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A short-note on 'Common fixed point theorems for non-compatible self-maps in generalized metric spaces'

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Abstract
The main aim of this short-note is point out that certain hypotheses assumed on some results in the very recent paper (Yang in J. Inequal. Appl. 2014:275, 2014) are unnecessary, and the results contained in that manuscript can easily be improved.
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1 Introduction and preliminaries

In recent times, due to its possible application to almost all branches of numerical sciences, the researchers' interest about fixed point theory has raised very much. Especially significant have been the fixed point results in partially ordered metric spaces (see [1, 2]), in G-metric spaces (see [3–6]), among other abstract metric spaces (see [7, 8] in partial metric spaces, [9–11] in fuzzy metric spaces, [12, 13] in intuitionistic fuzzy metric spaces, [14, 15] in probabilistic metric spaces and [16, 17] in Menger spaces), even in the multidimensional case (see [18–25]). In this paper we focus in the setting of G-metric spaces. Some basic notions and results about G-metric spaces (metric structure, convergence, completeness, *etc.*) can be found, for instance, in [5, 6, 26, 27].

In the sequel, let (X, G) be a G-metric space and let $f, g : X \to X$ be two self-mappings. In [28], the author introduced the following notions and basic facts.

Definition 1 The self-mappings f and g are said to be *compatible* if

$$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0 \quad \text{and} \tag{1}$$

$$\lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0 \tag{2}$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \quad \text{for some } t \in X.$$

Definition 2 The self-mappings f and g are said to be *R-weakly commuting mappings of type* (A_g) if there exists some positive real number R such that

$$G(gfx,ffx,ffx) \leq RG(gx,fx,fx) \quad \text{for all } x \in X.$$

One of the main results in [28] is the following one.

Theorem 3 (Yang [28, Theorem 2.1]) *Let* (X,G) *be a G-metric space and* (f,g) *be a pair of non-compatible self-mappings with* $\overline{fX} \subset gX$ (*here* $\overline{fX}$ *denotes the closure of* fX). *Assume the following conditions are satisfied*:

$$\begin{aligned} G(fx,fy,fz) \leq \alpha \max\biggl\{ & G(gx,gy,gz), \frac{G(fx,gx,gx)+G(fy,gy,gy)}{2}, \\ & \frac{G(fy,gy,gy)+G(fz,gz,gz)}{2}, \frac{G(fz,gz,gz)+G(fx,gx,gx)}{2}, \\ & \frac{G(fx,gy,gz)+G(gx,fy,gz)}{2}, \frac{G(gx,fy,gz)+G(gx,gy,fz)}{2}\biggr\} \end{aligned} \tag{3}$$

for all $x,y,z \in X$. *Here* $\alpha \in [0,1)$. *If* (f,g) *are a pair of R-weakly commuting mappings of type* (A_g), *then* f *and* g *have a unique common fixed point* (*say* t) *and both* f *and* g *are not G-continuous at* t.

2 Main remarks

First of all, about the definition given by the author of *compatible mappings*, we must clarify that conditions (1) and (2) are equivalent. In fact, in any G-metric space (X,G), one of the most useful properties is the well known inequality $G(x,x,y) \leq 2G(x,y,y)$ for all $x,y \in X$. As a result, the following statement is trivial.

Proposition 4 *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences of a G-metric space* (X,G). *Then*

$$\lim_{n\to\infty} G(x_n,x_n,y_n) = 0 \quad \textit{if and only if} \quad \lim_{n\to\infty} G(x_n,y_n,y_n) = 0.$$

On the other hand, the author assumed in Theorem 3 that f and g are not compatible. In such a case, there exists a sequence $\{x_n\} \subseteq X$, such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \quad \text{for some } t \in X$$

but either

$$\lim_{n\to\infty} G(fgx_n,gfx_n,gfx_n) \quad \text{or} \quad \lim_{n\to\infty} G(gfx_n,fgx_n,fgx_n) \tag{4}$$

does or does not exist and if it does it is different from zero. Even avoiding condition (4), this property was introduced by Aamri and El Moutawakil in [29] in the context of metric spaces.

