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Characterization of the support for the hypergeometric Fourier transform of the W-invariant functions and distributions on $\mathbb{R}^d$ and Roe's theorem

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Abstract
In this paper, we establish real Paley-Wiener theorems for the hypergeometric Fourier transform on $\mathbb{R}^d$. More precisely, we characterize the functions of the generalized Schwartz space $\mathcal{S}_2(\mathbb{R}^d)^W$ and of $L^p_{A_k}(\mathbb{R}^d)^W$, $1 \leq p \leq 2$, whose hypergeometric Fourier transform has bounded, unbounded, convex, and nonconvex support. Finally we study the spectral problem on the generalized tempered distributions $\mathcal{S}'_2(\mathbb{R}^d)^W$.

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1 Introduction

We consider the differential-difference operators T_j, $j = 1, 2, \ldots, d$, associated with a root system $\mathcal{R}$ and a multiplicity function k, introduced by Cherednik in [1], called Cherednik operators in the literature. These operators were helpful for the extension and simplification of the theory of Heckman-Opdam, which is a generalization of the harmonic analysis on the symmetric spaces G/K (*cf.* [2–4]).

The Paley-Wiener theorems for functions and distributions are most useful theorems in harmonic analysis. These theorems have as aim to characterize functions with compact support through the properties of the analytic extensions of their classical Fourier transform on $\mathbb{R}^d$. Recently there has been a great interest to real Paley-Wiener theorems.

The first theorem given by Bang (*cf.* [5]) can be stated as follows. Let f be a C^∞-function on $\mathbb{R}$ such that for all $n \in \mathbb{N}$, and let the function $\frac{d^n}{dx^n} f$ belong to the Lebesgue space $L^p(\mathbb{R})$, then the limit $R_f = \lim_{n\to\infty} \| \frac{d^n}{dx^n} f \|_p^{1/n}$ exists and we have

$$R_f := \sup\{|\lambda| : \lambda \in \operatorname{supp} \mathcal{F}(f)\},$$

where $\mathcal{F}(f)$ is the classical Fourier transform of f. Next the analogue of this theorem was established for many other integral transforms (*cf.* [6–9]).

Motivated by the treatment in the Euclidean setting, we will derive in this paper new real Paley-Wiener theorems for the hypergeometric Fourier transform, on some Lebesgue space $L^p_{A_k}(\mathbb{R}^d)^W$ and on a generalized tempered distribution space $\mathcal{S}'_2(\mathbb{R}^d)^W$.

The remaining part of the paper is organized as follows. In Section 2 we recall the main results as regards the harmonic analysis associated with the Cherednik operators and the Heckman-Opdam theory (*cf.* [1–3]). Section 3 is devoted to a study of the L^2-Schwartz functions such that the supports of their hypergeometric Fourier transform are compact. Next we prove a new real Paley-Wiener theorem for the hypergeometric Fourier transform on generalized Paley-Wiener spaces. In Section 4 we characterize the functions in the generalized Schwartz spaces such that their hypergeometric Fourier transform vanishes outside a polynomial domain. We give also a necessary and sufficient condition for functions in $L^2_{A_k}(\mathbb{R}^d)^W$ such that their hypergeometric Fourier transform vanishing in a neighborhood of the origin. In Section 5 we study the generalized tempered distributions with spectral gaps. Finally, in the last section we prove Roe's theorem for the hypergeometric Fourier transform.

2 Preliminaries

This section gives an introduction to the theory of Cherednik operators, hypergeometric Fourier transform, and hypergeometric convolution. Our main references are [1–4, 10].

2.1 Reflection groups, root systems, and multiplicity functions

The basic ingredients in the theory of Cherednik operators are root systems and finite reflection groups, acting on $\mathbb{R}^d$ with the standard Euclidean scalar product $\langle\cdot,\cdot\rangle$ and $\|x\| = \sqrt{\langle x,x\rangle}$. On $\mathbb{C}^d$, $\|\cdot\|$ denotes also the standard Hermitian norm, while $\langle z,w\rangle = \sum_{j=1}^d z_j\overline{w}_j$.

For $\alpha \in \mathbb{R}^d\backslash\{0\}$, let $\alpha^\vee = \frac{2}{\|\alpha\|}\alpha$ be the coroot associated to α and let

$$r_\alpha(x) = x - \langle\alpha^\vee, x\rangle\alpha, \tag{2.1}$$

be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α.

A finite set $\mathcal{R} \subset \mathbb{R}^d\backslash\{0\}$ is called a root system if $\mathcal{R} \cap \mathbb{R}.\ \alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha\mathcal{R} = \mathcal{R}$ for all $\alpha \in \mathcal{R}$. For a given root system $\mathcal{R}$ the reflections $\sigma_\alpha, \alpha \in \mathcal{R}$, generate a finite group $W \subset O(d)$, called the reflection group associated with $\mathcal{R}$. All reflections in W correspond to suitable pairs of roots. We fix a positive root system $\mathcal{R}_+ = \{\alpha \in \mathcal{R} : \langle\alpha,\beta\rangle > 0\}$ for some $\beta \in \mathbb{R}^d\backslash\bigcup_{\alpha\in\mathcal{R}} H_\alpha$.

Let

$$C_+ = \{x \in \mathbb{R}^d : \forall\alpha \in \mathcal{R}_+, \langle\alpha, x\rangle > 0\},$$

be the positive chamber. We denote by $\overline{C}_+$ its closure.

A function $k : \mathcal{R} \to [0,\infty)$ is called a multiplicity function if it is invariant under the action of the associated reflection group W. For abbreviation, we introduce the index

$$\gamma = \gamma(k) = \sum_{\alpha\in\mathcal{R}_+} k(\alpha). \tag{2.2}$$

Moreover, let A_k denote the weight function

$$\forall x \in \mathbb{R}^d, \quad A_k(x) = \prod_{\alpha\in\mathcal{R}_+} \left|\sinh\left\langle\frac{\alpha}{2}, x\right\rangle\right|^{2k(\alpha)}. \tag{2.3}$$

We note that this function is W invariant and satisfies

$$\forall x \in \overline{C}_+, \quad A_k(x) \leq \exp\bigl(2\langle \varrho, x\rangle\bigr), \tag{2.4}$$

where

$$\rho = \frac{1}{2}\sum_{\alpha\in\mathcal{R}_+} k(\alpha)\alpha.$$

2.2 The eigenfunctions of the Cherednik operators

The Cherednik operators T_j, $j = 1, \dots, d$, on $\mathbb{R}^d$ associated with the finite reflection group W and multiplicity function k are given by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha\in\mathcal{R}_+} \frac{k(\alpha)\alpha_j}{1 - e^{-\langle\alpha,x\rangle}} \bigl\{f(x) - f(r_\alpha x)\bigr\} - \rho_j f(x). \tag{2.5}$$

The operators T_j can also be written in the form

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \frac{1}{2}\sum_{\alpha\in\mathcal{R}_+} k(\alpha)\alpha_j \coth\biggl\langle\frac{\alpha}{2}, x\biggr\rangle\bigl\{f(x) - f(r_\alpha x)\bigr\} - \frac{1}{2} S_j f(x),$$

with

$$\forall x \in \mathbb{R}^d, \quad S_j f(x) = \sum_{\alpha\in\mathcal{R}_+} k(\alpha)\alpha_j f(r_\alpha x).$$

In the case $k(\alpha) = 0$, for all $\alpha \in \mathcal{R}_+$, the T_j, $j = 1, 2, \dots, d$, reduce to the corresponding partial derivatives.

Example 1 For $d = 1$, the root systems are $\mathcal{R} = \{-\alpha, \alpha\}$, $\mathcal{R} = \{-2\alpha, 2\alpha\}$ or $\mathcal{R} = \{-2\alpha, -\alpha, \alpha, 2\alpha\}$ with α the positive root. We take the normalization $\alpha = 2$.

For $\mathcal{R}_+ = \{\alpha\}$, we have the Cherednik operator

$$T_1 f(x) = \frac{d}{dx} f(x) + \frac{2k_\alpha}{1 - e^{-2x}} \bigl\{f(x) - f(-x)\bigr\} - \rho f(x),$$

with $\rho = k_\alpha$. This operator can also be written in the form

$$T_1 f(x) = \frac{d}{dx} f(x) + k_\alpha \coth(x)\bigl\{f(x) - f(-x)\bigr\} - k_\alpha f(-x). \tag{2.6}$$

For $\mathcal{R}_+ = \{2\alpha\}$, we have the Cherednik operator

$$T_1 f(x) = \frac{d}{dx} f(x) + \frac{4k_{2\alpha}}{1 - e^{-4x}} \bigl\{f(x) - f(-x)\bigr\} - \rho f(x).$$

This operator can also be written in the form

$$T_1 f(x) = \frac{d}{dx} f(x) + \bigl(k_{2\alpha} \coth(x) + k_{2\alpha} \tanh(x)\bigr)\bigl\{f(x) - f(x)\bigr\} - \rho f(-x), \tag{2.7}$$

with $\rho = 2k_{2\alpha}$.

For $\mathcal{R}_+ = \{\alpha, 2\alpha\}$, we have the Cherednik operator

$$T_1 f(x) = \frac{d}{dx} f(x) + \left(\frac{2k_\alpha}{1 - e^{-2x}} + \frac{4k_{2\alpha}}{1 - e^{-4x}}\right)\{f(x) - f(-x)\} - \rho f(x),$$

with $\rho = k_\alpha + 2k_{2\alpha}$. It is also equal to

$$T_1 f(x) = \frac{d}{dx} f(x) + \big((k_\alpha + k_{2\alpha}) \coth(x) + k_{2\alpha} \tanh(x)\big)\{f(x) - f(-x)\} - \rho f(-x). \tag{2.8}$$

The operators (2.6), (2.7), and (2.8) are particular cases of the differential-difference operator

$$\Lambda_{k,k'} f(x) = \frac{d}{dx} f(x) + \big(k \coth(x) + k' \tanh(x)\big)\{f(x) - f(-x)\} - \rho f(-x), \tag{2.9}$$

which is refereed to as the Jacobi-Cherednik operator (*cf.* [11, 12]).

The Heckman-Opdam Laplacian $\triangle_k$ is defined by

$$\begin{aligned} \triangle_k f(x) &:= \sum_{j=1}^{d} T_j^2 f(x) \\ &= \triangle f(x) + \sum_{\alpha \in \mathbb{R}_+} k(\alpha)\left(\coth\left\langle \frac{\alpha}{2}, x\right\rangle\right)\langle \nabla f(x), \alpha\rangle + \|\rho\|^2 f(x) \\ &\quad - \sum_{\alpha \in R_+} k(\alpha) \frac{\|\alpha\|^2}{4(\sinh\langle \frac{\alpha}{2}, x\rangle)^2}\{f(x) - f(r_\alpha x)\}, \end{aligned} \tag{2.10}$$

where $\triangle$ and ∇ are, respectively, the Laplacian and the gradient on $\mathbb{R}^d$.

