{-x_k} \quad \text{as } n \to \infty. \tag{2.3}$$

Thus, we have $E(N_n(B)) \sim n(d-c) \sum_{k=s}^{t}(\frac{e^{-x_k}}{n} + o(\frac{1}{n})) \to (d-c) \sum_{k=s}^{t} e^{-x_k}$. Since

$$\begin{aligned} E\big(N(B)\big) &= \sum_{k=s}^{t} E\big((d-c)\exp(-x_k - \gamma + \sqrt{2\gamma}\zeta)\big) \\ &= \sum_{k=s}^{t} (d-c)e^{-x_k-\gamma} \cdot e^{\frac{(\sqrt{2\gamma})^2}{2}} \\ &= \sum_{k=s}^{t} (d-c)e^{-x_k}, \end{aligned}$$

(a) follows. In order to prove (b), we must show that $P(N_n(B) = 0) \to P(N(B) = 0)$, where $B = \bigcup_1^m C_k$ with disjoint $C_k = (c_k, d_k] \times (r_k, s_k]$. It is convenient to discard any set C_k which does not intersect any of the lines $L_1, L_2, \dots, L_r$. Because there are intersections and differences of the intervals $(c_k, d_k]$, we may write B in the form $\bigcup_{k=1}^{s}(c_k, d_k] \times E_k$, where $(c_k, d_k]$ are disjoint and E_k is a finite union of semi-closed intervals. It therefore follows that

$$\{N_n(B) = 0\} = \bigcap_{k=1}^{s}\{N_n(F_k) = 0\}, \tag{2.4}$$

where $F_k = (c_k, d_k] \times E_k$. Denote the lowest L_j intersecting F_k by L_{l_k}. By the above thinning property, obviously

$$\{N_n(F_k) = 0\} = \{N_n^{(l_k)}((c_k, d_k]) = 0\} = \{M_n(c_k, d_k) \le u_n^{(l_k)}\}, \tag{2.5}$$

where $M_n(c_k, d_k)$ stands for the maximum of $\{\eta_i, i \ge 1\}$ with index k ($[cn] < k \le [dn]$). Consider the probabilities of (2.4) and (2.5) and obtain

$$P(N_n(B) = 0) = P\left(\bigcap_{k=1}^{s}\{M_n(c_k, d_k) \le u_n^{(l_k)}\}\right). \tag{2.6}$$

It is convenient to firstly prove the following result. Let $\{\bar{\xi}_i, i \ge 1\}$ be a standardized normal sequence with the correlation coefficient ρ. $M_n(c, d; \rho)$ stands for the maximum of $\{\bar{\xi}_k\}$ with index k ($[cn] < k \le [dn]$). It is well known that $M_n(c_1, d_1; \rho), \dots, M_n(c_k, d_k; \rho)$ have the same distribution as $(1-\rho)^{1/2}M_n(c_1, d_1; 0) + \rho^{1/2}\zeta, \dots, (1-\rho)^{1/2}M_n(c_k, d_k; 0) + \rho^{1/2}\zeta$, where $c = c_1 < d_1 < \cdots < c_k < d_k = d$ and ζ is a standard normal variable; see Leadbetter *et al.* [1]. Next we must estimate the bound of

$$\left|P\left(\bigcap_{k=1}^{s}\{M_n(c_k, d_k) \le u_n^{(l_k)}\}\right) - P\left(\bigcap_{k=1}^{s}\{M_n(c_k, d_k, \rho_n) \le u_n^{(l_k)}\}\right)\right|, \tag{2.7}$$

where $\rho_n = \gamma / \log n$.