Definition 5 (Aamri and El Moutawakil [29]) Let $f,g: X \to X$ be two self-mappings of a metric space (X,d). We say that f and g satisfy the *(E.A.) property* if there exist a sequence

$\{x_n\} \subseteq X$ and a point $t \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t.$$

In the framework of G-metric spaces, we have the following analog.

Definition 6 (Mustafa *et al.* [30]) Let $f, g : X \to X$ be two self-mappings of a G-metric space (X, G). We say that f and g satisfy the *(E.A.) property* if there exist a sequence $\{x_n\} \subseteq X$ and a point $t \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t.$$

Also in Theorem 3, the author assumed that $\overline{fX} \subset gX$. In such a case, the limit verifies

$$t \in \overline{fX} \subset gX.$$

As a consequence, there exists $u \in X$ such that $t = gu$. This idea yields the following notion, called *common limit in the range of g*, which originally was introduced by Sintunaravat and Kumam in [31] in the context of fuzzy metric spaces and, later, was particularized to G-metric spaces by Aydi *et al.* in [32].

Definition 7 (Aydi *et al.* [32]) Let $f, g : X \to X$ be two self-mappings of a G-metric space (X, G). We say that f and g satisfy the *'common limit in the range of g' property* (briefly, $(CLRg)$*-property*) if there exist a sequence $\{x_n\} \subseteq X$ and a point $u \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu \in gX. \tag{5}$$

This conclusion also holds when gX is closed. Then we have the following properties.

Lemma 8

(1) *(f, g) is not compatible $\Rightarrow$ (f, g) satisfies the $(E.A.)$-property.*
(2) *(f, g) satisfies the $(E.A.)$-property and gX is closed $\Rightarrow$ (f, g) satisfies the $(CLRg)$-property.*
(3) *(f, g) satisfies the $(E.A.)$-property and $\overline{fX} \subset gX \Rightarrow (f, g)$ satisfies the $(CLRg)$-property.*

The $(CLRg)$-property has two main advantages: (1) usually, it is not necessary to assume the completeness of the G-metric space; and (2) usually, the common limit gu is a point of coincidence of f and g, that is, $fu = gu$. We show it in the next section.

Before that, we must point out that the author did not appropriately take limit in the inequalities throughout the paper. Let us show some examples. Following the lines of Theorem 2.1 in [28], as $t \in \overline{fX} \subset gX$, there exists $u \in X$ such that $gu = t$. Applying the contractivity condition (3) to $x = u$ and $y = z = x_n$, the author wrote (see [28, p.4, lines 18-19]):

$$G(fu, fx_n, fx_n)$$
$$\leq \alpha \max\left\{ G(gu, gx_n, gx_n), \frac{G(fu, gu, gu) + G(fx_n, gx_n, gx_n)}{2}, \right.$$

$$\frac{G(fx_n,gx_n,gx_n)+G(fx_n,gx_n,gx_n)}{2},\frac{G(fx_n,gx_n,gx_n)+G(fu,gu,gu)}{2},$$
$$\left.\frac{G(fu,gx_n,gx_n)+G(gu,fx_n,gx_n)}{2},\frac{G(gu,fx_n,gx_n)+G(gu,gx_n,fx_n)}{2}\right\}. \tag{6}$$

Letting $n \to \infty$, the author wrote (see [28, p.4, lines 21-22]):

$$\begin{aligned} G(fu,gu,gu) \leq \alpha \max\biggl\{ & G(gu,gu,gu), \frac{G(fu,gu,gu)+G(gu,gu,gu)}{2}, \\ & \frac{G(fu,gu,gu)+G(fu,gu,gu)}{2}, \frac{G(fu,gu,gu)+G(fu,gu,gu)}{2}, \\ & \frac{G(fu,gu,gu)+G(gu,fu,gu)}{2}, \frac{G(gu,fu,gu)+G(gu,gu,fu)}{2}\biggr\}. \end{aligned} \tag{7}$$

