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Some results on a viscosity splitting algorithm in Hilbert spaces

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Abstract

In this paper, a viscosity splitting for common solution problems is proposed. Strong convergence theorems are obtained in the framework of Hilbert spaces. Applications are also provided to support the main results.

Keywords: zero point; fixed point; variational inclusion; nonexpansive mapping

1 Introduction; preliminaries

In this paper, we always assume that H is a real Hilbert space with the inner product $\langle x,y\rangle$ and the induced norm $\|x\| = \sqrt{\langle x,x\rangle}$ for $x,y \in H$. Recall that a set-valued mapping $M : H \rightrightarrows H$ is said to be *monotone* iff, for all $x,y \in H$, $f \in Mx$, and $g \in My$ imply $\langle x-y, f-g\rangle \geq 0$. In this paper, we use $M^{-1}(0)$ to denote the zero point set of M. A monotone mapping $M : H \rightrightarrows H$ is *maximal* iff the graph Graph(M) of M is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping M is maximal if and only if, for any $(x,f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$, for all $(y,g) \in$ Graph(M) implies $f \in Mx$. For a maximal monotone operator M on H, and $r > 0$, we may define the single-valued resolvent $J_r : H \to$ Dom(M), where Dom(M) denotes the domain of M. It is well known that J_r is firmly nonexpansive, and $M^{-1}(0) = F(J_r)$.

The proximal point algorithm, which was proposed by Martinet [1, 2] and generalized by Rockafellar [3, 4] is one of the classical methods for solving zero points of maximal monotone operators. In this paper, we investigate the problem of finding a zero of the sum of two monotone operators. The problem is very general in the sense that it includes, as special cases, convexly constrained linear inverse problems, split feasibility problem, convexly constrained minimization problems, fixed point problems, variational inequalities, Nash equilibrium problem in noncooperative games and others. Because of their importance, splitting methods, which were proposed by Lions and Mercier [5] and Passty [6], for zero problems have been studied extensively recently; see, for instance, [7–17] and the references therein.

Let C be a nonempty closed and convex subset of H. Let $A : C \to H$ be a mapping. Recall that the classical variational inequality problem is to find a point $x \in C$ such that

$$\langle y - x, Ax\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

Such a point $x \in C$ is called a solution of variational inequality (1.1). In this paper, we use $VI(C,A)$ to denote the solution set of variational inequality (1.1). Recall that A is said to be *monotone* iff

$$\langle Ax - Ay, x - y\rangle \ge 0, \quad \forall x, y \in C.$$

Recall that A is said to be *inverse-strongly monotone* iff there exists a constant $\kappa > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \kappa \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, we also call A is κ-inverse-strongly monotone. It is also not hard to see that every inverse-strongly monotone mapping is monotone and continuous.

Let $S : C \to C$ be a mapping. In this paper, we use $F(S)$ to denote the fixed point set of S. S is said to be *contractive* iff there exists a constant $\beta \in (0,1)$ such that

$$\|Sx - Sy\| \le \beta \|x - y\|, \quad \forall x, y \in C.$$

We also call S is β-contractive. S is said to be *nonexpansive* iff

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in C.$$

It is well known if C is nonempty closed convex of H, then $F(S)$ is not empty. S is said to be *firmly nonexpansive* iff

$$\|Sx - Sy\|^2 \le \|x - y\|^2 - \big\|(I - S)x - (I - S)y\big\|^2, \quad \forall x, y \in C.$$

In order to prove our main results, we also need the following lemmas.

Lemma 1.1 [18] *Let A be a maximal monotone operator on H. For $\lambda > 0$, $\mu > 0$, and $x \in E$, we have $J_\lambda x = J_\mu(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_\lambda x)$, where $J_\lambda = (I + \lambda A)^{-1}$ and $J_\mu = (I + \mu A)^{-1}$.*

Lemma 1.2 [19] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in H. Let $\{\beta_n\}$ be a sequence in $(0,1)$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$, $\forall n \ge 1$ and*

$$\limsup_{n\to\infty}\big(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\big) \le 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.3 [20] *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition $a_{n+1} \le (1 - t_n)a_n + t_n b_n$, $\forall n \ge 0$, where $\{t_n\}$ is a number sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=0}^{\infty} t_n = \infty$, $\{b_n\}$ is a number sequence such that $\limsup_{n\to\infty} b_n \le 0$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 1.4 [21] *Let C be a nonempty closed convex subset of H. Let $A : C \to H$ be a mapping and let $B : H \rightrightarrows H$ be a maximal monotone operator. Then $\|x - (I + sB)^{-1}x\| \le 2\|x - (I + rB)^{-1}x\|$ for all $0 < s \le r$.*

Lemma 1.5 [22] *Let $\{\lambda_n\}$ be a real sequence that does not decreasing at infinity, in the sense that there exists a subsequence $\{\lambda_{n_k}\}$ such that $\lambda_{n_k} \le \lambda_{n_k+1}$ for all $k \ge 0$. For every $n > n_0$, define an integer sequence $d(n)$ as $d(n) = \max\{n_0 \le k \le n | \lambda_{n_k} \le \lambda_{n_k+1}\}$. Then $\lim_{n\to\infty} d(n) = 0$ and for all $n > n_0$ $\max\{\lambda_{d(n)}, \lambda_n\} \le \lambda_{d(n)+1}$.*

Lemma 1.6 [23] *Let C be a nonempty closed convex subset of H. Let $S : C \to C$ be a nonexpansive mapping with a nonempty fixed point set. If $\{x_n\}$ converges weakly to x and $\{\|x_n - Tx_n\|\}$ converges to zero. Then $x \in F(S)$.*

2 Main results

Now, we are in a position to state our main results.