The Heckman-Opdam Laplacian on W-invariant functions is denoted by $\triangle_k^W$ and we have the expression

$$\triangle_k^W f(x) = \triangle f(x) + \sum_{\alpha \in R_+} k(\alpha)\left(\coth\left\langle \frac{\alpha}{2}, x\right\rangle\right)\langle \nabla f(x), \alpha\rangle + \|\rho\|^2 f(x).$$

Example 2 For $d = 1$, $W = \mathbb{Z}_2$ and $k \geq k' \geq 0$, $k \neq 0$, the Heckman-Opdam Laplacian $\triangle_k^W$ is the Jacobi operator defined for even functions f of class C^2 on $\mathbb{R}$ by

$$\triangle_k^W f(x) = \frac{d^2}{dx^2} f(x) + \big(2k \coth x + 2k' \tanh x\big)\frac{d}{dx} f(x) + \varrho^2 f(x),$$

with $\varrho = k + k'$.

We denote by G_λ the eigenfunction of the operators T_j, $j = 1, 2, \ldots, d$. It is the unique analytic function on $\mathbb{R}^d$ which satisfies the differential-difference system

$$\begin{cases} T_j u(x) = -i\lambda_j u(x), & j = 1, 2, \ldots, d, x \in \mathbb{R}^d, \\ u(0) = 1. \end{cases}$$

It is called the Opdam-Cherednik kernel.

We consider the function F_λ defined by

$$\forall x \in \mathbb{R}^d, \quad F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx).$$

This function is the unique analytic W-invariant function on $\mathbb{R}^d$, which satisfies the differential equations

$$\begin{cases} p(T)u(x) = p(-i\lambda)u(x), & x \in \mathbb{R}^d, \lambda \in \mathbb{R}^d, \\ u(0) = 1, \end{cases}$$

for all W-invariant polynomial p on $\mathbb{R}^d$ and $p(T) = p(T_1, \dots, T_d)$.

In particular for all $\lambda \in \mathbb{R}^d$ we have

$$\triangle_k^W F_\lambda(x) = -\|\lambda\|^2 F_\lambda(x).$$

The function F_λ is called the Heckman-Opdam kernel.

The functions G_λ and F_λ possess the following properties.

(i) For all $x \in \mathbb{R}^d$, the functions G_λ and F_λ are entire on $\mathbb{C}^d$.

(ii) We have

$$\forall x \in \mathbb{R}^d, \forall \lambda \in \mathbb{C}^d, \quad \left|G_\lambda(x)\right| \le G_{\mathrm{Im}(\lambda)}(x)$$

and

$$\forall x \in \mathbb{R}^d, \forall \lambda \in \mathbb{C}^d, \quad \left|F_\lambda(x)\right| \le F_{\mathrm{Im}(\lambda)}(x).$$

(iii) There exists a positive constant $M_0 := \sqrt{|W|}$ such that

$$\forall x \in \mathbb{R}^d, \forall \lambda \in \mathbb{R}^d, \quad \left|F_\lambda(x)\right| \le M_0 \tag{2.11}$$

and

$$\forall x \in \mathbb{R}^d, \forall \lambda \in \mathbb{R}^d, \quad \left|G_\lambda(x)\right| \le M_0.$$

(iv) We have

$$\forall x \in \overline{C}_+, \quad F_0(x) \asymp e^{-\langle \rho, x\rangle} \prod_{\alpha \in R_+^0} \bigl(1 + \langle \alpha, x\rangle\bigr).$$

(v) Let p and q be polynomials of degree m and n. Then there exists a positive constant M' such that for all $\lambda \in \mathbb{C}^d$ and for all $x \in \mathbb{R}^d$, we have

$$\left|p\left(\frac{\partial}{\partial \lambda}\right) q\left(\frac{\partial}{\partial x}\right) F_\lambda(x)\right| \le M' \bigl(1 + \|x\|\bigr)^n \bigl(1 + \|\lambda\|\bigr)^m F_0(x) e^{\max_{w \in W}(\mathrm{Im}\langle w\lambda, x\rangle)}. \tag{2.12}$$

(vi) The preceding estimate holds true for G_λ too.

Example 3 When $d = 1$ and $W = \mathbb{Z}_2$, and $k \geq k' \geq 0$, $k \neq 0$, the Opdam-Cherednik kernel $G_\lambda(x)$ is given for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$ by

$$G_\lambda(x) = \varphi_\lambda^{(k-\frac{1}{2},k'-\frac{1}{2})}(x) - \frac{1}{\rho - i\lambda}\frac{d}{dx}\varphi^{(k-\frac{1}{2},k'-\frac{1}{2})}(x),$$

where $\varphi_\lambda^{(\alpha,\beta)}(x)$ is the Jacobi function of index (α, β) defined by

$$\varphi_\lambda^{(\alpha,\beta)}(x) = {}_2F_1\left(\frac{1}{2}(\rho + i\lambda), \frac{1}{2}(\rho - i\lambda); \alpha + 1; -(\sinh x)^2\right),$$

with $\rho = \alpha + \beta + 1$ and ${}_2F_1$ is the Gauss hypergeometric function.

In this case the Heckman-Opdam kernel $F_\lambda(x)$ is given for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$ by

$$F_\lambda(x) = \varphi_\lambda^{(k-\frac{1}{2},k'-\frac{1}{2})}(x).$$

2.3 The hypergeometric Fourier transform on W-invariant function and distribution spaces

We denote by

$\mathcal{E}(\mathbb{R}^d)^W$ the space of C^∞-functions on $\mathbb{R}^d$, which are W-invariant;

$\mathcal{D}(\mathbb{R}^d)^W$ the space of C^∞-functions on $\mathbb{R}^d$, which are W-invariant and with compact support;

$\mathcal{S}(\mathbb{R}^d)^W$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$;

$\mathcal{S}_2(\mathbb{R}^d)^W$ the space of C^∞-functions on $\mathbb{R}^d$ which are W-invariant, and such that for all $\ell, n \in \mathbb{N}$, we have

$$\sup_{\substack{|\mu| \leq n \\ x \in \mathbb{R}^d}} \bigl(1 + \|x\|\bigr)^\ell F_0^{-1}(x)\bigl|D^\mu f(x)\bigr| < +\infty,$$

where

$$D^\mu = \frac{\partial^{|\mu|}}{\partial^{\mu_1}x_1 \cdots \partial^{\mu_d}x_d}, \quad \mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d;$$

$PW(\mathbb{C}^d)^W$ the space of entire functions on $\mathbb{C}^d$, which are W-invariant, rapidly decreasing and of exponential type;

$\mathcal{PW}(\mathbb{C}^d)^W$ the space of entire functions on $\mathbb{C}^d$, which are W-invariant, slowly increasing and of exponential type;

$\mathcal{D}'(\mathbb{R}^d)^W$ the space of distributions on $\mathbb{R}^d$, which are W-invariant;

$\mathcal{E}'(\mathbb{R}^d)^W$ the space of distributions on $\mathbb{R}^d$ which are W-invariant and with compact support;

$\mathcal{S}'(\mathbb{R}^d)^W$ the space of tempered distributions on $\mathbb{R}^d$, which are W-invariant. It is the topological dual of $\mathcal{S}(\mathbb{R}^d)^W$;

$\mathcal{S}'_2(\mathbb{R}^d)^W$ the topological dual of $\mathcal{S}_2(\mathbb{R}^d)^W$. We have

$$\begin{aligned} d\nu_k(\lambda) &:= C_k(\lambda)\,d\lambda \\ &= c\prod_{\alpha \in R_+}\frac{\Gamma(-i\langle\lambda,\alpha^\vee\rangle + k(\alpha) + \frac{1}{2}k(\frac{\alpha}{2}))\Gamma(i\langle\lambda,\alpha^\vee\rangle + k(\alpha) + \frac{1}{2}k(\frac{\alpha}{2}))}{\Gamma(-i\langle\lambda,\alpha^\vee\rangle + \frac{1}{2}k(\frac{\alpha}{2}))\Gamma(i\langle\lambda,\alpha^\vee\rangle + \frac{1}{2}k(\frac{\alpha}{2}))}\,d\lambda, \end{aligned}$$

with c a normalizing constant and $k(\frac{\alpha}{2}) = 0$ if $\frac{\alpha}{2} \notin R_+$.

The measure $d\nu_k(\lambda)$ is called the symmetric Plancherel measure or the Harish-Chandra measure (*cf.* [2, 4]).

Remark 1 The function C_k is positive, continuous on $\mathbb{R}^d$, and it satisfies the estimate

$$\forall\lambda \in \mathbb{R}^d, \quad \bigl|C_k(\lambda)\bigr| \le const.\bigl(1 + \|\lambda\|\bigr)^b,$$

for some $b > 0$.

$L^p_{A_k}(\mathbb{R}^d)^W$, $1 \le p \le \infty$, is the space of measurable functions f on $\mathbb{R}^d$ which are W-invariant and satisfy

$$\|f\|_{L^p_{A_k}(\mathbb{R}^d)^W} = \left(\int_{\mathbb{R}^d} \bigl|f(x)\bigr|^p A_k(x)\,dx\right)^{1/p} < \infty, \quad \text{if } 1 \le p < \infty,$$

$$\|f\|_{L^\infty_{A_k}(\mathbb{R}^d)^W} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \bigl|f(x)\bigr| < +\infty.$$

$L^p_{\nu_k}(\mathbb{R}^d)^W$, $1 \le p \le \infty$, is the space of measurable functions f on $\mathbb{R}^d$ which are W-invariant and satisfy

$$\|f\|_{L^p_{\nu_k}(\mathbb{R}^d)^W} = \left(\int_{\mathbb{R}^d} \bigl|f(x)\bigr|^p\,d\nu_k(x)\right)^{1/p} < \infty, \quad \text{if } 1 \le p < \infty,$$

$$\|f\|_{L^\infty_{\nu_k}(\mathbb{R}^d)^W} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \bigl|f(x)\bigr| < \infty.$$

The hypergeometric Fourier transform of a function f in $D(\mathbb{R}^d)^W$ is given by

$$\mathcal{H}^W(f)(\lambda) = \int_{\mathbb{R}^d} f(x)F_\lambda(x)A_k(x)\,dx, \quad \text{for all } \lambda \in \mathbb{R}^d. \tag{2.13}$$

Proposition 1 *The transform $\mathcal{H}^W$ is a topological isomorphism from*
(i) $D(\mathbb{R}^d)^W$ *onto* $PW(\mathbb{C}^d)^W$,
(ii) $\mathcal{S}_2(\mathbb{R}^d)^W$ *onto* $\mathcal{S}(\mathbb{R}^d)^W$.
The inverse transform is given by

$$\forall x \in \mathbb{R}^d, \quad \bigl(\mathcal{H}^W\bigr)^{-1}(h)(x) = \int_{\mathbb{R}^d} h(\lambda)F_\lambda(-x)\,d\nu_k(\lambda).$$

Proposition 2 *For f in $L^1_{A_k}(\mathbb{R}^d)^W$ the function $\mathcal{H}^W(f)$ is continuous on $\mathbb{R}^d$ and we have*

$$\bigl\|\mathcal{H}^W(f)\bigr\|_{L^\infty_{\nu_k}(\mathbb{R}^d)^W} \le M_0\|f\|_{L^1_{A_k}(\mathbb{R}^d)^W},$$

where M_0 is the constant given by the relation (2.11).