Unfortunately, inequality (7) is false, because the author seems to apply that $\{x_n\} \to u$ and f and g are continuous. This is not the case, because we only know that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t = gu.$$

In such a case, letting $n \to \infty$ in (6), we obtain

$$\begin{aligned} G(fu,gu,gu) \leq \alpha \max\biggl\{ & G(gu,gu,gu), \frac{G(fu,gu,gu)+G(gu,gu,gu)}{2}, \\ & \frac{G(gu,gu,gu)+G(gu,gu,gu)}{2}, \frac{G(gu,gu,gu)+G(fu,gu,gu)}{2}, \\ & \frac{G(fu,gu,gu)+G(gu,gu,gu)}{2}, \frac{G(gu,gu,gu)+G(gu,gu,gu)}{2}\biggr\} \\ = \alpha \max\biggl\{ & 0, \frac{G(fu,gu,gu)}{2}, 0, \frac{G(fu,gu,gu)}{2}, \frac{G(fu,gu,gu)}{2}, 0\biggr\} \\ = \frac{\alpha}{2} & G(fu,gu,gu). \end{aligned}$$

This correct inequality is better because we may assume that $\alpha \in [0,2)$ to deduce that $fu = gu$. In other words, as the reader can easily see, we can refine the arguments in [28] to get sharper results. This is the main aim of the present manuscript.

3 Common fixed point theorems

In the following result, we improve Theorem 3 in two senses: (1) our contractivity condition is weaker; and (2) we do not assume that f and g are not compatible.

Theorem 9 *Let* (X,G) *be a G-metric space and let* $f,g : X \to X$ *be two self-mappings satisfying the* $(CLRg)$*-property. Suppose that there exists* $\alpha \in [0,1)$ *such that*

$$\begin{aligned} G(fx,fy,fy) \leq \alpha \max\bigl\{ & G(gx,gy,gy), G(fx,gx,gx)+G(fy,gy,gy), 2G(fy,gy,gy), \\ & G(fx,gy,gy)+G(gx,fy,gy), 2G(gx,fy,gy)\bigr\} \end{aligned} \tag{8}$$

for all $x,y,z \in X$. *Then any point* $u \in X$ *as in* (5) *is a coincidence point of* f *and* g, *that is,* $fu = gu$.

Proof As (f,g) satisfies the $(CLRg)$-property, there exist a sequence $\{x_n\} \subseteq X$ and a point $u \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu \in gX. \tag{9}$$

Let us apply the contractivity condition using $x = u$ and $y = x_n$. Then, for all $n \in \mathbb{N}$, it follows that

$$\begin{aligned} G(fu,fx_n,fx_n) \leq \alpha \max\bigl\{&G(gu,gx_n,gx_n), G(fu,gu,gu) + G(fx_n,gx_n,gx_n), 2G(fx_n,gx_n,gx_n), \\ &G(fu,gx_n,gx_n) + G(gu,fx_n,gx_n), 2G(gu,fx_n,gx_n)\bigr\}. \end{aligned} \tag{10}$$

Taking into account (9) and the fact that G is jointly continuous on its three variables, then, letting $n \to \infty$ in (10), we deduce that

$$\begin{aligned} G(fu,gu,gu) &\leq \alpha \max\bigl\{G(gu,gu,gu), G(fu,gu,gu) + G(gu,gu,gu), 2G(gu,gu,gu), \\ &\qquad G(fu,gu,gu) + G(gu,gu,gu), 2G(gu,gu,gu)\bigr\} \\ &= \alpha G(fu,gu,gu). \end{aligned}$$

As $\alpha \in [0,1)$, then $G(fu,gu,gu) = 0$, so $fu = gu$. □

If the contractivity condition is slightly stronger, then it is easy to show a second part.