Theorem 2.1 *Let C be a nonempty closed convex subset of H. Let $S : C \to C$ be a nonexpansive mapping with fixed points and let $f : C \to C$ be a β-contractive mapping. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let B be a maximal monotone operator on H. Assume that $\mathrm{Dom}(B) \subset C$ and $F(S) \cap (A+B)^{-1}(0)$ is not empty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in $(0,1)$ and let $\{r_n\}$ be a positive real number sequence in $(0, 2\alpha)$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S(I + r_n B)^{-1}(y_n - r_n A y_n), \quad \forall n \ge 1. \end{cases}$$

Assume that the control sequences satisfy the following restrictions:
- (a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (b) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
- (c) $0 < a \le r_n \le b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$,

where a and b are two real numbers. Then $\{x_n\}$ converges strongly to a point $\bar{x} \in F(S) \cap (A + B)^{-1}(0)$, which is also a unique solution to the following variational inequality:

$$\langle f(\bar{x}) - \bar{x}, p - \bar{x} \rangle \le 0, \quad \forall p \in F(S) \cap (A+B)^{-1}(0).$$

Proof Note that the mapping $I - r_n A$ is nonexpansive. Indeed, we have

$$\begin{aligned} &\|(I - r_n A)x - (I - r_n A)y\|^2 \\ &\quad = \|x-y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + {r_n}^2\|Ax - Ay\|^2 \\ &\quad \le \|x-y\|^2 - r_n(2\alpha - r_n)\|Ax - Ay\|^2. \end{aligned}$$

In light of restriction (c), one finds that $I - r_n A$ is nonexpansive. It is obvious that $F((I + r_n B)^{-1}(I - r_n A)) = (A+B)^{-1}(0)$. Fix $p \in (A+B)^{-1}(0) \cap F(S)$. It follows that

$$\begin{aligned} \|y_n - p\| &\le \alpha_n \|f(x_n) - p\| + (1-\alpha_n)\|x_n - p\| \\ &\le \bigl(1 - \alpha_n(1-\beta)\bigr)\|x_n - p\| + \alpha_n \|f(p) - p\|. \end{aligned}$$

Putting $J_{r_n} = (I + r_n B)^{-1}$, we see that

$$\begin{aligned}\|x_{n+1} - p\| &\le \beta_n\|x_n - p\| + (1-\beta_n)\big\|SJ_{r_n}(y_n - r_nAy_n) - p\big\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\big\|(y_n - r_nAy_n) - p\big\| \\ &\le \big(1 - \alpha_n(1-\beta_n)(1-\beta)\big)\|x_n - p\| + \alpha_n(1-\beta_n)(1-\beta)\frac{\|f(p) - p\|}{1-\beta}.\end{aligned}$$

By mathematical induction, we find that the sequence $\{x_n\}$ is bounded. Note that

$$\begin{aligned}\|y_n - y_{n-1}\| &\le \alpha_n\big\|f(x_n) - f(x_{n-1})\big\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}|\big\|f(x_{n-1}) - x_{n-1}\big\| \\ &\le \big(1 - \alpha_n(1-\beta)\big)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\big\|f(x_{n-1}) - x_{n-1}\big\|.\end{aligned}$$

Putting $z_n = y_n - r_nAy_n$, we find from Lemma 1.1 that

$$\begin{aligned}\|J_{r_n}z_n - J_{r_{n-1}}z_{n-1}\| &\le \left\|\frac{r_{n-1}}{r_n}(z_n - z_{n-1}) + \left(1 - \frac{r_{n-1}}{r_n}\right)(J_{r_n}z_n - z_{n-1})\right\| \\ &\le \|z_n - z_{n-1}\| + \frac{|r_n - r_{n-1}|}{a}\|J_{r_n}z_n - z_n\| \\ &\le \|y_n - y_{n-1}\| + |r_n - r_{n-1}|\|Ay_{n-1}\| + \frac{|r_n - r_{n-1}|}{a}\|J_{r_n}z_n - z_n\| \\ &\le \big(1 - \alpha_n(1-\beta)\big)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\big\|f(x_{n-1}) - x_{n-1}\big\| \\ &\quad + |r_n - r_{n-1}|\|Ay_{n-1}\| + \frac{|r_n - r_{n-1}|}{a}\|J_{r_n}z_n - z_n\|.\end{aligned}$$

This yields

$$\begin{aligned}\|SJ_{r_n}z_n - SJ_{r_{n-1}}z_{n-1}\| &\le \|J_{r_n}z_n - J_{r_{n-1}}z_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\big\|f(x_{n-1}) - x_{n-1}\big\| \\ &\quad + |r_n - r_{n-1}|\|Ay_{n-1}\| + \frac{|r_n - r_{n-1}|}{a}\|J_{r_n}z_n - z_n\|.\end{aligned}$$