Proposition 3 (i) Plancherel formula. *For all f, g in $D(\mathbb{R}^d)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^d)^W$) we have*

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)}A_k(x)\,dx = \int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda)\overline{\mathcal{H}^W(g)(\lambda)}\,d\nu_k(\lambda). \tag{2.14}$$

(ii) Plancherel theorem. *The transform $\mathcal{H}^W$ extends uniquely to an isomorphism from $L^2_{A_k}(\mathbb{R}^d)^W$ onto $L^2_{\nu_k}(\mathbb{R}^d)^W$.*

Proposition 4 *For all f in $L^2_{A_k}(\mathbb{R}^d)^W$ such that $\mathcal{H}^W(f)$ belongs to $L^1_{\nu_k}(\mathbb{R}^d)^W$, we have the inversion formula*

$$f(x) = \int_{\mathbb{R}^d} \mathcal{H}^W(f)(\lambda)F_\lambda(-x)\,d\nu_k(\lambda), \quad a.e. \tag{2.15}$$

2.4 The hypergeometric convolution

Definition 1 Let y be in $\mathbb{R}^d$. The hypergeometric translation operator $f \mapsto \tau_y f$ is defined on $\mathcal{S}_2(\mathbb{R}^d)^W$ by

$$\mathcal{H}^W(\tau_y f)(x) = F_x(y)\mathcal{H}^W(f)(x), \quad \text{for all } x \in \mathbb{R}^d. \tag{2.16}$$

Using the hypergeometric translation operator, we define the hypergeometric convolution product of functions as follows.

Definition 2 The hypergeometric convolution product of f and g in $\mathcal{S}_2(\mathbb{R}^d)^W$ is the function $f *_k g$ defined by

$$f *_k g(x) = \int_{\mathbb{R}^d} \tau_x f(-y)g(y)A_k(y)\,dy, \quad \text{for all } x \in \mathbb{R}^d. \tag{2.17}$$

Proposition 5 ([10]) (i) *Let $1 \le p < 2 < q \le \infty$. Then*

$$L^p_{A_k}\big(\mathbb{R}^d\big)^W *_k L^2_{A_k}\big(\mathbb{R}^d\big)^W \subset L^2_{A_k}\big(\mathbb{R}^d\big)^W \tag{2.18}$$

and

$$L^2_{A_k}\big(\mathbb{R}^d\big)^W *_k L^2_{A_k}\big(\mathbb{R}^d\big)^W \subset L^q_{A_k}\big(\mathbb{R}^d\big)^W. \tag{2.19}$$

(ii) *Let f be in $L^2_{A_k}(\mathbb{R}^d)^W$ and g in $L^1_{A_k}(\mathbb{R}^d)^W$. Then*

$$\mathcal{H}^W(f *_k g) = \mathcal{H}^W(f) \cdot \mathcal{H}^W(g). \tag{2.20}$$

Definition 3 (i) We define the hypergeometric Fourier transform $\mathcal{H}^W$ of a distribution S in $\mathcal{E}'(\mathbb{R}^d)^W$ by

$$\forall \lambda \in \mathbb{R}^d, \quad \mathcal{H}^W(S)(\lambda) = \langle S, F_\lambda\rangle.$$

(ii) The hypergeometric Fourier transform of a distribution S in $\mathcal{S}'_2(\mathbb{R}^d)$ is defined by

$$\big\langle \mathcal{H}^W(S), \psi\big\rangle = \big\langle S, \big(\mathcal{H}^W\big)^{-1}(\psi)\big\rangle, \quad \psi \in \mathcal{S}\big(\mathbb{R}^d\big)^W.$$

Theorem 1 *The transform $\mathcal{H}^W$ is a topological isomorphism from*
(i) *$\mathcal{E}'(\mathbb{R}^d)^W$ onto $\mathcal{PW}(\mathbb{C}^d)^W$,*
(ii) *$\mathcal{S}'_2(\mathbb{R}^d)^W$ onto $\mathcal{S}'(\mathbb{R}^d)^W$.*

Let τ be in $\mathcal{S}_2'(\mathbb{R}^d)^W$. We define the distribution $\triangle_k\tau$, by

$$\langle \triangle_k\tau, \psi\rangle = \langle \tau, \triangle_k\psi\rangle, \quad \text{for all } \psi \in \mathcal{S}_2\big(\mathbb{R}^d\big)^W. \tag{2.21}$$

This distribution satisfies the following property:

$$\mathcal{H}_k^W(\triangle_k\tau) = -\|y\|^2\mathcal{H}_k^W(\tau). \tag{2.22}$$

3 Functions with compact spectrum

We consider f in $L^2_{A_k}(\mathbb{R}^d)^W$. We define the distribution $\mathcal{T}_f$ in $\mathcal{S}_2'(\mathbb{R}^d)^W$ by

$$\langle \mathcal{T}_f, \varphi\rangle = \int_{\mathbb{R}^d} f(x)\varphi(x)A_k(x)\,dx, \quad \varphi \in \mathcal{S}_2\big(\mathbb{R}^d\big)^W.$$

Notations We denote by

$L^2_{A_k,c}(\mathbb{R}^d)^W$ the space of functions in $L^2_{A_k}(\mathbb{R}^d)^W$ with compact support;

$\mathcal{H}_{L^2_k}(\mathbb{C}^d)$ the space of entire functions f on $\mathbb{C}^d$ of exponential type such that $f_{|\mathbb{R}^d}$ belongs to $L^2_{\nu_k}(\mathbb{R}^d)^W$.

Theorem 2 *The hypergeometric Fourier transform $\mathcal{H}^W$ is bijective from $L^2_{A_k,c}(\mathbb{R}^d)^W$ onto $\mathcal{H}_{L^2_k}(\mathbb{C}^d)$.*

Proof (i) We consider the function f on $\mathbb{C}^d$ given by

$$\forall z \in \mathbb{C}^d, \quad f(z) = \int_{\mathbb{R}^d} g(x)F_z(x)A_k(x)\,dx, \tag{3.1}$$

with $g \in L^2_{A_k,c}(\mathbb{R}^d)^W$.

By derivation under the integral sign and by using the inequality (2.12), we deduce that the function f is entire on $\mathbb{C}^d$ and of exponential type. On the other hand the relation (3.1) can also be written in the form

$$\forall y \in \mathbb{R}^d, \quad f(y) = \mathcal{H}^W(g)(y).$$

Thus from Proposition 3 the function $f_{|\mathbb{R}^d}$ belongs to $L^2_{\nu_k}(\mathbb{R}^d)^W$. Thus $f \in \mathcal{H}_{L^2_k}(\mathbb{C}^d)$.

(ii) Reciprocally let ψ be in $\mathcal{H}_{L^2_k}(\mathbb{C}^d)$. From [10] there exists $S \in \mathcal{E}'(\mathbb{R}^d)^W$ with support in the boule $B(0,a)$, such that

$$\forall y \in \mathbb{R}^d, \quad \psi(y) = \big\langle S_x, F_y(x)\big\rangle. \tag{3.2}$$

On the other hand as $\psi_{|\mathbb{R}^d}$ belongs to $L^2_{\nu_k}(\mathbb{R}^d)^W$, from Proposition 3 there exists h in $L^2_{A_k}(\mathbb{R}^d)^W$ such that

$$\psi_{|\mathbb{R}^d} = \mathcal{H}^W(h). \tag{3.3}$$

Thus from (3.2), for all $\varphi \in D(\mathbb{R}^d)^W$ we have

$$\int_{\mathbb{R}^d} \psi(y)\overline{\mathcal{H}^W(\varphi)(y)}\,d\nu_k(y) = \left\langle S_x, \int_{\mathbb{R}^d} F_y(x)\overline{\mathcal{H}^W(\varphi)(y)}\,d\nu_k(y)\right\rangle.$$

Thus using (2.14) we deduce that

$$\int_{\mathbb{R}^d} \psi(y)\overline{\mathcal{H}^W(\varphi)(y)}\,d\nu_k(y) = \langle S, \overline{\varphi}\rangle. \tag{3.4}$$

On the other hand (3.3) implies

$$\int_{\mathbb{R}^d} \psi(y)\overline{\mathcal{H}^W(\varphi)(y)}\,d\nu_k(y) = \int_{\mathbb{R}^d} \mathcal{H}^W(h)(y)\overline{\mathcal{H}^W(\varphi)(y)}\,d\nu_k(y).$$

But from Proposition 3 we deduce that

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{H}^W(h)(y)\overline{\big(\mathcal{H}^W\big)(\varphi)(y)}\,d\nu_k(y) &= \int_{\mathbb{R}^d} h(y)\overline{\varphi(y)}A_k(y)\,dy \\ &= \langle \mathcal{T}_h, \overline{\varphi}\rangle.\end{aligned}$$

Thus the previous relation and the relation (3.4) imply

$$S = \mathcal{T}_h.$$

This relation shows that the support of h is compact. Then $h \in L^2_{A_k,c}(\mathbb{R}^d)^W$. □

In the following $\mathcal{T}_f$ will be denoted by f.

Definition 4 (i) We define the support of $g \in L^2_{\nu_k}(\mathbb{R}^d)^W$ and we denote it by $\operatorname{supp} g$, the smallest closed set, outside of which the function g vanishes almost everywhere.

(ii) We denote by

$$R_g := \sup_{\lambda \in \operatorname{supp} g} \|\lambda\|,$$

the radius of the support of g.

Remark 2 It is clear that R_g is finite if and only if g has compact support.

Proposition 6 *Let $g \in L^2_{\nu_k}(\mathbb{R}^d)^W$ such that for all $n \in \mathbb{N}$, the function $\|\lambda\|^{2n} g(\lambda)$ belongs to $L^2_{\nu_k}(\mathbb{R}^d)^W$. Then*

$$R_g = \lim_{n\to\infty} \left\{ \int_{\mathbb{R}^d} \|\lambda\|^{4n} \big|g(\lambda)\big|^2\,d\nu_k(\lambda) \right\}^{\frac{1}{4n}}. \tag{3.5}$$

Proof We suppose that $\|g\|_{L^2_{\nu_k}(\mathbb{R}^d)^W} \neq 0$, otherwise $R_g = 0$ and (3.5) is trivial.