Theorem 10 *Let (X,G) be a G-metric space and let $f,g : X \to X$ be two self-mappings satisfying the $(CLRg)$-property. Suppose that there exists $\alpha \in [0,1)$ such that*

$$\begin{aligned} G(fx,fy,fy) \leq \alpha \max\Biggl\{&G(gx,gy,gy), G(fx,gx,gx) + G(fy,gy,gy), 2G(fy,gy,gy), \\ &\frac{G(fx,gy,gy) + G(gx,fy,gy)}{2}, G(gx,fy,gy)\Biggr\} \end{aligned} \tag{11}$$

for all $x,y \in X$. Then any point $u \in X$ as in (5) is a coincidence point of f and g, that is, $fu = gu$.

Furthermore, if (f,g) is a pair of R-weakly commuting mappings of type (A_g), then f and g have a unique common fixed point, which is $\omega = fu = gu$.

And if we additionally assume that f is G-continuous at ω, then

$$\lim_{n\to\infty} G(fgx_n,gfx_n,gfx_n) = \lim_{n\to\infty} G(gfx_n,fgx_n,fgx_n) = 0$$

whatever the sequence $\{x_n\}$ as in (5).

Proof Taking into account that

$$\frac{r+s}{2} \leq \max\{r,s\} \quad \text{for all } r,s \in \mathbb{R},$$

then condition (11) implies condition (8). As a consequence, Theorem 9 guarantees that any point $u \in X$ as in (5) is a coincidence point of f and g, that is,

$$gu = fu. \tag{12}$$

Next, assume that (f,g) is a pair of R-weakly commuting mappings of type (A_g). In such a case,

$$G(gfu,ffu,ffu) \leq RG(gu,fu,fu) = 0.$$

Therefore,

$$gfu = ffu. \tag{13}$$

Let us apply the contractivity condition (11) to $x = u$ and $y = fu$. Then we deduce

$$\begin{aligned} G(fu,ffu,ffu) \leq \alpha \max\biggl\{ & G(gu,gfu,gfu), G(fu,gu,gu) + G(ffu,gfu,gfu), 2G(ffu,gfu,gfu), \\ & \frac{G(fu,gfu,gfu) + G(gu,ffu,gfu)}{2}, G(gu,ffu,gfu)\biggr\}. \end{aligned}$$

By (12) and (13), it follows that

$$\begin{aligned} G(fu,ffu,ffu) &\leq \alpha \max\biggl\{ G(fu,ffu,ffu), G(fu,fu,fu) + G(ffu,ffu,ffu), 2G(ffu,ffu,ffu), \\ &\qquad \frac{G(fu,ffu,ffu) + G(fu,ffu,ffu)}{2}, G(fu,ffu,ffu)\biggr\} \\ &= \alpha G(fu,ffu,ffu), \end{aligned}$$

which means that

$$ffu = fu.$$

If we take $\omega = fu = gu$, then

$$fu = ffu = gfu \quad \Rightarrow \quad \omega = f\omega = g\omega,$$

so ω is a common fixed point of f and g.

Next we show that the common fixed point ω is unique. Actually, suppose that $z \in X$ is also a common fixed point of f and g. Then, by the contractivity condition (11) applied to $x = \omega$ and $y = z$, we derive that

$$\begin{aligned} G(\omega,z,z) &= G(f\omega,fz,fz) \\ &\leq \alpha \max\biggl\{ G(g\omega,gz,gz), G(f\omega,g\omega,g\omega) + G(fz,gz,gz), 2G(fz,gz,gz), \\ &\qquad \frac{G(f\omega,gz,gz) + G(g\omega,fz,gz)}{2}, G(g\omega,fz,gz)\biggr\} \\ &= \alpha G(\omega,z,z), \end{aligned}$$

which means that $\omega = z$.