It follows from restrictions (a) and (c) that

$$\limsup_{n\to\infty}\big(\|SJ_{r_n}z_n - SJ_{r_{n-1}}z_{n-1}\| - \|x_n - x_{n-1}\|\big) \le 0.$$

Using Lemma 1.2, we have $\lim_{n\to\infty}\|SJ_{r_n}z_n - x_n\| = 0$. It follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.1}$$

Since $\|y_n - x_n\| = \alpha_n\|f(x_n) - x_n\|$, we find that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{2.2}$$

Since $\|\cdot\|^2$ is convex, we find that

$$\|y_n - p\|^2 \le \alpha_n \big\|f(x_n) - p\big\|^2 + (1-\alpha_n)\|x_n - p\|^2. \tag{2.3}$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|SJ_{r_n}z_n - p\|^2\\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\big\|J_{r_n}(I - r_nA)y_n - p\big\|^2\\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\big\|(I - r_nA)y_n - p\big\|^2\\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 - r_n(1-\beta_n)(2\alpha - r_n)\|Ay_n - Ap\|^2\\ &\le \|x_n - p\|^2 + \alpha_n\big\|f(x_n) - p\big\|^2 - r_n(1-\beta_n)(2\alpha - r_n)\|Ay_n - Ap\|^2.\end{aligned}$$

Hence, we have

$$\begin{aligned}&r_n(1-\beta_n)(2\alpha - r_n)\|Ay_n - Ap\|^2\\ &\quad\le \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_{n+1} - x_n\| + \alpha_n\big\|f(x_n) - p\big\|^2.\end{aligned}$$

In view of restrictions (a), (b), and (c), we find from (2.1) that

$$\lim_{n\to\infty}\|Ay_n - Ap\| = 0. \tag{2.4}$$

Since J_{r_n} is firmly nonexpansive, we have

$$\begin{aligned}\|J_{r_n}z_n - p\|^2 &\le \big\langle J_{r_n}z_n - p, (y_n - r_nAy_n) - (p - r_nAp)\big\rangle\\ &= \frac{1}{2}\Big(\|J_{r_n}z_n - p\|^2 + \big\|(y_n - r_nAy_n) - (p - r_nAp)\big\|^2\\ &\quad - \big\|(J_{r_n}z_n - p) - \big((y_n - r_nAy_n) - (p - r_nAp)\big)\big\|^2\Big)\\ &\le \frac{1}{2}\Big(\|J_{r_n}z_n - p\|^2 + \|y_n - p\|^2 - \|J_{r_n}z_n - y_n\|^2\\ &\quad + 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\|\Big).\end{aligned}$$

This implies from (2.3) that

$$\begin{aligned}\|J_{r_n}z_n - p\|^2 &\le \|y_n - p\|^2 - \|J_{r_n}z_n - y_n\|^2 + 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\|\\ &\le \alpha_n\big\|f(x_n) - p\big\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \|J_{r_n}z_n - y_n\|^2\\ &\quad + 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\|.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|SJ_{r_n}z_n - p\|^2\\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|J_{r_n}z_n - p\|^2\end{aligned}$$

$$\begin{aligned}&\leq \|x_n - p\|^2 + \alpha_n \big\|f(x_n) - p\big\|^2 - (1-\beta_n)\|J_{r_n}z_n - y_n\|^2\\&\quad + 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\|.\end{aligned}$$

This implies that

$$\begin{aligned}(1-\beta_n)\|J_{r_n}z_n - y_n\|^2 &\leq \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_n - x_{n+1}\| + \alpha_n\big\|f(x_n) - p\big\|^2\\&\quad + 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\|.\end{aligned}$$

In view of restrictions (a) and (b), we find from (2.1) and (2.4) that

$$\lim_{n\to\infty}\|J_{r_n}z_n - y_n\| = 0. \tag{2.5}$$

Next, we show that $\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, y_n - \bar{x}\rangle \leq 0$, where $\bar{x} = \mathrm{Proj}_{F(S)\cap(A+B)^{-1}(0)}f(\bar{x})$. To show it, we can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\big\langle f(\bar{x}) - \bar{x}, y_n - \bar{x}\big\rangle = \lim_{i\to\infty}\big\langle f(\bar{x}) - \bar{x}, y_{n_i} - \bar{x}\big\rangle.$$

Since $\{y_{n_i}\}$ is bounded, we can choose a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly some point x. We may assume, without loss of generality, that y_{n_i} converges weakly to x.

Now, we are in a position to show that $x \in (A+B)^{-1}(0)$. Set $\lambda_n = J_{r_n}(y_n - r_nAy_n)$. It follows that $\frac{y_n-\lambda_n}{r_n} - Ay_n \in B\lambda_n$. Since B is monotone, we get, for any $(\mu,\nu)\in B$,

$$\left\langle \lambda_n - \mu, \frac{y_n - \lambda_n}{r_n} - Ay_n - \nu\right\rangle \geq 0.$$

Replacing n by n_i and letting $i\to\infty$, we obtain from (2.5) that

$$\langle x - \mu, -Ax - \nu\rangle \geq 0.$$

This gives $-Ax \in Bx$, that is, $0 \in (A+B)(x)$. This proves that $x \in (A+B)^{-1}(0)$.