Assume now that g has compact support with $R_g > 0$. Then

$$\left\{ \int_{\mathbb{R}^d} \|\lambda\|^{4n} \big|g(\lambda)\big|^2\,d\nu_k(\lambda) \right\}^{\frac{1}{4n}} \le \left\{ \int_{\|\lambda\| \le R_g} \big|g(\lambda)\big|^2\,d\nu_k(\lambda) \right\}^{\frac{1}{4n}} R_g.$$

Thus we deduce that

$$\limsup_{n\to\infty} \left\{ \int_{\mathbb{R}^d} \|\lambda\|^{4n} \big|g(\lambda)\big|^2\,d\nu_k(\lambda) \right\}^{\frac{1}{4n}} \le \limsup_{n\to\infty} \left\{ \int_{\|\lambda\| \le R_g} \big|g(\lambda)\big|^2\,d\nu_k(\lambda) \right\}^{\frac{1}{4n}} R_g = R_g.$$

On the other hand, for any positive ε we have

$$\int_{R_g-\varepsilon\leq\|\lambda\|\leq R_g}\left|g(\lambda)\right|^2\,d\nu_k(\lambda)>0.$$

Hence

$$\begin{aligned}\liminf_{n\to\infty}\left\{\int_{\mathbb{R}^d}\|\lambda\|^{4n}\left|g(\lambda)\right|^2\,d\nu_k(\lambda)\right\}^{\frac{1}{4n}}&\geq\liminf_{n\to\infty}\left\{\int_{R_g-\varepsilon\leq\|\lambda\|\leq R_g}\|\lambda\|^{4n}\left|g(\lambda)\right|^2\,d\nu_k(\lambda)\right\}^{\frac{1}{4n}}\\&\geq R_g-\varepsilon.\end{aligned}$$

Thus

$$R_g=\lim_{n\to\infty}\left\{\int_{\mathbb{R}^d}\|\lambda\|^{4n}\left|g(\lambda)\right|^2\,d\nu_k(\lambda)\right\}^{\frac{1}{4n}}.$$

We prove now the assertion in the case where g has unbounded support. Indeed for any positive N, we have

$$\int_{\|\lambda\|\geq N}\left|g(\lambda)\right|^2\,d\nu_k(\lambda)>0.$$

Thus

$$\liminf_{n\to\infty}\left\{\int_{\mathbb{R}^d}\|\lambda\|^{4n}\left|g(\lambda)\right|^2\,d\nu_k(\lambda)\right\}^{\frac{1}{4n}}\geq\liminf_{n\to\infty}\left\{\int_{\|\lambda\|\geq N}\|\lambda\|^{4n}\left|g(\lambda)\right|^2\,d\nu_k(\lambda)\right\}^{\frac{1}{4n}}\geq N.$$

This implies that

$$\liminf_{n\to\infty}\left\{\int_{\mathbb{R}^d}\|\lambda\|^{4n}\left|g(\lambda)\right|^2\,d\nu_k(\lambda)\right\}^{\frac{1}{4n}}=\infty.$$

□

Notations We denote by
$L^2_{\nu_k,c}(\mathbb{R}^d)^W$ the space of functions in $L^2_{\nu_k}(\mathbb{R}^d)^W$ with compact support;
$L^2_{\nu_k,c,R}(\mathbb{R}^d)^W := \{g\in L^2_{\nu_k,c}(\mathbb{R}^d)^W : R_g = R\}$, for $R\geq 0$;
$D_R(\mathbb{R}^d)^W := \{g\in D(\mathbb{R}^d)^W : R_g = R\}$, for $R\geq 0$.

Definition 5 We define the generalized Paley-Wiener spaces $PW^2_k(\mathbb{R}^d)^W$ and $PW^2_{k,R}(\mathbb{R}^d)^W$ as follows.

(i) $PW^2_k(\mathbb{R}^d)^W$ is the space of functions $f\in\mathcal{E}(\mathbb{R}^d)^W$ satisfying
  (a) $\triangle^n_k f\in L^2_{A_k}(\mathbb{R}^d)^W$ for all $n\in\mathbb{N}$;
  (b) $R^{\triangle_k}_f := \lim_{n\to\infty}\|\triangle^n_k f\|^{\frac{1}{2n}}_{L^2_{A_k}(\mathbb{R}^d)^W}<\infty$.

(ii) $PW^2_{k,R}(\mathbb{R}^d)^W := \{f\in PW^2_k(\mathbb{R}^d)^W : R^{\triangle_k}_f = R\}$.

The real L^2-Paley-Wiener theorem for the hypergeometric Fourier transform can be formulated as follows.

Theorem 3 *The hypergeometric Fourier transform* $\mathcal{H}^W$ *is a bijection*

(i) *from* $PW_{k,R}^2(\mathbb{R}^d)^W$ *onto* $L^2_{\nu_k,c,R}(\mathbb{R}^d)^W$;
(ii) *from* $PW_k^2(\mathbb{R}^d)^W$ *onto* $L^2_{\nu_k,c}(\mathbb{R}^d)^W$.

Proof Let $g \in PW_k^2(\mathbb{R}^d)^W$. Using (2.22), we see that the function

$$\mathcal{H}^W\big(\triangle_k^n g\big)(\xi) = (-1)^n\|\xi\|^{2n}\mathcal{H}^W(g)(\xi) \in L^2_{\nu_k}\big(\mathbb{R}^d\big)^W, \quad \forall n \in \mathbb{N}.$$

On the other hand from Proposition 3 we deduce that

$$\begin{aligned}\lim_{n\to\infty}\left\{\int_{\mathbb{R}^d}\|\xi\|^{4n}\big|\mathcal{H}^W(g)(\xi)\big|^2\,d\nu_k(\xi)\right\}^{\frac{1}{4n}} &= \lim_{n\to\infty}\left\{\int_{\mathbb{R}^d}\big|\triangle_k^n g(x)\big|^2 A_k(x)\,dx\right\}^{\frac{1}{4n}} \\ &= R_g^{\triangle_k} < \infty.\end{aligned}$$

Thus using Proposition 6 we conclude that $\mathcal{H}^W(g)$ has compact support with

$$R_{\mathcal{H}^W(g)} = R_g^{\triangle_k}.$$

Conversely let $f \in L^2_{\nu_k,c,R}(\mathbb{R}^d)^W$. Then $\xi^n f(\xi) \in L^1_{\nu_k}(\mathbb{R}^d)^W$ for any $n \in \mathbb{N}$, and $(\mathcal{H}^W)^{-1}(f)$ belongs to $\mathcal{E}(\mathbb{R}^d)^W$. On the other hand from Proposition 3 we have

$$\begin{aligned}\lim_{n\to\infty}\left\{\int_{\mathbb{R}^d}\big|\triangle_k^n\big(\big(\mathcal{H}^W\big)^{-1}f\big)(x)\big|^2 A_k(x)\,dx\right\}^{\frac{1}{4n}} &= \lim_{n\to\infty}\left\{\int_{\mathbb{R}^d}\|\xi\|^{4n}\big|f(\xi)\big|^2\,d\nu_k(\xi)\right\}^{\frac{1}{4n}} \\ &= R.\end{aligned}$$

Thus $(\mathcal{H}^W)^{-1}(f) \in PW_k^2(\mathbb{R}^d)^W$.

(ii) We deduce the result from (i). □

We finish this section with an application on the generalized Schrödinger equation, which was introduced and studied in [13].

Corollary 1 *Let* $f \in L^2_{A_k}(\mathbb{R}^d)^W$. *Then* f *belongs to* $PW_k^2(\mathbb{R}^d)^W$ *if and only if the solution* $u(t,\cdot)$ *of the Cauchy problem for the generalized Schrödinger equation*

$$(\mathrm{S}) \quad \begin{cases}\partial_t u - i\triangle_k u = 0,\\ u_{|t=0} = f\end{cases}$$

has the following properties:
(i) *as a function of* t, *it has an analytic extension* $u(z,\cdot)$, $z \in \mathbb{C}^d$ *to the complex plane* $\mathbb{C}^d$ *as an entire function*,
(ii) *it has exponential type* σ *in the variable* z, *that is*,

$$\big\|u(z,\cdot)\big\|_{L^2_{A_k}(\mathbb{R}^d)^W} \le e^{\sigma|z|}\|f\|_{L^2_{A_k}(\mathbb{R}^d)^W},$$

and it is bounded on the real line.

4 Hypergeometric Fourier transform of functions with polynomial domain support

Notation We denote by $\mathbb{C}[x]$ the set of polynomials on $\mathbb{R}$ with complex coefficients.

Definition 6 Let u be a distribution on $\mathbb{R}^d$ and P a polynomial. Then we let

$$R(P,u) = \sup\{|P(\|y\|^2)| : y \in \operatorname{supp} u\} \in [0,\infty],$$

where by convention $R(P,u) = 0$ if $u = 0$.

Theorem 4 *Let $P \in \mathbb{C}[x]$. For any function $f \in \mathcal{S}_2(\mathbb{R}^d)^W$ the following relation holds:*

$$\lim_{n\to\infty} \|P^n(-\triangle_k)f\|^{\frac{1}{n}}_{L^2_{A_k}(\mathbb{R}^d)^W} = \sup_{y\in\operatorname{supp}\mathcal{H}^W(f)} |P(\|y\|^2)|. \tag{4.1}$$

Proof We consider $f \neq 0$ in $\mathcal{S}_2(\mathbb{R}^d)^W$. The proof is divided in two steps. In the following step we suppose that

$$\sup_{y\in\operatorname{supp}\mathcal{H}^W(f)} |P(\|y\|^2)| < \infty. \tag{4.2}$$

First step: In this step we shall prove that

$$\limsup_{n\to\infty} \|P^n(-\triangle_k)f\|^{\frac{1}{n}}_{L^2_{A_k}(\mathbb{R}^d)^W} \leq \sup_{y\in\operatorname{supp}\mathcal{H}^W(f)} |P(\|y\|^2)|.$$

In this case we assume firstly that $\mathcal{H}^W(f)$ has compact support. Hölder's inequality gives

$$\begin{aligned}\|f\|^2_{L^2_{A_k}(\mathbb{R}^d)^W} &= |W| \int_{C_+} (1+\|x\|^2)^{-2m} |(1+\|x\|^2)^m f(x)|^2 A_k(x)\,dx \\ &\leq |W| \left(\int_{C_+} (1+\|x\|^2)^{-2m}\,dx\right) \sup_{x\in\mathbb{R}^d} |A_k(x)(1+\|x\|^2)^m f(x)|^2,\end{aligned}$$

for $m \geq \frac{d+1}{4}$. Using the relation (2.4), we obtain

$$\|f\|_{L^2_{A_k}(\mathbb{R}^d)^W} \leq C \sup_{x\in\mathbb{R}^d} |e^{\langle\varrho,x\rangle}(1+\|x\|^2)^m f(x)|.$$