Finally, assume that f is G-continuous at ω. Therefore, as $\{fx_n\} \to gu = \omega$ and $\{gx_n\} \to gu = \omega$,

$$\{ffx_n\} \to f\omega = \omega \quad \text{and} \quad \{fgx_n\} \to f\omega = \omega.$$

Moreover, as (f,g) is a pair of R-weakly commuting mappings of type (A_g),

$$G(gfx_n,ffx_n,ffx_n) \leq RG(gx_n,fx_n,fx_n).$$

Hence, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} G(fgx_n,gfx_n,gfx_n) &\leq G(fgx_n,ffx_n,ffx_n) + G(ffx_n,gfx_n,gfx_n) \\ &\leq G(fgx_n,ffx_n,ffx_n) + 2G(gfx_n,ffx_n,ffx_n) \\ &\leq G(fgx_n,ffx_n,ffx_n) + 2RG(gx_n,fx_n,fx_n). \end{aligned}$$

Taking the limit as $k \to \infty$ we deduce that

$$\lim_{n\to\infty} G(fgx_n,gfx_n,gfx_n) = 0,$$

and, by Proposition 4, we conclude that

$$\lim_{n\to\infty} G(gfx_n,fgx_n,fgx_n) = 0,$$

which means that f and g are compatible. □

Remark 11 In Theorem 3, the author assumed that f and g are not compatible, and it is announced that f and g are not G-continuous at ω. By the previous theorem, if f and g are not compatible, then f cannot be G-continuous at ω. However, the argument given by the author to prove that g is not G-continuous at ω is not correct: assuming that g is continuous at ω, it is proved that $\{ffx_n\}$ converges to $\omega = f\omega$, but this does not mean that f is G-continuous at ω (this property must be demonstrated for all sequence $\{y_n\}$ converging to ω).

Corollary 12 *Theorem* 3 (*avoiding the unproved fact that g is not G-continuous at the unique common fixed point*) *is an immediate consequence of Theorem* 10.

Proof It follows from the fact that (3) implies (11) using $y = z$. □

In the sequel, we extend the previous results. Let

$$\mathcal{F} = \left\{ \phi : [0,\infty) \to [0,\infty) : \phi(t) < t \text{ and } \lim_{s\to t^+} \phi(s) < t \text{ for all } t > 0 \right\}.$$

It is clear that, given $\alpha \in [0,1)$, the mapping $\phi_\alpha : [0,\infty) \to [0,\infty)$ defined by $\phi_\alpha(s) = \alpha s$ for all $s \in [0,\infty)$, belongs to $\mathcal{F}$.

Theorem 13 *Let* (X,G) *be a G-metric space and let* $f,g : X \to X$ *be two self-mappings satisfying the* (*CLRg*)*-property. Suppose that there exists* $\phi \in \mathcal{F}$ *such that*

$$\begin{aligned} G(fx,fy,fy) \leq \phi\big(\max\big\{&G(gx,gy,gy), G(fx,gx,gx) + G(fy,gy,gy), 2G(fy,gy,gy), \\ &G(fx,gy,gy), G(gx,fy,gy), G(gx,gy,fy)\big\}\big) \end{aligned} \tag{14}$$

for all $x, y \in X$. Then any point $u \in X$ as in (5) *is a coincidence point of f and g, that is, $fu = gu$.*

Furthermore, if (f,g) is a pair of R-weakly commuting mappings of type (A_g), then f and g have a unique common fixed point, which is $\omega = fu = gu$.

And if we additionally assume that f is G-continuous at ω, then

$$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = \lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0$$

whatever the sequence $\{x_n\}$ as in (5), *that is, f and g are compatible.*

Proof For convenience, let us define, for all $x, y \in X$,

$$\begin{aligned} M(x,y) = \max\bigl\{&G(gx,gy,gy), G(fx,gx,gx) + G(fy,gy,gy), 2G(fy,gy,gy), \\ &G(fx,gy,gy), G(gx,fy,gy), G(gx,gy,fy)\bigr\}. \end{aligned}$$

Hence, the contractivity condition (14) can be rewritten as

$$G(fx,fy,fy) \le \phi\bigl(M(x,y)\bigr) \quad \text{for all } x, y \in X.$$

As (f,g) satisfies the $(CLRg)$-property, there exist a sequence $\{x_n\} \subseteq X$ and a point $u \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu \in gX. \tag{15}$$