Now, we are in a position to prove that $x \in F(S)$. Notice that

$$\|S\lambda_n - y_n\| \leq \frac{1}{1-\beta_n}\|x_{n+1} - y_n\| + \frac{\beta_n}{1-\beta_n}\|y_n - x_n\|.$$

This implies that $\lim_{n\to\infty}\|S\lambda_n - y_n\| = 0$. This implies from (2.5) that $\|S\lambda_n - \lambda_n\| \to 0$. Since $I - S$ is demiclosed at zero, we find that $x \in F(S)$. This complete the proof that $x \in F(S)\cap(A+B)^{-1}(0)$. It follows that

$$\limsup_{n\to\infty}\big\langle f(\bar{x}) - \bar{x}, y_n - \bar{x}\big\rangle \leq 0.$$

Finally, we show that $x_n \to \bar{x}$. Notice that

$$\begin{aligned}\|y_n - \bar{x}\|^2 &\leq \alpha_n\big\langle f(x_n) - \bar{x}, y_n - \bar{x}\big\rangle + (1-\alpha_n)\|x_n - \bar{x}\|\|y_n - \bar{x}\|\\&\leq \big(1 - \alpha_n(1-\beta)\big)\|x_n - \bar{x}\|\|y_n - \bar{x}\| + \alpha_n\big\langle f(\bar{x}) - \bar{x}, y_n - \bar{x}\big\rangle.\end{aligned}$$

This implies that

$$\|y_n - \bar{x}\|^2 \le 2\alpha_n \langle f(\bar{x}) - \bar{x}, y_n - \bar{x} \rangle + \big(1 - \alpha_n(1-\beta)\big)\|x_n - \bar{x}\|^2.$$

It follows that

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &\le \beta_n\|x_n - \bar{x}\|^2 + (1-\beta_n)\big\|SJ_{r_n}(I - r_nA)y_n - \bar{x}\big\|^2 \\ &\le \beta_n\|x_n - \bar{x}\|^2 + (1-\beta_n)\|y_n - \bar{x}\|^2 \\ &\le \big(1 - \alpha_n(1-\beta_n)(1-\beta)\big)\|x_n - \bar{x}\|^2 + 2\alpha_n(1-\beta_n)\langle f(\bar{x}) - \bar{x}, y_n - \bar{x}\rangle.\end{aligned}$$

In view of restrictions (a) and (b), we find from Lemma 1.3 that $x_n \to \bar{x}$. This completes the proof. □

From Theorem 2.1, we have the following results immediately.

Corollary 2.2 *Let C be a nonempty closed convex subset of H. Let $S : C \to C$ be a nonexpansive mapping with fixed points and let $f : C \to C$ be a β-contractive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in* $(0,1)$. *Let $\{x_n\}$ be a sequence generated in the following process*: $x_1 \in C$ *and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)Sy_n, \quad \forall n \ge 1. \end{cases}$$

Assume that the control sequences satisfy the following restrictions:

(a) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(b) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$.

Then $\{x_n\}$ converges strongly to a point $\bar{x} \in F(S)$, which is also a unique solution to the following variational inequality:

$$\langle f(\bar{x}) - \bar{x}, p - \bar{x}\rangle \le 0, \quad \forall p \in F(S).$$

Corollary 2.3 *Let C be a nonempty closed convex subset of H. Let $f : C \to C$ be a β-contractive mapping. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let B be a maximal monotone operator on H. Assume that* $\operatorname{Dom}(B) \subset C$ *and* $(A+B)^{-1}(0)$ *is not empty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in* $(0,1)$ *and $\{r_n\}$ be a positive real number sequence in* $(0,2\alpha)$. *Let $\{x_n\}$ be a sequence generated in the following process*: $x_1 \in C$ *and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(I + r_nB)^{-1}(y_n - r_nAy_n), \quad \forall n \ge 1. \end{cases}$$

Assume that the control sequences satisfy the following restrictions:

(a) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(b) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(c) $0 < a \le r_n \le b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$,

where a and b are two real numbers. Then $\{x_n\}$ *converges strongly to a point* $\bar{x} \in (A+B)^{-1}(0)$, *which is also a unique solution to the following variational inequality*:

$$\langle f(\bar{x}) - \bar{x}, p - \bar{x} \rangle \le 0, \quad \forall p \in (A+B)^{-1}(0).$$

Next, we give a result on the zeros of the sum of the operators A and B based on a different method.