Consequently for all $n \in \mathbb{N}$, we deduce that

$$\begin{aligned}\|P^n(-\triangle_k)f\|_{L^2_{A_k}(\mathbb{R}^d)^W} &\leq C \sup_{x\in\mathbb{R}^d} |e^{\langle\varrho,x\rangle}(1+\|x\|^2)^m P^n(-\triangle_k)f(x)| \\ &\leq C \sup_{x\in\mathbb{R}^d} |e^{\langle\varrho,x\rangle}(1+\|x\|^2)^m [(\mathcal{H}^W)^{-1}(P^n(\|\xi\|^2)\mathcal{H}^W(f))](x)|.\end{aligned}$$

Using the continuity of $(\mathcal{H}^W)^{-1}$ we can show that

$$\|P^n(-\triangle_k)f\|_{L^2_{A_k}(\mathbb{R}^d)^W} \leq C \sup_{\xi\in\mathbb{R}^d} \left| \sum_{1\leq l,j\leq M} (1+\|\xi\|^2)^j \frac{d^l}{d\xi^l}[P^n(\|\xi\|^2)\mathcal{H}^W(f)(\xi)]\right|, \tag{4.3}$$

with positive constants C and integers M, m, independent of n. Using Leibniz's rule we deduce that

$$\|P^n(-\triangle_k)f\|_{L^2_{A_k}(\mathbb{R}^d)^W} \leq Cn^M \sup_{y\in\operatorname{supp}\mathcal{H}^W(f)} |P(\|y\|^2)|^{n-M},$$

with C is a constant independent of n. Hence, from the previous inequalities we obtain

$$\limsup_{n\to\infty}\left\|P^n(-\triangle_k)f\right\|_{L^2_{A_k}(\mathbb{R}^d)^W}^{\frac{1}{n}}\leq \sup_{y\in\operatorname{supp}\mathcal{H}^W(f)}\left|P\left(\|y\|^2\right)\right|. \tag{4.4}$$

In particular, if $\sup_{y\in\operatorname{supp}\mathcal{H}^W(f)}|P(\|y\|^2)| = 0$, the identity (4.1) follows at once.

Second step: In this step we shall prove that

$$\liminf_{n\to\infty}\left\|P^n(-\triangle_k)f\right\|_{L^2_{A_k}(\mathbb{R}^d)^W}^{\frac{1}{n}}\geq \sup_{y\in\operatorname{supp}\mathcal{H}^W(f)}\left|P\left(\|y\|^2\right)\right|.$$

Fix $\xi_0\in\operatorname{supp}\mathcal{H}^W(f)$. We can assume that $P(\|\xi_0\|^2)\neq 0$. We will show that

$$\liminf_{n\to\infty}\left\|P^n(-\triangle_k)f\right\|_{L^2_{A_k}(\mathbb{R}^d)^W}^{\frac{1}{n}}\geq \left|P\left(\|\xi_0\|^2\right)\right|-\varepsilon,$$

for any fixed $\varepsilon > 0$ such that $0 < 2\varepsilon < |P(\|\xi_0\|^2)|$.

To this end, choose and fix $\chi\in D(\mathbb{R}^d)^W$ such that $\langle\mathcal{H}^W(f),\chi\rangle\neq 0$, and

$$\operatorname{supp}\chi\subset\left\{\xi\in\mathbb{R}^d:\left|P\left(\|\xi_0\|^2\right)\right|-\varepsilon<\left|P\left(\|\xi\|^2\right)\right|<\left|P\left(\|\xi_0\|^2\right)\right|+\varepsilon\right\}.$$

For $n\in\mathbb{N}$, let $\chi_n(\xi) = P^{-n}(\|\xi_0\|^2)\chi(\xi)$. In the following we want to estimate $\|(\mathcal{H}^W)^{-1}(\chi_n)\|_{L^2_{A_k}(\mathbb{R}^d)^W}$. Indeed as above we have

$$\begin{aligned}\left\|\left(\mathcal{H}^W\right)^{-1}(\chi_n)\right\|_{L^2_{A_k}(\mathbb{R}^d)^W}&\leq C\sup_{x\in\mathbb{R}^d}\left|e^{\langle\varrho,x\rangle}\left(1+\|x\|^2\right)^m\left(\mathcal{H}^W\right)^{-1}(\chi_n)(x)\right|\\&\leq C\sup_{x\in\mathbb{R}^d}\left|e^{\langle\varrho,x\rangle}\left(1+\|x\|^2\right)^m\left[\left(\mathcal{H}^W\right)^{-1}\left(P^{-n}\left(\|\xi_0\|^2\right)\chi\right)\right](x)\right|,\end{aligned}$$

with $m\geq\frac{d+1}{2}$. Using the continuity of $(\mathcal{H}^W)^{-1}$ we can show that

$$\left\|\left(\mathcal{H}^W\right)^{-1}(\chi_n)\right\|_{L^2_{A_k}(\mathbb{R}^d)^W}\leq C\sup_{\xi\in\mathbb{R}^d}\left|\sum_{1\leq l,j\leq M}\left(1+\|\xi\|^2\right)^j\frac{d^l}{d\xi^l}\left[P^{-n}\left(\|\xi\|^2\right)\chi(\xi)\right]\right|, \tag{4.5}$$

with positive constants C and integers M, m, independent of n. Using Leibniz's rule we deduce that

$$\left\|\left(\mathcal{H}^W\right)^{-1}(\chi_n)\right\|_{L^2_{A_k}(\mathbb{R}^d)^W}\leq Cn^M\left(\left|P\left(\|\xi_0\|^2\right)\right|-\varepsilon\right)^{-n}.$$

Then

$$\begin{aligned}\left\langle\mathcal{H}^W(f),\chi\right\rangle&=\left\langle\mathcal{H}^W(f),P^n\left(\|\xi\|^2\right)\chi_n\right\rangle\\&=\left\langle P^n\left(\|\xi\|^2\right)\mathcal{H}^W(f),\chi_n\right\rangle\\&=\left\langle\mathcal{H}^W\left(P^n(-\triangle_k)f\right),\chi_n\right\rangle\\&=\left\langle\left(P^n(-\triangle_k)f\right),\left(\mathcal{H}^W\right)^{-1}(\chi_n)\right\rangle.\end{aligned}$$

Hence, from the Hölder inequality we obtain

$$\begin{aligned} \left|\langle \mathcal{H}^W(f), \chi\rangle\right| &\le \left\|P^n(-\triangle_k)f\right\|_{L^2_{A_k}(\mathbb{R}^d)^W} \left\|\left(\mathcal{H}^W\right)^{-1}(\chi_n)\right\|_{L^2_{A_k}(\mathbb{R}^d)^W} \\ &\le Cn^M\left(\left|P\left(\|\xi_0\|^2\right)\right| - \varepsilon\right)^{-n}\left\|P^n(-\triangle_k)f\right\|_{L^2_{A_k}(\mathbb{R}^d)^W}. \end{aligned}$$

Since $|\langle \mathcal{H}^W(f), \chi\rangle| > 0$, we deduce that

$$\liminf_{n\to\infty}\left\|P^n(-\triangle_k)f\right\|^{\frac{1}{n}}_{L^2_{A_k}(\mathbb{R}^d)^W} \ge \left|P\left(\|\xi_0\|^2\right)\right| - \varepsilon.$$

Thus

$$\liminf_{n\to\infty}\left\|P^n(-\triangle_k)f\right\|^{\frac{1}{n}}_{L^2_{A_k}(\mathbb{R}^d)^W} \ge \sup_{y\in\operatorname{supp}\mathcal{H}^W(f)}\left|P\left(\|y\|^2\right)\right|.$$

In particular, if $\sup_{y\in\operatorname{supp}\mathcal{H}^W(f)}|P(\|y\|^2)| = \infty$, the identity (4.1) follows at once.

Hence the proof of the theorem is finished. □

Definition 7 Let P be in $\mathbb{C}[x]$. We define the domain U_P by

$$U_P := \left\{x \in \mathbb{R}^d : \left|P\left(\|x\|^2\right)\right| \le 1\right\}.$$

We have the following result.

Corollary 2 *Let $f \in \mathcal{S}_2(\mathbb{R}^d)^W$. The hypergeometric Fourier transform $\mathcal{H}^W(f)$ vanishes outside a polynomial domain U_P, if and only if*

$$\limsup_{n\to\infty}\left\|P^n(-\triangle_k)f\right\|^{\frac{1}{n}}_{L^2_{A_k}(\mathbb{R}^d)^W} \le 1. \tag{4.6}$$

Remark 3 If we take $P(y) = -y$, then $P(-\triangle_k) = \triangle_k$, and Theorem 4 and Corollary 2 characterize functions such that the support of their hypergeometric Fourier transform is $B(0,1)$.

Theorem 5 *Let $P \in \mathbb{C}[x]$. Let f be in $\mathcal{E}(\mathbb{R}^d) \cap L^p_{A_k}(\mathbb{R}^d)^W$, for some $p \in [1,2[$, such that for all $n \in \mathbb{N}^*$, the function $P^n(-\triangle_k)f$ belongs to $L^2_{A_k}(\mathbb{R}^d)^W$. Then*

$$\lim_{n\to\infty}\left\|P^n(-\triangle_k)f\right\|^{1/n}_{L^2_{A_k}(\mathbb{R}^d)^W} = \sup_{y\in\operatorname{supp}\mathcal{H}^W(f)}\left|P\left(\|y\|^2\right)\right|.$$

Proof Let now f be in $\mathcal{E}(\mathbb{R}^d) \cap L^p_{A_k}(\mathbb{R}^d)^W$, for some $p \in [1,2[$ such that for all $n \in \mathbb{N}^*$, the function $P^n(-\triangle_k)f$ belongs to $L^2_{A_k}(\mathbb{R}^d)^W$ and $\mathcal{H}^W(f)$ has compact support.