We prove that $fu = gu$ by *reductio ad absurdum*, that is, we assume that $fu \neq gu$ and we will get a contradiction. In such a case,

$$G(fu,gu,gu) > 0.$$

Let us apply the contractivity condition (14) using $x = u$ and $y = x_n$. Then, for all $n \in \mathbb{N}$, it follows that

$$G(fu,fx_n,fx_n) \le \phi\bigl(M(u,x_n)\bigr), \tag{16}$$

where

$$\begin{aligned} M(u,x_n) = \max\bigl\{&G(gu,gx_n,gx_n), G(fu,gu,gu) + G(fx_n,gx_n,gx_n), 2G(fx_n,gx_n,gx_n), \\ &G(fu,gx_n,gx_n), G(gu,fx_n,gx_n), G(gu,gx_n,fx_n)\bigr\}. \end{aligned}$$

We can distinguish two cases.

- *Case* 1. *Assume that there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $M(u,x_{n(k)}) \le G(fu,gu,gu)$ for all $k \in \mathbb{N}$.* In such a case, as

$$\begin{aligned} G(fu,gu,gu) &\le G(fu,gu,gu) + G(fx_{n(k)},gx_{n(k)},gx_{n(k)}) \\ &\le M(u,x_{n(k)}) \le G(fu,gu,gu), \end{aligned}$$

we deduce that $M(u,x_{n(k)}) = G(fu,gu,gu)$ for all $k \in \mathbb{N}$. Using (16), we have

$$G(fu,fx_{n(k)},fx_{n(k)}) \leq \phi\big(M(u,x_{n(k)})\big) = \phi\big(G(fu,gu,gu)\big).$$

Taking the limit as $k \to \infty$, we deduce that

$$G(fu,gu,gu) = \lim_{n\to\infty} G(fu,fx_n,fx_n) \leq \phi\big(G(fu,gu,gu)\big).$$

Since $\phi \in \mathcal{F}$ and $G(fu,gu,gu) > 0$, it follows that

$$G(fu,gu,gu) \leq \phi\big(G(fu,gu,gu)\big) < G(fu,gu,gu),$$

which is a contradiction.

- *Case* 2. *Assume that there exists* $n_0 \in \mathbb{N}$ *such that* $M(u,x_n) > G(fu,gu,gu)$ *for all* $n \geq n_0$. In such a case, we have

$$\lim_{n\to\infty} M(u,x_n) = G(fu,gu,gu) \quad \text{and} \quad M(u,x_n) > G(fu,gu,gu) \quad \text{for all } n \geq n_0.$$

Hence, as $\phi \in \mathcal{F}$, it follows from (16) that

$$\begin{aligned} G(fu,gu,gu) &= \lim_{n\to\infty} G(fu,fx_n,fx_n) \leq \lim_{n\to\infty} \phi\big(M(u,x_n)\big) \\ &= \lim_{s\to G(fu,gu,gu)^+} \phi(s) < G(fu,gu,gu), \end{aligned}$$

which is also a contradiction.

In any case, we get a contradiction, so we must admit that $fu = gu$, that is, u is a coincidence point of f and g.

Next, assume that (f,g) is a pair of R-weakly commuting mappings of type (A_g). In such a case,

$$G(gfu,ffu,ffu) \leq RG(gu,fu,fu) = 0.$$

Therefore,

$$gfu = ffu.$$

Let us apply the contractivity condition (14) to $x = u$ and $y = fu$. Then we deduce

$$G(fu,ffu,ffu) \leq \phi\big(M(u,fu)\big),$$

where

$$\begin{aligned} M(u,fu) = {} & \max\big\{G(gu,gfu,gfu), G(fu,gu,gu) + G(ffu,gfu,gfu), 2G(ffu,gfu,gfu), \\ & G(fu,gfu,gfu), G(gu,ffu,gfu), G(gu,gfu,ffu)\big\} \\ = {} & \max\big\{G(fu,ffu,ffu), G(fu,fu,fu) + G(ffu,ffu,ffu), 2G(ffu,ffu,ffu), \end{aligned}$$

$$\begin{aligned} &G(fu,ffu,ffu),G(fu,ffu,ffu),G(fu,ffu,ffu)\bigr\} \\ &= G(fu,ffu,ffu). \end{aligned}$$