Theorem 2.4 *Let* C *be a nonempty closed convex subset of* H. *Let* $f : C \to C$ *be a* β-*contractive mapping. Let* $A : C \to H$ *be an* α-*inverse-strongly monotone mapping and let* B *be a maximal monotone operator on* H. *Assume that* $\mathrm{Dom}(B) \subset C$ *and* $(A+B)^{-1}(0)$ *is not empty. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real number sequences in* $(0,1)$ *and* $\{r_n\}$ *be a positive real number sequence in* $(0, 2\alpha)$. *Let* $\{x_n\}$ *be a sequence generated in the following process*: $x_1 \in C$ *and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(I + r_n B)^{-1}(y_n - r_n A y_n), \quad \forall n \ge 1. \end{cases}$$

Assume that the control sequences satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $0 \le \beta_n \le \bar{\beta} < 1$;

(c) $0 < a \le r_n \le b < 2\alpha$,

where $\bar{\beta}$, *a, and b are real numbers. Then* $\{x_n\}$ *converges strongly to a point* $\bar{x} \in (A+B)^{-1}(0)$, *which is also a unique solution to the following variational inequality*:

$$\langle f(\bar{x}) - \bar{x}, p - \bar{x} \rangle \le 0, \quad \forall p \in (A+B)^{-1}(0).$$

Proof From the proof of Theorem 2.1, we find that $\{x_n\}$ is bounded. Since $P_{F(S)\cap(A+B)^{-1}(0)}f$ is contractive, it has a unique fixed point. Next, we use $\bar{x}$ to denote the unique fixed point. Note that

$$\begin{aligned} \|y_n - \bar{x}\|^2 &= \alpha_n \langle f(x_n) - \bar{x}, y_n - \bar{x} \rangle + (1-\alpha_n)\langle x_n - \bar{x}, y_n - \bar{x} \rangle \\ &\le \bigl(1 - \alpha_n(1-\beta)\bigr)\|x_n - \bar{x}\| \|y_n - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, y_n - \bar{x} \rangle \\ &\le \frac{(1 - \alpha_n(1-\beta)}{2}\bigl(\|x_n - \bar{x}\|^2 + \|y_n - \bar{x}\|^2\bigr) + \alpha_n \langle f(\bar{x}) - \bar{x}, y_n - \bar{x} \rangle. \end{aligned}$$

It follows that

$$\|y_n - \bar{x}\|^2 \le \left(1 - \frac{2\alpha_n(1-\beta)}{1 + \alpha_n(1-\beta)}\right)\|x_n - \bar{x}\|^2 + \frac{2\alpha_n}{1 + \alpha_n(1-\beta)}\langle f(\bar{x}) - \bar{x}, y_n - \bar{x} \rangle. \tag{2.6}$$

Since J_{r_n} is firmly nonexpansive and A is inverse-strongly monotone, we find that

$$\begin{aligned} \bigl\|J_{r_n}(y_n - r_n A y_n) - \bar{x}\bigr\|^2 &\le \bigl\|(y_n - r_n A y_n) - (\bar{x} - r_n A\bar{x})\bigr\|^2 \\ &\quad - \bigl\|(I - J_{r_n})(y_n - r_n A y_n) - (I - J_{r_n})(\bar{x} - r_n A\bar{x})\bigr\|^2 \end{aligned}$$

$$\begin{aligned}&\leq \|y_n - \bar{x}\|^2 - r_n(2\alpha - r_n)\|Ay_n - A\bar{x}\|^2 \\ &\quad - \big\|(I - J_{r_n})(y_n - r_nAy_n) - (I - J_{r_n})(\bar{x} - r_nA\bar{x})\big\|^2. \qquad (2.7)\end{aligned}$$

Substituting (2.6) into (2.7), we find that

$$\begin{aligned}&\big\|J_{r_n}(y_n - r_nAy_n) - \bar{x}\big\|^2 \\ &\quad\leq \left(1 - \frac{2\alpha_n(1-\beta)}{1+\alpha_n(1-\beta)}\right)\|x_n - \bar{x}\|^2 + \frac{2\alpha_n}{1+\alpha_n(1-\beta)}\big\langle f(\bar{x}) - \bar{x}, y_n - \bar{x}\big\rangle \\ &\qquad - r_n(2\alpha - r_n)\|Ay_n - A\bar{x}\|^2 - \big\|(I - J_{r_n})(y_n - r_nAy_n) - (I - J_{r_n})(\bar{x} - r_nA\bar{x})\big\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &\leq \beta_n\|x_n - \bar{x}\|^2 + (1-\beta_n)\big\|J_{r_n}(y_n - r_nAy_n) - \bar{x}\big\|^2 \\ &\leq \left(1 - \frac{2\alpha_n(1-\beta_n)(1-\beta)}{1+\alpha_n(1-\beta)}\right)\|x_n - \bar{x}\|^2 + \frac{2\alpha_n(1-\beta_n)}{1+\alpha_n(1-\beta)}\big\langle f(\bar{x}) - \bar{x}, y_n - \bar{x}\big\rangle \\ &\quad - (1-\beta_n)r_n(2\alpha - r_n)\|Ay_n - A\bar{x}\|^2 \\ &\quad - (1-\beta_n)\big\|(I - J_{r_n})(y_n - r_nAy_n) - (I - J_{r_n})(\bar{x} - r_nA\bar{x})\big\|^2. \qquad (2.8)\end{aligned}$$

Next, we consider the following possible two cases.