Using Berman's inequality, the bound of (2.7) does not exceed

$$\frac{1}{2\pi}\sum |r_{ij} - \rho_n|(1-\rho_n^2)^{-1/2}\exp\left(-\frac{\frac{1}{2}((u_n^{(i)})^2 + (u_n^{(j)})^2)}{1+\omega_{ij}}\right), \tag{2.8}$$

where the sum is carried out over $i < j$ and $i, j \in \bigcup_{k=1}^{s}([c_k n], [d_k n]]$, $u_n^{(i)}$ or $u_n^{(j)}$ stands for $x_i/a_n + b_n$ or $x_j/a_n + b_n$, and $\omega_{ij} = \max\{|r_{ij}|, \rho_n\}$. Furthermore, (2.8) does not exceed

$$\begin{aligned} & C\sum_{1\le i<j\le n}|r_{ij} - \rho_n|\exp\left(-\frac{\frac{1}{2}((x_i/a_n + b_n)^2 + (x_j/a_n + b_n)^2)}{1+\omega_{ij}}\right) \\ & < Cn\sum_{k=1}^{n}|r_k - \rho_n|\exp\left(-\frac{((\min_{1\le i\le n} x_i)/a_n + b_n)^2}{1+\omega_k}\right) \\ & \to 0. \end{aligned}$$

Noting $n(1 - \Phi((\min_{1\le i\le n} x_i)/a_n + b_n))$ is bounded, the last '$\to$' attributes to Lemma 2.1. So it suffices to prove

$$P\left(\bigcap_{k=1}^{s}\{M_n(c_k, d_k, \rho_n) \le u_n^{(l_k)}\}\right) \to P\left(\bigcap_{k=1}^{s}\{N(B) = 0\}\right).$$

Noting the definition of $M_n(c_k, d_k, \rho_n)$, clearly, it follows that

$$\begin{aligned}
&P\left(\bigcap_{k=1}^{s}\left\{M_n(c_k, d_k, \rho_n) \le u_n^{(l_k)}\right\}\right) \\
&= P\left(\bigcap_{k=1}^{s}\left\{(1-\rho_n)^{\frac{1}{2}} M_n(c_k, d_k, 0) + \rho_n^{\frac{1}{2}}\zeta \le u_n^{(l_k)}\right\}\right) \\
&= \int_{-\infty}^{+\infty} P\left(\bigcap_{k=1}^{s}\left\{M_n(c_k, d_k, 0) \le (1-\rho_n)^{-\frac{1}{2}}\left(u_n^{(l_k)} - \rho_n^{\frac{1}{2}} z\right)\right\}\right)\phi(z)\,dz,
\end{aligned}$$

where the proof of the last '=' can be completed by using the argument on the first line from the bottom on p.136 in Leadbetter *et al.* [1]. Since $a_n = (2\log n)^{\frac{1}{2}}$, $b_n = a_n + O(a_n^{-1}\log\log n)$, and $\rho_n = \gamma/\log n$, it is easy to show

$$(1-\rho_n)^{-\frac{1}{2}}\left(u_n^{(l_k)} - \rho_n^{\frac{1}{2}} z\right) = \frac{x_{l_k} + \gamma - \sqrt{2\gamma}z}{a_n} + b_n + o\left(a_n^{-1}\right),$$

see also the proof of Theorem 6.5.1 on p.137 in Leadbetter *et al.* [1]. Furthermore, we may obtain the following result:

$$\begin{aligned}
&P\left(\bigcap_{k=1}^{s}\left\{M_n(c_k, d_k, 0) \le (1-\rho_n)^{-\frac{1}{2}}\left(u_n^{(l_k)} - \rho_n^{\frac{1}{2}} z\right)\right\}\right) \\
&= P\left(\bigcap_{k=1}^{s}\left\{\tilde{\zeta}_{[c_k n]+1} \le (1-\rho_n)^{-\frac{1}{2}}\left(u_n^{(l_k)} - \rho_n^{\frac{1}{2}} z\right), \ldots,\right.\right. \\
&\qquad \left.\left.\tilde{\zeta}_{[d_k n]} \le (1-\rho_n)^{-\frac{1}{2}}\left(u_n^{(l_k)} - \rho_n^{\frac{1}{2}} z\right)\right\}\right) \\
&\to \prod_{k=1}^{s}\exp\left(-(d_k - c_k)e^{-x_{l_k} - \gamma + \sqrt{2\gamma}z}\right),
\end{aligned}$$

where $\tilde{\zeta}_k$ is a sequence of independent standard normal variables and we used the same arguments as (2.3) for the last step. Using the dominated convergence theorem, it follows that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} P\left(\bigcap_{k=1}^{s}\left\{M_n(c_k, d_k, 0) \le (1-\rho_n)^{-\frac{1}{2}}\left(u_n^{(l_k)} - \rho_n^{\frac{1}{2}} z\right)\right\}\right)\phi(z)\,dz \\
&\to \int_{-\infty}^{+\infty}\prod_{k=1}^{s}\exp\left(-(d_k - c_k)e^{-x_{l_k} - \gamma + \sqrt{2\gamma}z}\right)\phi(z)\,dz \\
&= P\left(N(B) = 0\right).
\end{aligned}$$