As a consequence,

$$G(fu,ffu,ffu) \leq \phi\bigl(M(u,fu)\bigr) = \phi\bigl(G(fu,ffu,ffu)\bigr).$$

If $fu \neq ffu$, then

$$G(fu,ffu,ffu) \leq \phi\bigl(G(fu,ffu,ffu)\bigr) < G(fu,ffu,ffu),$$

which is impossible. Then, necessarily,

$$ffu = fu.$$

If we take $\omega = fu = gu$, then

$$fu = ffu = gfu \quad \Rightarrow \quad \omega = f\omega = g\omega,$$

so ω is a common fixed point of f and g.

Next we show that the common fixed point ω is unique. Actually, suppose that $z \in X$ is also a common fixed point of f and g. Then, by the contractivity condition (14) applied to $x = \omega$ and $y = z$, we derive that

$$G(\omega,z,z) = G(f\omega,fz,fz) \leq \phi\bigl(M(\omega,z)\bigr),$$

where

$$\begin{aligned} M(\omega,z) = {}& \max\bigl\{G(g\omega,gz,gz),G(f\omega,g\omega,g\omega)+G(fz,gz,gz),2G(fz,gz,gz), \\ &G(f\omega,gz,gz),G(g\omega,fz,gz),G(g\omega,gz,fz)\bigr\} \\ = {}& G(\omega,z,z). \end{aligned}$$

The condition

$$G(\omega,z,z) \leq \phi\bigl(M(\omega,z)\bigr) = \phi\bigl(G(\omega,z,z)\bigr)$$

implies that $G(\omega,z,z) = 0$, which means that $\omega = z$.

Finally, assume that f is G-continuous at ω. Therefore, as $\{fx_n\} \to gu = \omega$ and $\{gx_n\} \to gu = \omega$,

$$\{ffx_n\} \to f\omega = \omega \quad \text{and} \quad \{fgx_n\} \to f\omega = \omega.$$

Moreover, as (f,g) is a pair of R-weakly commuting mappings of type (A_g),

$$G(gfx_n,ffx_n,ffx_n) \leq RG(gx_n,fx_n,fx_n).$$

Hence, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} G(fgx_n, gfx_n, gfx_n) &\leq G(fgx_n, ffx_n, ffx_n) + G(ffx_n, gfx_n, gfx_n) \\ &\leq G(fgx_n, ffx_n, ffx_n) + 2G(gfx_n, ffx_n, ffx_n) \\ &\leq G(fgx_n, ffx_n, ffx_n) + 2RG(gx_n, fx_n, fx_n) \\ &\leq G(fgx_n, \omega, \omega) + G(\omega, ffx_n, ffx_n) + 2RG(gx_n, fx_n, fx_n). \end{aligned}$$

Taking the limit as $k \to \infty$ we deduce that

$$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0,$$

and, by Proposition 4, we conclude that

$$\lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0,$$

which means that f and g are compatible. □

Taking into account that

$$\frac{r+s}{2} \leq \max\{r, s\} \quad \text{for all } r, s \in \mathbb{R},$$

then Theorem 3 (avoiding the unproved fact that g is not G-continuous at the unique common fixed point) is an immediate consequence of Theorem 13.

One of the conclusions of Theorem 3 is that f and g are not continuous at ω. Such a result is not applicable when f and g are continuous mappings, which is a very common hypothesis in fixed point theory, as in the following example.