Case 1. Suppose that there exists some nonnegative integer m such that the sequence $\{\|x_n - \bar{x}\|\}$ is eventually decreasing. Then $\lim_{n\to\infty}\|x_n - \bar{x}\|$ exists. By (2.8), we find that

$$\begin{aligned}&(1-\beta_n)r_n(2\alpha - r_n)\|Ay_n - A\bar{x}\|^2 \\ &\quad\leq \|x_n - \bar{x}\|^2 - \|x_{n+1} - \bar{x}\|^2 + 2\alpha_n\big\|f(\bar{x}) - \bar{x}\big\|\|y_n - \bar{x}\|.\end{aligned}$$

By use of restrictions (b) and (c), we have $\lim_{n\to\infty}\|Ay_n - A\bar{x}\| = 0$. It also follows from (2.8) that

$$\begin{aligned}&(1-\beta_n)\big\|(I - J_{r_n})(y_n - r_nAy_n) - (I - J_{r_n})(\bar{x} - r_nA\bar{x})\big\|^2 \\ &\quad\leq \|x_n - \bar{x}\|^2 - \|x_{n+1} - \bar{x}\|^2 + 2\alpha_n\big\|f(\bar{x}) - \bar{x}\big\|\|y_n - \bar{x}\|.\end{aligned}$$

From restrictions (b) and (c), we obtain

$$\lim_{n\to\infty}\big\|(I - J_{r_n})(y_n - r_nAy_n) - (I - J_{r_n})(\bar{x} - r_nA\bar{x})\big\| = 0.$$

Hence, we have $\lim_{n\to\infty}\|y_n - J_{r_n}(y_n - r_nAy_n)\| = 0$. From Lemma 1.4, we find that $\|y_n - J_r(y_n - rAy_n)\| \leq 2\|y_n - J_{r_n}(y_n - r_nAy_n)\|$. This implies that $\lim_{n\to\infty}\|y_n - J_r(y_n - rAy_n)\| = 0$. Next, we show that $\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, y_n - \bar{x}\rangle \leq 0$. To show it, we can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\big\langle f(\bar{x}) - \bar{x}, y_n - \bar{x}\big\rangle = \lim_{i\to\infty}\big\langle f(\bar{x}) - \bar{x}, y_{n_i} - \bar{x}\big\rangle.$$

Since $\{y_{n_i}\}$ is bounded, we can choose a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to some point x. We may assume, without loss of generality, that y_{n_i} converges weakly to x.

Since the mapping $J_r(I - rA)$ is nonexpansive, we find that $x \in F(J_r(I - rA)) = (A + B)^{-1}(0)$. It follows that $\limsup_{n\to\infty} \langle f(\bar{x}) - \bar{x}, y_n - \bar{x}\rangle \leq 0$. In view of (2.8), we find from Lemma 1.3 that $x_n \to \bar{x}$.

Next, we consider another case.

Case 2. Suppose that the sequence $\{\|x_n - \bar{x}\|\}$ is not eventually decreasing. There exists a subsequence $\{\|x_{n_i} - \bar{x}\|\}$ such that $\|x_{n_i} - \bar{x}\| \leq \|x_{n_i+1} - \bar{x}\|$ for all $i \geq 0$. We define an integer sequence $k(n)$ as in Lemma 1.5. By use of (2.8), we have

$$\begin{aligned}
&\|x_{k(n)+1} - \bar{x}\|^2 \\
&\quad\leq \left(1 - \frac{2\alpha_{k(n)}(1-\beta_{k(n)})(1-\beta)}{1+\alpha_{k(n)}(1-\beta)}\right)\|x_{k(n)} - \bar{x}\|^2 \\
&\qquad + \frac{2\alpha_{k(n)}(1-\beta_{k(n)})}{1+\alpha_{k(n)}(1-\beta)}\langle f(\bar{x}) - \bar{x}, y_{k(n)} - \bar{x}\rangle \\
&\qquad - (1-\beta_{k(n)})r_{k(n)}(2\alpha - r_{k(n)})\|Ay_{k(n)} - A\bar{x}\|^2 \\
&\qquad - (1-\beta_{k(n)})\left\|(I - J_{r_{k(n)}})(y_{k(n)} - r_{k(n)}Ay_{k(n)}) - (I - J_{r_{k(n)}})(\bar{x} - r_{k(n)}A\bar{x})\right\|^2.
\end{aligned} \tag{2.9}$$

It follows that

$$\lim_{n\to\infty}\left\|y_{k(n)} - J_{r_{k(n)}}(y_{k(n)} - r_{k(n)}Ay_{k(n)})\right\| = 0. \tag{2.10}$$

Hence, we have $\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, y_{k(n)} - \bar{x}\rangle \leq 0$. In view of (2.9), we find that

$$\lim_{n\to\infty}\|x_{k(n)} - \bar{x}\| = 0.$$

Note that

$$\begin{aligned}
\|y_{k(n)+1} - \bar{x}\| &\leq \|y_{k(n)+1} - x_{k(n)+1}\| + \|x_{k(n)+1} - x_{k(n)}\| + \|x_{k(n)} - \bar{x}\| \\
&\leq \alpha_{k(n)+1}\left\|f(x_{k(n)+1}) - x_{k(n)+1}\right\| \\
&\quad + (1-\beta_{k(n)})\left\|J_{r_{k(n)}}(y_{k(n)} - r_{k(n)}Ay_{k(n)}) - x_{k(n)}\right\| + \|x_{k(n)} - \bar{x}\| \\
&\leq \alpha_{k(n)+1}\left\|f(x_{k(n)+1}) - x_{k(n)+1}\right\| + \left\|J_{r_{k(n)}}(y_{k(n)} - r_{k(n)}Ay_{k(n)}) - y_{k(n)}\right\| \\
&\quad + \alpha_{k(n)}\left\|f(x_{k(n)}) - x_{k(n)}\right\| + \|x_{k(n)} - \bar{x}\|.
\end{aligned}$$