The proof of (b) is completed. □

Corollary 2.1 *Suppose $\{\xi_i, i \ge 1\}$ satisfies the conditions of Theorem* 2.1. *Let $B_1, \ldots, B_s$ be Borel subsets of the unit interval, whose boundaries have zero Lebesgue measure. Then for* *integers* $m_j^{(k)}$,

$$P\big(N_n^{(k)}(B_j)=m_j^{(k)}, j=1,2,\dots,s; k=1,2,\dots,r\big) \to P\big(N^{(k)}(B_j)=m_j^{(k)}, j=1,2,\dots,s; k=1,2,\dots,r\big).$$

Proof Combining Theorem 2.1 and the proof of Corollary 5.5.2 in Leadbetter *et al.* [1], we can complete the proof. □

Theorem 2.2 *Let the levels* $u_n^{(k)}$ $(1 \le k \le r)$ *satisfy*

$$P\Big(\max_{1\le i\le n}\xi_i \le u_n^{(k)}\Big) \to \int_{-\infty}^{+\infty} \exp\big(-e^{-x_k-\gamma+\sqrt{2\gamma}z}\big)\phi(z)\,dz, \quad n\to\infty,$$

with $u_n^{(1)} \ge u_n^{(2)} \ge \cdots \ge u_n^{(r)}$. *Let* $S_n^{(k)}$ *be the number of exceedances of* $u_n^{(k)}$ *by* $\{\xi_i, 1\le i\le n\}$. *Then for* $k_1\ge 0, k_2\ge 0,\dots,k_r\ge 0$,

$$\begin{aligned} &P\big(S_n^{(1)}=k_1, S_n^{(2)}=k_1+k_2,\dots,S_n^{(r)}=k_1+k_2+\cdots+k_r\big) \\ &\quad\to \frac{\tau_1^{k_1}(\tau_2-\tau_1)^{k_2}\cdots(\tau_r-\tau_{r-1})^{k_r}}{k_1!k_2!\cdots k_r!} \\ &\qquad\cdot \int_{-\infty}^{+\infty}\big(\exp(\sqrt{2\gamma}z-\gamma)\big)^{k_1+k_2+\cdots+k_r}\cdot\exp\big(-e^{-x_k-\gamma+\sqrt{2\gamma}z}\big)\phi(z)\,dz. \end{aligned} \tag{2.9}$$

Proof By Corollary 2.1, the left-hand side of (2.9) converges to

$$P\big(S^{(1)}=k_1, S^{(2)}=k_1+k_2,\dots,S^{(r)}=k_1+k_2+\cdots+k_r\big), \tag{2.10}$$

where $S^{(i)}=N^{(i)}([0,1])$. In our paper, the definition of the Cox process is similar to that of the in plane Poisson process in Leadbetter *et al.* [1]. So we can refer to the proof of Theorem 5.6.1 in Leadbetter *et al.* [1] and find that (2.10) equals

$$\begin{aligned} &\frac{(k_1+k_2+\cdots+k_r)!}{k_1!k_2!\cdots k_r!}\left(\frac{\tau_1}{\tau_r}\right)^{k_1}\left(\frac{\tau_2-\tau_1}{\tau_r}\right)^{k_2}\cdots\left(\frac{\tau_r-\tau_{r-1}}{\tau_r}\right)^{k_r} \\ &\quad\cdot P\big(N^{(r)}\big((0,1]\big)=k_1+k_2+\cdots+k_r\big). \end{aligned}$$