Example 14 Let $X = [0, \infty)$ be endowed with the complete G-metric $G(x, y, z) = |x - y| + |x - z| + |y - z|$ for all $x, y, z \in X$, and let us consider the mappings $f, g : X \to X$ defined by $fx = x$ and $gx = 2x$ for all $x \in X$. The sequence $x_n = 1/n$ for all $n \geq 1$ shows that f and g satisfy the $(CLRg)$-property. Furthermore, for all $x, y \in X$, we have

$$\begin{aligned} G(fx, fy, fy) &= G(x, y, y) = 2|x - y| = \frac{1}{2} 4|x - y| \\ &= \frac{1}{2} G(2x, 2y, 2y) = \frac{1}{2} G(gx, gy, gy). \end{aligned}$$

Then Theorem 10 guarantees that f and g have a coincidence point (and so does Theorem 13). In fact, as f is the identity mapping on X, trivially f and g are R-weakly commuting mappings of type (A_g) for $R = 1$, so f and g have a unique common fixed point, which is $\omega = 0$. In addition to this, as f is continuous, f and g are compatible. Nevertheless, as f and g are compatible and continuous, Theorem 3 is not applicable.

In the following example we illustrate the applicability of Theorems 10 and 13, and we also show that the contractivity conditions (11) and (14) are easier to prove than (3) because they only involve two variables ($\{x, y\}$ rather than $\{x, y, z\}$).

Example 15 Let $X = [0,5]$ be endowed with the complete G-metric $G(x,y,z) = |x-y| + |x-z| + |y-z|$ for all $x,y,z \in X$, and let us consider the mappings $f,g : X \to X$ defined, for all $x \in X$, by

$$fx = \begin{cases} 1, & \text{if } 0 < x \leq 1, \\ 0, & \text{otherwise;} \end{cases} \qquad gx = \begin{cases} 0, & \text{if } x = 0, \\ 2, & \text{if } 0 < x \leq 1, \\ \frac{x-1}{2}, & \text{if } 1 < x \leq 5. \end{cases}$$

The sequence $x_n = 1 + 1/n$ for all $n \geq 1$ shows that f and g satisfy the $(CLRg)$-property because

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 = g0 \in gX.$$

We claim that the contractivity condition (11) is satisfied using $\alpha = 1/2$. Indeed, on the one hand, we have, for all $x,y \in X$,

$$G(fx,fy,fy) = 2|fx - fy| = \begin{cases} 2, & \text{if } [0 < x \leq 1 \text{ and } y \in \{0\} \cup (1,5]] \\ & \text{or } [0 < y \leq 1 \text{ and } x \in \{0\} \cup (1,5]], \\ 0, & \text{otherwise.} \end{cases}$$

We only have to discuss the cases in which $G(fx,fy,fy)$ takes the value 2. We distinguish the following possibilities.

- If $0 < x \leq 1$ and $y = 0$, then

$$G(gx,gy,gy) = G(2,0,0) = 4 = \frac{1}{2}2 = \alpha G(fx,fy,fy).$$

- If $0 < x \leq 1$ and $y \in (1,5]$, then

$$\begin{aligned} G(gx,fy,gy) &= G\left(2,0,\frac{y-1}{2}\right) = 2 + \left|2 - \frac{y-1}{2}\right| + \left|\frac{y-1}{2}\right| \\ &= 2 + 2 - \frac{y-1}{2} + \frac{y-1}{2} = 4 = \frac{1}{2}2 = \alpha G(fx,fy,fy). \end{aligned}$$

- If $0 < y \leq 1$ and $x \in \{0\} \cup (1,5]$, then

$$2G(fy,gy,gy) = 2G(1,2,2) = 4 = \frac{1}{2}2 = \alpha G(fx,fy,fy).$$

In any case, the contractivity condition (11) holds. Furthermore, as

$$G(gfx,ffx,ffx) = \begin{cases} 2, & \text{if } 0 < x \leq 1, \\ 0, & \text{otherwise;} \end{cases} \quad \text{and} \quad G(gx,fx,fx) = \begin{cases} 0, & \text{if } x = 0, \\ 2, & \text{if } 0 < x \leq 1, \\ x-1, & \text{if } 1 < x \leq 5, \end{cases}$$

f and g are R-weakly commuting mappings of type (A_g), where $R = 1$. As a consequence, Theorem 10 guarantees that f and g have a unique common fixed point (and so does Theorem 13).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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