By use of (2.10), we find that $\lim_{n\to\infty}\|y_n - \bar{x}\| = 0$. Since $\|x_n - \bar{x}\| \leq \alpha_n\|f(x_n) - x_n\| + \|y_n - \bar{x}\|$, we find that $\lim_{n\to\infty}\|x_n - \bar{x}\| = 0$. This completes the proof. □

Remark 2.5 Comparing Theorem 2.4 with the recent results announced in [7, 11] and [24], we have the following:

(i) Our proofs are different from theirs.

(ii) We remove the additional restriction $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$.

3 Applications

In this section, we investigate solutions of equilibrium problems, variational inequalities and convex minimization problems, respectively.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x,y) \geq 0, \quad \forall y \in C. \tag{3.1}$$

In this paper, we use $EP(F)$ to denote the solution set of the equilibrium problem.

To study equilibrium problems (3.1), we may assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F\bigl(tz + (1-t)x, y\bigr) \leq F(x,y);$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semi-continuous.

Lemma 3.1 [24] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4) *and let B be a multivalued mapping of H into itself defined by*

$$Bx = \begin{cases} \{z \in H : F(x,y) \geq \langle y - x, z \rangle, \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{3.2}$$

Then B is a maximal monotone operator with the domain $D(B_F) \subset C$, $EP(F) = B^{-1}(0)$, and $T_r x = (I + rB)^{-1}x$, $\forall x \in H$, $r > 0$, where T_r is defined as

$$T_r x = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}.$$

Theorem 3.2 *Let C be a nonempty closed convex subset of H. Let $S : C \to C$ be a nonexpansive mapping with fixed points and let $f : C \to C$ be a β-contractive mapping. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4). *Assume that $F(S) \cap EP(F)$ is not empty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in* $(0,1)$ *and let $\{r_n\}$ be a positive real number sequence in* $(0, 2\alpha)$. *Let $\{x_n\}$ be a sequence generated in the following process*: $x_1 \in C$ *and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1 - \alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S(I + r_n B)^{-1}(y_n - r_n A y_n), \quad \forall n \geq 1, \end{cases}$$

where B is a mapping defined as in (3.2). *Assume that the control sequences satisfy the following restrictions*:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(c) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$, *where a and b are two real numbers. Then* $\{x_n\}$ *converges strongly to a point* $\bar{x} \in F(S) \cap EP(F)$*, which is a unique solution to the following variational inequality*:

$$\langle f(\bar{x}) - \bar{x}, p - \bar{x} \rangle \leq 0, \quad \forall p \in F(S) \cap EP(F).$$

Theorem 3.3 *Let* C *be a nonempty closed convex subset of* H. *Let* $f : C \to C$ *be a* β-*contractive mapping. Let* $A : C \to H$ *be an* α-*inverse-strongly monotone mapping and Let* F *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies* (A1)-(A4). *Assume that* $EP(F)$ *is not empty. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real number sequences in* $(0,1)$ *and let* $\{r_n\}$ *be a positive real number sequence in* $(0, 2\alpha)$. *Let* $\{x_n\}$ *be a sequence generated in the following process*: $x_1 \in C$ *and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1 - \alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)(I + r_n B)^{-1}(y_n - r_n A y_n), \quad \forall n \geq 1. \end{cases}$$

Assume that the control sequences satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(b) $0 \leq \beta_n \leq \bar{\beta} < 1$;

(c) $0 < a \leq r_n \leq b < 2\alpha$,

where $\bar{\beta}$, a, *and* b *are real numbers. Then* $\{x_n\}$ *converges strongly to a point* $\bar{x} \in (A + B)^{-1}(0)$, *which is also a unique solution to the following variational inequality*:

$$\langle f(\bar{x}) - \bar{x}, p - \bar{x} \rangle \leq 0, \quad \forall p \in (A + B)^{-1}(0).$$

Let $g : H \to (-\infty, +\infty]$ be a proper convex lower semi-continuous function. Then the subdifferential ∂g of g is defined as follows:

$$\partial g(x) = \{ y \in H : g(z) \geq g(x) + \langle z - x, y \rangle, z \in H \}, \quad \forall x \in H.$$

From Rockafellar [4], we know that ∂g is maximal monotone. It is easy to verify that $0 \in \partial g(x)$ if and only if $g(x) = \min_{y \in H} g(y)$. Let I_C be the indicator function of C, *i.e.*,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases} \tag{3.3}$$

Since I_C is a proper lower semi-continuous convex function on H, we see that the subdifferential ∂I_C of I_C is a maximal monotone operator.