The proof is completed since

$$\begin{aligned} &P\big(N^{(r)}\big((0,1]\big)=k_1+k_2+\cdots+k_r\big) \\ &\quad=\int_{-\infty}^{+\infty}\frac{(\exp(-x_r-\gamma+\sqrt{2\gamma}z))^{k_1+k_2+\cdots+k_r}}{(k_1+k_2+\cdots+k_r)!}\cdot\exp\big(-e^{-x_r-\gamma+\sqrt{2\gamma}z}\big)\phi(z)\,dz \\ &\quad=\frac{(\exp(-x_r))^{k_1+k_2+\cdots+k_r}}{(k_1+k_2+\cdots+k_r)!}\int_{-\infty}^{+\infty}\big(\exp(-\gamma+\sqrt{2\gamma}z)\big)^{k_1+k_2+\cdots+k_r} \\ &\qquad\cdot\exp\big(-e^{-x_r-\gamma+\sqrt{2\gamma}z}\big)\phi(z)\,dz. \end{aligned}$$

□

3 The joint distributions of some order statistics

This section contains two important results which concerns the joint distributions of order statistics of $\{\xi_i, i\ge 1\}$.

Theorem 3.1 *Suppose $\{\xi_i, i \geq 1\}$ is a standard normal sequence satisfying the conditions of Theorem 2.1. Let $u_n^{(k)} = x_k/a_n + b_n$ and $M_n^{(2)}$, $L_n^{(2)}$ be the second largest maxima of $\xi_1, \xi_2, \ldots, \xi_n$ and its location. Then for $x_1 > x_2$, as $n \to \infty$,*

$$\begin{aligned} &P\big(a_n\big(M_n^{(1)} - b_n\big) \leq x_1, a_n\big(M_n^{(2)} - b_n\big) \leq x_2\big) \\ &\quad \to \int_{-\infty}^{+\infty} \big(\exp(-x_2 - \gamma + \sqrt{2\gamma}z) - \exp(-x_1 - \gamma + \sqrt{2\gamma}z) + 1\big) \\ &\qquad \cdot \exp\big(-e^{-x_2-\gamma+\sqrt{2\gamma}z}\big)\phi(z)\,dz \end{aligned} \tag{3.1}$$

and

$$P\left(a_n\big(M_n^{(2)} - b_n\big) \leq x, \frac{1}{n}L_n^{(2)} \leq t\right) \to \int_{-\infty}^{x} H(y,t)\,dy, \tag{3.2}$$

where

$$\begin{aligned} H(y,t) = {}& \int_{-\infty}^{+\infty} (1-t)\exp(-y-\gamma+\sqrt{2\gamma}z)\exp\big(-(1-t)e^{-y-\gamma+\sqrt{2\gamma}z}\big)\phi(z)\,dz \\ &\cdot \int_{-\infty}^{+\infty} t\exp(-y-\gamma+\sqrt{2\gamma}z)\exp\big(-te^{-y-\gamma+\sqrt{2\gamma}z}\big)\phi(z)\,dz \\ &+ \int_{-\infty}^{+\infty} \exp\big(-(1-t)e^{-y-\gamma+\sqrt{2\gamma}z}\big)\phi(z)\,dz \\ &\cdot \int_{-\infty}^{+\infty} t^2\exp(-2y-2\gamma+2\sqrt{2\gamma}z)\exp\big(-te^{-y-\gamma+\sqrt{2\gamma}z}\big)\phi(z)\,dz. \end{aligned}$$

Proof Clearly the left-hand side of (3.1) is equal to

$$\begin{aligned} &P\big(a_n\big(M_n^{(1)} - b_n\big) \leq x_1, a_n\big(M_n^{(2)} - b_n\big) \leq x_2\big) \\ &\quad = P\big(S_n^{(2)} = 0\big) + P\big(S_n^{(1)} = 0, S_n^{(2)} = 1\big), \end{aligned}$$

where $S_n^{(i)}$ is the number of exceedances of $u_n^{(i)}$ by $\eta_1, \eta_2, \ldots, \eta_n$. Using Theorem 2.2 in the special case yields (3.1). In order to prove (3.2), write I and J for intervals $\{1, 2, \ldots, [nt]\}$ and $\{[nt]+1, \ldots, n\}$, respectively. $M^{(1)}(I), M^{(2)}(I), M^{(1)}(J), M^{(2)}(J)$ stand for the maxima and second largest maxima of $\xi_1, \xi_2, \ldots, \xi_n$ in the intervals I, J, respectively. Let $H_n(x_1, x_2, x_3, x_4)$ be the joint d.f. of the normalized r.v.