Let $\varepsilon > 0$. We choose $\Phi \in \mathcal{S}_2(\mathbb{R}^d)^W$ such that $\mathcal{H}^W(\Phi) \equiv 1$ on a neighborhood of $\operatorname{supp}\mathcal{H}^W(f)$ and $|P(\|\xi\|^2)| < R(P, \mathcal{H}^W(f)) + \varepsilon$ for all $\xi \in \operatorname{supp}\mathcal{H}^W(\Phi)$. Moreover it is easy to see that $f *_k \Phi = f$, hence from (2.18) we deduce

$$\begin{aligned} \left\|P^n(-\triangle_k)f\right\|_{L^2_{A_k}(\mathbb{R}^d)^W} &= \left\|P^n(-\triangle_k)(f *_k \Phi)\right\|_{L^2_{A_k}(\mathbb{R}^d)^W} \\ &= \left\|f *_k P^n(-\triangle_k)\Phi\right\|_{L^2_{A_k}(\mathbb{R}^d)^W} \\ &\le C\|f\|_{L^p_{A_k}(\mathbb{R}^d)^W}\left\|P^n(-\triangle_k)\Phi\right\|_{L^2_{A_k}(\mathbb{R}^d)^W}. \end{aligned}$$

Thus from Theorem 4

$$\lim_{n\to\infty}\sup\big\|P^n(-\triangle_k)f\big\|^{1/n}_{L^2_{A_k}(\mathbb{R}^d)^W}\le R\big(P,\mathcal{H}^W(\phi)\big)\le R\big(P,\mathcal{H}^W(f)\big)+\varepsilon.$$

Since the case $R(P,\mathcal{H}^W(f))=\infty$ is trivially true, we obtain

$$\lim_{n\to\infty}\sup\big\|P^n(-\triangle_k)f\big\|^{1/n}_{L^2_{A_k}(\mathbb{R}^d)^W}\le R\big(P,\mathcal{H}^W(f)\big).$$

Now we consider f in $\mathcal{E}(\mathbb{R}^d)$ such that the function $P^n(-\triangle_k)f$ belongs to $L^2_{A_k}(\mathbb{R}^d)^W$ and we proceed as in Theorem 4, step 2 to obtain

$$\lim_{n\to\infty}\inf\big\|P^n(-\triangle_k)f\big\|^{1/n}_{L^2_{A_k}(\mathbb{R}^d)^W}\ge R\big(P,\mathcal{H}^W(f)\big),$$

and the theorem follows. □

The following theorem gives the radius of the large disc on which the hypergeometric Fourier transform of functions in $L^2_{A_k}(\mathbb{R}^d)^W$ vanishes everywhere.

Theorem 6 *Let $f\in L^2_{A_k}(\mathbb{R}^d)^W$ be non-negligible. We consider the sequence*

$$f_n(x)=E^{(k)}_n *_k f(x),\quad x\in\mathbb{R}^d, n\in\mathbb{N}\backslash\{0\},\tag{4.7}$$

where

$$E^{(k)}_n(y)=\big(\mathcal{H}^W\big)^{-1}\big(e^{-n\|x\|^2}\big)(y).$$

Then

$$\lim_{n\to\infty}\sqrt{-\frac{1}{n}\ln\|f_n\|_{L^2_{A_k}(\mathbb{R}^d)^W}}=\lambda_{\mathcal{H}^W(f)},\tag{4.8}$$

where

$$\lambda_{\mathcal{H}^W(f)}:=\inf\big\{\|\xi\|:\xi\in\operatorname{supp}\mathcal{H}^W(f)\big\}.\tag{4.9}$$

Proof To prove (4.8) it is sufficient to verify the equivalent identity

$$\lim_{n\to\infty}\|f_n\|^{\frac{1}{n}}_{L^2_{A_k}(\mathbb{R}^d)^W}=\exp\big(-\lambda^2_{\mathcal{H}^W(f)}\big).\tag{4.10}$$

Using (2.20) we deduce that the hypergeometric Fourier transform of $f_n(x)$ is $\exp(-n\|\xi\|^2)\times\mathcal{H}^W(f)(\xi)$. Then by applying Proposition 3 we obtain

$$\begin{aligned}\|f_n\|^2_{L^2_{A_k}(\mathbb{R}^d)^W}&=\big\|\exp\big(-n\|\xi\|^2\big)\mathcal{H}^W(f)(\xi)\big\|^2_{L^2_{\nu_k}(\mathbb{R}^d)^W}\\&=\|f\|^2_{L^2_{A_k}(\mathbb{R}^d)^W}\bigg[\int_{\operatorname{supp}\mathcal{H}^W(f)}\exp\big(-2n\|\xi\|^2\big)\frac{(|\mathcal{H}^W(f)(\xi)|^2)}{\|f\|^2_{L^2_{A_k}(\mathbb{R}^d)^W}}\,d\nu_k(\xi)\bigg].\end{aligned}\tag{4.11}$$

On the other hand it is well known that if m is the Lebesgue measure on $\mathbb{R}^d$ and U a subset of $\mathbb{R}^d$ such that $m(U) = 1$, then for all ϕ in the Lebesgue space $L^p(U, dm)$, $1 \leq p \leq \infty$, we have

$$\lim_{p\to\infty} \|\phi\|_{L^p(U;dm)} = \|\phi\|_{L^\infty(U;dm)}. \tag{4.12}$$

By applying formula (4.12) with

$$U = \operatorname{supp} \mathcal{H}^W(f), \qquad \phi = \exp\bigl(-\|\xi\|^2\bigr) : p = 2n \quad \text{and} \quad dm(\xi) = \frac{|\mathcal{H}^W(f)(\xi)|^2}{\|f\|^2_{L^2_{A_k}(\mathbb{R}^d)^W}} d\nu_k(\xi),$$

and using the fact that $\lim_{n\to\infty}(\|f\|_{L^2_{A_k}(\mathbb{R}^d)^W})^{\frac{1}{n}} = 1$, we obtain

$$\lim_{n\to\infty} \|f_n\|^{\frac{1}{n}}_{L^2_{A_k}(\mathbb{R}^d)^W} = \sup_{\xi\in\operatorname{supp}\mathcal{H}^W(f)} \exp\bigl(-\|\xi\|^2\bigr) = \exp\bigl(-\lambda^2_{\mathcal{H}^W(f)}\bigr). \tag{4.13}$$

This is the relation (4.10). □

A function $f \in L^2_{A_k}(\mathbb{R}^d)^W$ is the hypergeometric Fourier transform of a function vanishing in a neighborhood of the origin, if and only if, $\lambda_{\mathcal{H}^W(f)} > 0$, or equivalently, if and only if the limit (4.10) is less than 1. Thus we have proved the following result.

Corollary 3 *The condition*

$$\lim_{n\to\infty} \|f_n\|^{\frac{1}{n}}_{L^2_{A_k}(\mathbb{R}^d)^W} < 1, \tag{4.14}$$

is necessary and sufficient for a function $f \in L^2_{A_k}(\mathbb{R}^d)^W$ to have its hypergeometric Fourier transform vanishing in a neighborhood of the origin.

Remark 4 From Theorem 3 and Corollary 3 it follows that the support of the hypergeometric Fourier transform of a function in $L^2_{A_k}(\mathbb{R}^d)^W$ is in the torus $\lambda_{\mathcal{H}^W(f)} \leq \|\xi\| \leq R_{\mathcal{H}^W(f)}$, if and only if,

$$\lambda_{\mathcal{H}^W(f)} \leq \lim_{n\to\infty} \sqrt{-\frac{1}{n} \ln \|f_n\|_{L^2_{A_k}(\mathbb{R}^d)^W}} \leq \lim_{n\to\infty} \bigl\|\triangle^n_k f\bigr\|^{\frac{1}{2n}}_{L^2_{A_k}(\mathbb{R}^d)^W} \leq R_{\mathcal{H}^W(f)}. \tag{4.15}$$

Theorem 7 *For any function $f \in \mathcal{S}^2(\mathbb{R}^d)^W$ the following relation holds*:

$$\lim_{n\to\infty} \left\| \sum_{m=0}^{\infty} \frac{(n\triangle_k)^m f}{m!} \right\|^{\frac{1}{n}}_{L^2_{A_k}(\mathbb{R}^d)^W} = \exp\bigl(-\lambda^2_{\mathcal{H}^W(f)}\bigr). \tag{4.16}$$

In particular, a function $f \in \mathcal{S}_2(\mathbb{R}^d)^W$ is the inverse hypergeometric Fourier transform of a function in $\mathcal{S}(\mathbb{R}^d)^W$ vanishing in $B(0,R)$, if and only if we have

$$\lim_{n\to\infty} \left\| \sum_{m=0}^{\infty} \frac{(n\triangle_k)^m f}{m!} \right\|^{\frac{1}{n}}_{L^p_{A_k}(\mathbb{R}^d)^W} \leq \exp\bigl(-R^2\bigr). \tag{4.17}$$

Proof A similar proof to that of Theorem 4 gives the result. □

5 Real Paley-Wiener theorems for the hypergeometric Fourier transform on $\mathcal{S}_2'(\mathbb{R}^d)^W$

Theorem 8 *Let $u \in \mathcal{E}(\mathbb{R}^d)^W \cap \mathcal{S}_2'(\mathbb{R}^d)^W$, and suppose the set $V_r := \{\xi \in \mathbb{R}^d : |P(\|\xi\|^2)| \le r\}$ is compact for a polynomial $P \in \mathbb{C}[x]$ and a constant $r \ge 0$. Then the support of $\mathcal{H}^W(u)$ is contained in V_r, if and only if for each $R > r$, there exist N_R and a positive constant $C(R)$ such that*

$$\left|P^n(-\triangle_k)(u)(x)\right| \le C(R) n^{N_R} R^n \left(1 + \|x\|\right)^N e^{-\langle \varrho, x\rangle}, \tag{5.1}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$.

Proof Assume that the support of $\mathcal{H}^W(u)$ is contained in the compact V_r. We choose $\chi \in D(\mathbb{R}^d)^W$ such that $\chi \equiv 1$ on an open neighborhood of support of $\mathcal{H}^W(u)$, and $\chi \equiv 0$ outside V_r. As $\mathcal{H}^W(u)$ is of order N, there exists a positive constant C such that for all $x \in \mathbb{R}^d$

$$\begin{aligned}
\left|P^n(-\triangle_k)(u)(x)\right| &= \left|\left(\mathcal{H}^W\right)^{-1}\left(P^n\left(\|\xi\|^2\right)\mathcal{H}^W(u)\right)(x)\right| \\
&= \left|\left(\mathcal{H}^W\right)^{-1}\left(\chi(\xi)P^n\left(\|\xi\|^2\right)\mathcal{H}^W(u)\right)(x)\right| \\
&= \left|\left\langle \chi(\xi)P^n\left(\|\xi\|^2\right)\mathcal{H}^W(u)(\xi), F_\xi(-x)\right\rangle\right| \\
&= \left|\left\langle \mathcal{H}^W(u)(\xi), \chi(\xi)P^n\left(\|\xi\|^2\right)F_\xi(-x)\right\rangle\right| \\
&\le C \sum_{0 \le j \le N} \left\|D^j\left(\chi(\xi)P^n\left(\|\xi\|^2\right)F_\xi(-x)\right)\right\|_{L^\infty_{v_k}(\mathbb{R}^d)^W}.
\end{aligned}$$

Thus from the Leibniz formula, relation (2.5), we deduce the result.

Conversely we assume that we have (5.1).