Lemma 3.4 [24] *Let* C *be a nonempty closed convex subset of* H *and let* Proj_C *be the metric projection from* H *onto* C. *Let* ∂I_C *be the subdifferential of* I_C, *where* I_C *is as defined in* (3.3). *Then* $y = (I + \lambda \partial I_C)^{-1} x \Longleftrightarrow y = \mathrm{Proj}_C x$, $\forall x \in H$, $y \in C$,

Theorem 3.5 *Let* C *be a nonempty closed convex subset of* H. *Let* $S : C \to C$ *be a nonexpansive mapping with fixed points and let* $f : C \to C$ *be a* β-*contractive mapping. Let* $A : C \to H$ *be an* α-*inverse-strongly monotone mapping. Assume that* $F(S) \cap VI(C, A)$ *is not empty. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real number sequences in* $(0,1)$ *and let* $\{r_n\}$ *be a positive* *real number sequence in* $(0, 2\alpha)$. *Let* $\{x_n\}$ *be a sequence generated in the following process*: $x_1 \in C$ *and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S\operatorname{Proj}_C(y_n - r_n A y_n), \quad \forall n \geq 1. \end{cases}$$

Assume that the control sequences satisfy the following restrictions:

(1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(3) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$,

where a *and* b *are two real numbers. Then* $\{x_n\}$ *converges strongly to a point* $\bar{x} \in F(S) \cap VI(C, A)$, *which is a unique solution to the following variational inequality*:

$$\langle f(\bar{x}) - \bar{x}, p - \bar{x} \rangle \leq 0, \quad \forall p \in F(S) \cap VI(C, A).$$

Proof Putting $Bx = \partial I_C$, we find from Theorem 2.1 and Lemma 3.4 the desired conclusion immediately. □

Theorem 3.6 *Let* C *be a nonempty closed convex subset of* H. *Let* $f : C \to C$ *be a* β-*contractive mapping and let* $A : C \to H$ *be an* α-*inverse-strongly monotone mapping. Assume that* $FVI(C, A)$ *is not empty. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real number sequences in* $(0, 1)$ *and let* $\{r_n\}$ *be a positive real number sequence in* $(0, 2\alpha)$. *Let* $\{x_n\}$ *be a sequence generated in the following process*: $x_1 \in C$ *and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)\operatorname{Proj}_C(y_n - r_n A y_n), \quad \forall n \geq 1. \end{cases}$$

Assume that the control sequences satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(b) $0 \leq \beta_n \leq \bar{\beta} < 1$;
(c) $0 < a \leq r_n \leq b < 2\alpha$,

where $\bar{\beta}$, a, *and* b *are real numbers. Then* $\{x_n\}$ *converges strongly to a point* $\bar{x} \in (A + B)^{-1}(0)$, *which is also a unique solution to the following variational inequality*:

$$\langle f(\bar{x}) - \bar{x}, p - \bar{x} \rangle \leq 0, \quad \forall p \in (A + B)^{-1}(0).$$

Proof Putting $B = \partial I_C$, we find from Theorem 2.4 and Lemma 3.4 the desired conclusion immediately. □

Let $W : H \to \mathbb{R}$ be a convex and differentiable function and $M : H \to \mathbb{R}$ is a convex function. Consider the convex minimization problem $\min_{x\in H}(W(x) + M(x))$. From [25], we know if ∇W is $\frac{1}{L}$-Lipschitz continuous, then it is L-inverse-strongly monotone. Hence, we have the following results.

Theorem 3.7 *Let* $W : H \to \mathbb{R}$ *be a convex and differentiable function such that* ∇W *is* $\frac{1}{L}$-*Lipschitz continuous and let* $M : H \to \mathbb{R}$ *be a convex and lower semi-continuous function* *such that* $(\nabla W + \partial M)^{-1}(0)$ *is not empty. Let* f *be a* β*-contractive mapping on* H*. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real number sequences in* $(0,1)$ *and let* $\{r_n\}$ *be a positive real number sequence in* $(0,2\alpha)$*. Let* $\{x_n\}$ *be a sequence generated in the following process*: $x_1 \in C$ *and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(I + r_n M)^{-1}(y_n - r_n \nabla W y_n), \quad \forall n \geq 1. \end{cases}$$

Assume that the control sequences satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(b) $0 \leq \beta_n \leq \bar{\beta} < 1$;
(c) $0 < a \leq r_n \leq b < 2\alpha$,

where $\bar{\beta}$, a, *and* b *are real numbers. Then* $\{x_n\}$ *converges strongly to a point* $\bar{x} \in (\nabla W + \partial M)^{-1}$, *which is also a unique solution to the following variational inequality*:

$$\langle f(\bar{x}) - \bar{x}, p - \bar{x} \rangle \leq 0, \quad \forall p \in (\nabla W + \partial M)^{-1}.$$

Proof Putting $A = \nabla W$ and $B = \partial M$, we find from Theorem 2.4 the desired conclusion immediately. □

4 Conclusions

In this paper, we study a convex feasibility problem via two monotone mappings and a nonexpansive mapping. The common solution is also a unique solution of another variational inequality. The restrictions imposed on the sequence $\{r_n\}$ are mild. The results presented in this paper mainly improve the corresponding results in [7] and [11].

Competing interests
The author declares that they have no competing interests.

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