$$\begin{aligned} X_n^{(1)} &= a_n\big(M_n^{(1)}(I) - b_n\big), & X_n^{(2)} &= a_n\big(M_n^{(2)}(I) - b_n\big), \\ Y_n^{(1)} &= a_n\big(M_n^{(1)}(J) - b_n\big), & Y_n^{(2)} &= a_n\big(M_n^{(2)}(J) - b_n\big). \end{aligned}$$

Generally let $x_1 > x_2$ and $x_3 > x_4$, that is,

$$\begin{aligned} &H_n(x_1, x_2, x_3, x_4) \\ &\quad = P\big(M_n^{(1)}(I) \leq u_n^{(1)}, M_n^{(2)}(I) \leq u_n^{(2)}, M_n^{(1)}(J) \leq u_n^{(3)}, M_n^{(2)}(J) \leq u_n^{(4)}\big) \\ &\quad = P\big(N_n^{(1)}(I') = 0, N_n^{(2)}(I') \leq 1, N_n^{(3)}(J') = 0, N_n^{(4)}(J') \leq 1\big), \end{aligned}$$

where $I' = (0,t]$ and $J' = (t,1]$. By using Corollary 2.1 with $B_1 = I'$ and $B_2 = J'$, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} H_n(x_1,x_2,x_3,x_4) \\
&\quad= \lim_{n\to\infty} P\bigl(N_n^{(1)}\bigl(I'\bigr) = 0, N_n^{(2)}\bigl(I'\bigr) \le 1\bigr) \cdot P\bigl(N_n^{(3)}\bigl(J'\bigr) = 0, N_n^{(4)}\bigl(J'\bigr) \le 1\bigr) \\
&\quad= \int_{-\infty}^{+\infty} \bigl(\bigl(e^{-x_2} - e^{-x_1}\bigr)t\exp(\sqrt{2\gamma} - \gamma) + 1\bigr)\exp\bigl(-te^{-x_2-\gamma+\sqrt{2\gamma}z}\bigr)\phi(z)\,dz \\
&\qquad\cdot \int_{-\infty}^{+\infty} \bigl(\bigl(e^{-x_4} - e^{-x_3}\bigr)(1-t)\exp(\sqrt{2\gamma} - \gamma) + 1\bigr)\exp\bigl(-(1-t)e^{-x_4-\gamma+\sqrt{2\gamma}z}\bigr)\phi(z)\,dz \\
&\quad= H_t(x_1,x_2)H_{1-t}(x_3,x_4) = H(x_1,x_2,x_3,x_4).
\end{aligned}$$

Now the left-hand side of (3.2) is equal to

$$\begin{aligned}
&P\bigl(M_n^{(2)}(I) \le u_n^{(2)}, M_n^{(2)}(I) \ge M_n^{(1)}(J)\bigr) \\
&\quad+ P\bigl(M_n^{(1)}(I) \le u_n^{(2)}, M_n^{(1)}(J) > M_n^{(1)}(I) \ge M_n^{(2)}(J)\bigr).
\end{aligned} \tag{3.3}$$

Obviously H is absolutely continuous and the boundaries of sets in R^4 such as $\{(w_1,w_2,w_3,w_4) : w_2 \le x_2, w_2 > w_3\}$ and $\{(w_1,w_2,w_3,w_4) : w_1 \le x_2, w_3 > w_1 \ge w_4\}$ have zero Lebesgue measure. Thus using Corollary 2.1, it follows that (3.3) converges to

$$P(X_2 \le x_2, X_2 \ge Y_1) + P(X_1 \le x_2, Y_1 > X_1 \ge Y_2).$$

According to the joint distribution $H(x_1,x_2,x_3,x_4)$ of X_1, X_2, Y_1, and Y_2, a simple evaluation completes the proof. □

Remark 3.1 We may obtain the joint asymptotic distribution of $M_n^{(1)}, M_n^{(2)}, \ldots, M_n^{(k)}$ by using the same method as in Theorem 3.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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