Suppose $\xi_0 \in \mathbb{R}^d$ is fixed and such that $|P(\|\xi_0\|^2)| \ge R + \varepsilon$, for some $\varepsilon > 0$. Choose and fix $\chi \in D(\mathbb{R}^d)^W$ such that $\operatorname{supp} \chi \subset \{\xi \in \mathbb{R}^d : |P(\|\xi\|^2)| \ge R + \frac{\varepsilon}{3}\}$, and put $\chi_n = P^{-n}(\|\xi\|^2)\chi$. We have

$$\begin{aligned}
\left\langle \mathcal{H}^W(u), \chi\right\rangle &= \left\langle \mathcal{H}^W(u), P^n\left(\|\xi\|^2\right)\chi_n\right\rangle = \left\langle P^n\left(\|\xi\|^2\right)\mathcal{H}^W(u), \chi_n\right\rangle \\
&= \left\langle \mathcal{H}^W\left(P^n(-\triangle_k)u\right), \chi_n\right\rangle \\
&= \left\langle \left(e^{\langle \varrho, x\rangle}\left(1 + \|x\|\right)^{-N} P^n(-\triangle_k)u\right), e^{-\langle \varrho, x\rangle}\left(1 + \|x\|\right)^N \left(\mathcal{H}^W\right)^{-1}(\chi_n)\right\rangle.
\end{aligned}$$

Hence, from the Hölder inequality we obtain

$$\begin{aligned}
\left|\left\langle \mathcal{H}^W(u), \chi\right\rangle\right| \le{}& C\left\|e^{\langle \varrho, x\rangle}\left(1 + \|x\|\right)^{-N+d+1} P^n(-\triangle_k)u\right\|_{L^\infty_{A_k}(\mathbb{R}^d)^W} \\
&\times \left\|e^{-\langle \varrho, x\rangle}\left(1 + \|x\|\right)^N \left(\mathcal{H}^W\right)^{-1}(\chi_n)\right\|_{L^2_{A_k}(\mathbb{R}^d)^W}.
\end{aligned}$$

We proceed as in Theorem 4, step 2, and we prove that

$$\left\|e^{-\langle \varrho, x\rangle}\left(1 + \|x\|\right)^N \left(\mathcal{H}^W\right)^{-1}(\chi_n)\right\|_{L^2_{A_k}(\mathbb{R}^d)^W} \le C n^M \left(\left|P\left(\|\xi_0\|^2\right)\right| + \frac{\varepsilon}{3}\right)^{-n} \le C n^M \left(R + \frac{\varepsilon}{3}\right)^{-n}.$$

Thus

$$\left|\langle \mathcal{H}^W(u), \chi\rangle\right| \le C(R) n^{M+N}\left(\frac{R}{R+\frac{\varepsilon}{3}}\right)^n.$$

Thus we deduce $\langle \mathcal{H}^W(u), \chi\rangle = 0$, which implies that $\xi_0 \notin \operatorname{supp} \mathcal{H}^W(u)$. □

Corollary 4 *Let $u \in \mathcal{E}(\mathbb{R}^d)^W \cap \mathcal{S}_2'(\mathbb{R}^d)^W$ such that* $\operatorname{supp} \mathcal{H}^W(u)$ *is compact. Let $P \in \mathbb{C}[x]$. Then*

$$\sup_{y \in \operatorname{supp} \mathcal{H}^W(f)} \left|P\left(\|y\|^2\right)\right| = \mathcal{R}_u,$$

where $\mathcal{R}_u$ is defined as the infimum of all $R \ge 0$ for which there exist N and $C(N,R) \ge 0$, such that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$

$$\left|P^n(-\triangle_k)(u)(x)\right| \le C(R,N) n^N R^n \left(1+\|x\|\right)^N e^{-\langle \varrho, x\rangle}.$$

Remark 5 We note that the analogue of Theorem 8 and Corollary 4, studied in [14, 15], is missing it the term $e^{-\varrho|x|}$ (as a typing error).

Notations Let $u \in \mathcal{S}_2'(\mathbb{R}^d)^W$. We denote by

$$\Gamma_u := \inf\left\{r \in (0,\infty] : \operatorname{supp}\left(\mathcal{H}^W(u)\right) \subset B(0,r)\right\},$$
$$\gamma_u := \sup\left\{r \in [0,\infty) : \operatorname{supp}\left(\mathcal{H}^W(u)\right) \subset B(0,r)^c\right\},$$

where $B(0,r) = \{x \in \mathbb{R}^d : \|x\| < r\}$ and $B(0,r)^c = \{x \in \mathbb{R}^d : \|x\| \ge r\}$.

Theorem 9 *Let $u \in \mathcal{S}_2'(R^d)^W$. Then the support of $\mathcal{H}^W(u)$ is included in $B(0,M)$, with $M > 0$, if and only if for all $R > M$ we have*

$$\lim_{n\to\infty} R^{-2n} \triangle_k^n u = 0, \quad \text{in } \mathcal{S}_2'\left(\mathbb{R}^d\right)^W.$$

Proof Let $u \in \mathcal{S}_2'(\mathbb{R}^d)^W$ and $M > 0$ such that

$$\lim_{n\to\infty} R^{-2n} \triangle_k^n u = 0, \quad \text{for all } R > M.$$

Let $\varphi \in D(\mathbb{R}^d)^W$ satisfying $\operatorname{supp}(\varphi) \subset B(0,M)^c$. We have to prove that

$$\left\langle \mathcal{H}^W(u), \varphi\right\rangle = 0.$$

Let $r > M$ satisfying $\varphi(x) = 0$ for all $x \in B(0,r)$ and $R \in]M, r[$. Then for all $n \in \mathbb{N}$ the function $\|x\|^{-2n}\varphi$ is in $D(\mathbb{R}^d)^W$ and we can write

$$\left\langle \mathcal{H}^W(u), \varphi\right\rangle = \left\langle \left(-\|x\|^2\right)^n R^{-2n} \mathcal{H}^W(u), \left(-\|x\|^2\right)^{-n} R^{2n}\varphi\right\rangle = 0,$$

and by (2.22), we have

$$\left\langle \mathcal{H}^W(u), \varphi\right\rangle = \left\langle \mathcal{H}^W\left(R^{-2n}\triangle_k^n(u)\right), \left(-\|x\|^2\right)^{-n} R^{2n}\varphi\right\rangle = 0.$$

The hypothesis implies that $\mathcal{H}^W(R^{-2n}\triangle_k^n(u)) \to 0$ in $\mathcal{S}'(\mathbb{R}^d)^W$. Moreover from the Leibniz formula we deduce that $(-\|x\|^2)^{-n}R^{2n}\varphi \to 0$ in $\mathcal{S}(\mathbb{R}^d)^W$. So using the Banach-Steinhaus theorem we prove that

$$\langle \mathcal{H}^W(u), \varphi\rangle = 0.$$

Conversely, let $u \in \mathcal{S}'_2(\mathbb{R}^d)^W$ and $M > 0$ such that $\operatorname{supp}\mathcal{H}^W(u) \subset B(0,M)$. We are going to prove that for all $R > M$

$$\lim_{n\to\infty} R^{-2n}\triangle_k^n u = 0, \quad \text{in } \mathcal{S}'_2(\mathbb{R}^d)^W.$$

Let $M < R$ and choose $\varrho \in]M, r[$ and $\psi \in D(R)$ satisfying $\psi \equiv 1$ on a neighborhood of $B(0,M)$ and $\psi(x) = 0$ for all $x \notin B(0,\varrho)$. Then for all $\varphi \in D(\mathbb{R}^d)^W$ we have

$$\langle \mathcal{H}^W(u), \varphi\rangle = \langle \mathcal{H}^W(u), \psi\varphi\rangle,$$

and then

$$\langle \mathcal{H}^W\bigl(R^{-2n}\triangle_k^n(u)\bigr), \varphi\rangle = \langle \mathcal{H}^W(u), \bigl(-\|x\|^2\bigr)^n R^{-2n}\psi\varphi\rangle.$$

Finally we deduce the result by using the fact that $(-\|x\|^2)^n R^{-2n}\psi\varphi \to 0$ in $\mathcal{S}(\mathbb{R}^d)^W$. □

From the previous theorem we obtain the following.

Corollary 5 *We have*

$$\Gamma_u = \inf\Bigl\{R > 0 : \lim_{n\to\infty} R^{-2n}\triangle_k^n u = 0, \mathit{in}\ \mathcal{S}'_2\bigl(\mathbb{R}^d\bigr)^W\Bigr\}.$$

Theorem 10 *Let $u \in \mathcal{S}'_2(\mathbb{R}^d)^W$ such that $(-\|x\|^2)^{-n}\mathcal{H}^W(u) \in \mathcal{S}'(\mathbb{R}^d)^W$ for all $n \in \mathbb{N}$. Let $u_n = (\mathcal{H}^W)^{-1}((-\|x\|^2)^{-n}\mathcal{H}^W(u))$. Then the support of $\mathcal{H}^W(u)$ is included in $B(0,M)^c$, $M > 0$, if and only if for all $R < M$ we have*

$$\lim_{n\to\infty} R^{2n}u_n = 0, \quad \mathit{in}\ \mathcal{S}'_2\bigl(\mathbb{R}^d\bigr)^W.$$

Proof Let $u \in \mathcal{S}'_2(\mathbb{R}^d)^W$ and $M > 0$ such that

$$\lim_{n\to\infty} R^{2n}u_n = 0, \quad \text{for all } R < M.$$

Let $\varphi \in D(\mathbb{R}^d)^W$ satisfying $\operatorname{supp}(\varphi) \subset B(0,M)$. We have to prove that

$$\langle \mathcal{H}^W(u), \varphi\rangle = 0.$$

Let $\varphi \in D(\mathbb{R}^d)^W$ satisfying $\operatorname{supp}\varphi \subset B(0,M)$ and $R \in]r, M[$. Then for all $n \in \mathbb{N}$ the function $\|x\|^{2n}\varphi$ is in $D(\mathbb{R}^d)^W$ and we can write

$$\langle \mathcal{H}^W(u), \varphi\rangle = \langle \bigl(-\|x\|^2\bigr)^{-n}R^{2n}\mathcal{H}^W(u), \bigl(-\|x\|^2\bigr)^n R^{-2n}\varphi\rangle = 0,$$

and by (2.22), we have

$$\langle \mathcal{H}^W(u), \varphi \rangle = \langle \mathcal{H}^W(R^{2n}u_n), (-\|x\|^2)^n R^{-2n}\varphi \rangle = 0.$$

The hypothesis implies that $\mathcal{H}^W(R^{2n}u_n) \to 0$ in $\mathcal{S}'(\mathbb{R}^d)^W$. Moreover, from the Leibniz formula we deduce that $(-\|x\|^2)^n R^{-2n}\varphi \to 0$ in $\mathcal{S}(\mathbb{R}^d)^W$. So applying the Banach-Steinhaus theorem we prove that

$$\langle \mathcal{H}^W(u), \varphi \rangle = 0.$$

Conversely, let $u \in \mathcal{S}'_2(\mathbb{R}^d)^W$ and $M > 0$ such that $\operatorname{supp} \mathcal{H}^W(u) \subset B(0, M)^c$. We are going to prove that for all $R < M$

$$\lim_{n\to\infty} R^{2n}u_n = 0, \quad \text{in } \mathcal{S}'_2(\mathbb{R}^d)^W.$$

Let $M > R$ and choose $\varrho \in]M, r[$ and $\psi \in D(R)$ satisfying $\psi(x) \equiv 1$ for $\|x\| \geq \frac{M+\varrho}{2}$ and $\psi(x) = 0$ for all $\|x\| \leq \varrho$. Then for all $\varphi \in D(\mathbb{R}^d)^W$ we have

$$\langle \mathcal{H}^W(u), \varphi \rangle = \langle \mathcal{H}^W(u), \psi\varphi \rangle,$$

and then

$$\langle \mathcal{H}^W(R^n u_n), \varphi \rangle = \langle \mathcal{H}^W(u), (-\|x\|^2)^{-n} R^{2n}\psi\varphi \rangle.$$

Finally we deduce the result by using the fact that $(-\|x\|^2)^{-n}R^{2n}\psi\varphi \to 0$ in $\mathcal{S}(\mathbb{R}^d)^W$. □

From the previous theorem we obtain the following.

Corollary 6 *We have*

$$\gamma_u = \sup\left\{R > 0 : \lim_{n\to\infty} R^{2n}u_n = 0, in\ \mathcal{S}'_2(\mathbb{R}^d)^W\right\}.$$

6 Roe's theorem associated with the Heckman-Opdam Laplacian operator

In [16] Roe proved that if a doubly infinite sequence $(f_j)_{j\in\mathbb{Z}}$ of functions on $\mathbb{R}$ satisfies $\frac{df_j}{dx} = f_{j+1}$ and $|f_j(x)| \leq M$ for all $j = 0, \pm1, \pm2, \ldots$ and $x \in \mathbb{R}$, then $f_0(x) = a\sin(x + b)$ where a and b are real constants. This result was extended to $\mathbb{R}^d$ by Strichartz [17] where $\frac{d}{dx}$ is substituted for by the Laplacian on $\mathbb{R}^d$ as follows.

Theorem (Strichartz) *Let $(f_j)_{j\in\mathbb{Z}}$ be a doubly infinite sequence of measurable functions on $\mathbb{R}^d$ such that for all $j \in \mathbb{Z}$,* (i) *$\|f_j\|_{L^\infty(\mathbb{R}^d)} \leq C$ for some constant $C > 0$ and* (ii) *for some $a > 0$, $\triangle f_j = af_{j+1}$. Then $\triangle f_0 = -af_0$.*

The purpose of this section is to generalize this theorem for the Heckman-Opdam Laplacian operator. We now state our main result.

Theorem 11 *Suppose $P(\xi) = \sum_n a_n\xi^n$ is real-valued. Let $a \geq 0$ and let $\{f_j\}_{-\infty}^{\infty}$ be a sequence of W-invariant complex-valued functions on $\mathbb{R}^d$ so that*

$$P(-\triangle_k)f_j = f_{j+1}$$

and

$$\forall x \in C_+, \quad \left|f_j(x)\right| \leq M_j\left(1 + \|x\|\right)^a e^{-\langle \varrho, x\rangle}, \tag{6.1}$$

where $(M_j)_{j\in\mathbb{Z}}$ *satisfies the sublinear growth condition,*

$$\lim_{j\to\infty} \frac{M_{|j|}}{j} = 0. \tag{6.2}$$

Then $f = f_+ + f_-$ *where* $P(-\triangle_k)f_+ = f_+$ *and* $P(-\triangle_k)f_- = -f_-$. *If* 1 (*or* -1) *is not in the range of* P *then* $f_+ = 0$ (*or* $f_- = 0$).

We break the proof up into three steps. In the first step we consider the hypergeometric Fourier transform $\mathcal{H}^W(f_0)$ of f_0, which exists as a distribution.

Lemma 1 *Let* $a \geq 0$. *Let* $(f_j)_{j\in\mathbb{Z}}$ *is a sequence of* W*-invariant functions on* $\mathbb{R}^d$ *satisfying*

$$P(-\triangle_k)f_j = f_{j+1}, \tag{6.3}$$

$$\forall x \in C_+, \quad \left|f_j(x)\right| \leq M_j\left(1 + \|x\|\right)^a e^{-\langle \varrho, x\rangle} \tag{6.4}$$

and

$$\lim_{j\to\infty} \frac{M_j}{(1+\varepsilon)^j} = 0, \tag{6.5}$$

for all $\varepsilon > 0$, *then*

$$\operatorname{support}\left(\mathcal{H}^W(f_0)\right) \subset S := \left\{\xi : \left|P\left(\|\xi\|^2\right)\right| = 1\right\}.$$

Proof First we show that $\mathcal{H}^W(f_0)$ is supported in $\{\xi : |P(\|\xi\|^2)| \leq 1\}$. To do this we need to show that $\langle \mathcal{H}^W(f_0), \phi\rangle = 0$ if $\phi \in D(\mathbb{R}^d)$ and $\operatorname{support}(\phi) \cap \{\xi : |P(\|\xi\|^2)| \leq 1\} = \emptyset$. Since $\operatorname{support}(\phi)$ is compact, there is some $r < 1$ so that $\frac{1}{|P(\|\xi\|^2)|} \leq r$, for all $\xi \in \operatorname{support}(\phi)$. Then

$$\begin{aligned}
\left\langle \mathcal{H}^W(f_0), \phi\right\rangle &= \left\langle P^j \mathcal{H}^W(f_0), \frac{\phi}{P^j}\right\rangle \\
&= \left\langle \left(\mathcal{H}^W\right)\left(P(-\triangle_k)^j f_0\right), \frac{\phi}{P^j}\right\rangle \\
&= \left\langle P(-\triangle_k)^j f_0, \left(\mathcal{H}^W\right)^{-1}\left(\frac{\phi}{P^j}\right)\right\rangle.
\end{aligned}$$

Choose an integer m with $2m \geq 2a + d + 1$. A calculation, using the hypothesis of the lemma and the Cauchy-Schwartz inequality, implies

$$\begin{aligned}
\left|\left\langle \mathcal{H}^W(f_0), \phi\right\rangle\right| &\leq \int_{\mathbb{R}^d} \left|P(-\triangle_k)^j f_0(x)\right| \left|\left(\mathcal{H}^W\right)^{-1}\left(\frac{\phi}{P^j}\right)(x)\right| A_k(x)\,dx \\
&\leq CM_j \sup_{x\in\mathbb{R}^d} \left|e^{\langle \varrho, x\rangle}\left(1 + \|x\|^2\right)^m \left(\mathcal{H}^W\right)^{-1}\left(\frac{\phi}{P^j}\right)(x)\right|.
\end{aligned}$$

Using the continuity of $(\mathcal{H}^W)^{-1}$ and the fact that ϕ is supported in $\{\xi : |P(\|\xi\|^2)| \geq 1+\varepsilon\}$ for some fixed $\varepsilon > 0$, it is not hard to prove that the right-hand side of this goes to zero as $j \to \infty$ and so $\langle \mathcal{H}^W(f_0), \phi\rangle = 0$. To complete the proof we need to show that $\mathcal{H}^W(f_0)$ is also supported in $\{\xi : |P(\|\xi\|^2)| \geq 1\}$, which means $\langle \mathcal{H}^W(f_0), \phi\rangle = 0$ if ϕ is supported in $\{\xi : |P(\|\xi\|^2)| \leq 1\}$. Here we use (6.3) to obtain

$$\langle \mathcal{H}^W(f_0), \phi\rangle = \langle \mathcal{H}^W(f_{-j}), P^j\phi\rangle,$$

and the argument proceeds as before. □

In the next step in the proof we assume firstly that -1 is not a value of $P(\|\xi\|^2)$, and we show that $P(-\triangle_k)f_0 = f_0$.

Lemma 2 *There exists an integer N such that*

$$(P-1)^{N+1}\mathcal{H}^W(f_0) = 0. \tag{6.6}$$

Proof From the growth conditions on the sequence $(f_j)_{j\in\mathbb{Z}}$, Lemma 1, and the assumption that $P(\|\xi\|^2) \neq -1$, we obtain

$$\text{support}\big(\mathcal{H}^W(f_0)\big) \subset \big\{\xi : P\big(\|\xi\|^2\big) = 1\big\}.$$

As $\mathcal{H}^W(f_0)$ is a continuous linear functional on $\mathcal{S}(\mathbb{R}^d)$, there is a constant C and there are integers m and N so that

$$\big|\langle \mathcal{H}^W(f_0), \phi\rangle\big| \leq C\|\phi\|_{N,m}, \tag{6.7}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ when the topology on the space $\mathcal{S}(\mathbb{R}^d)$ is defined by the seminorms

$$\|\phi\|_{N,m} = \sup_{x\in\mathbb{R}^d} \sum_{n\leq N} \big(1+\|x\|^2\big)^m \big|\triangle_k^n \phi(x)\big|.$$

Thus the distribution $\mathcal{H}^W(f_0)$ is of order $\leq N$. For this N we want to prove that

$$(P-1)^{N+1}\mathcal{H}^W(f_0) = 0. \tag{6.8}$$

To simplify the notation set $Q := (P-1)$. Then we need to show, for any compactly supported C^∞ function ϕ, that

$$\langle Q^{N+1}\mathcal{H}^W(f_0), \phi\rangle = \langle \mathcal{H}^W(f_0), Q^{N+1}\phi\rangle = 0.$$

Let $g : \mathbb{R} \to [0,1]$ be a C^∞ function with $g = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $g = 0$ outside $(-1,1)$.

Set $g_r(t) := g(\frac{t}{r})$, $Q_r = g_r(Q)Q^{N+1}\phi$. Then $Q_r = Q^{N+1}\phi$ in a neighborhood of

$$\text{support}\, \mathcal{H}^W(f_0) \subset \big\{\xi : Q(\xi) = 0\big\} = \big\{\xi : P\big(\|\xi\|^2\big) = 1\big\}.$$

Thus by (6.7) we have

$$\left|\left\langle \mathcal{H}^W(f_0), Q^{N+1}\phi\right\rangle\right| = \left|\left\langle \mathcal{H}^W(f_0), Q_r\right\rangle\right| \leq C\|Q_r\|_{N,m}.$$

We proceed as [18] to prove that $\|Q_r\|_{N,m} \to 0$ as $r \to 0$. Thus (6.6) is proved.

Inverting the hypergeometric Fourier transform in (6.6) yields

$$\bigl(P(-\triangle_k) - 1\bigr)^{N+1} f_0 = 0. \tag{6.9}$$

This equation implies

$$\begin{aligned}\operatorname{span}\{f_0, f_1, f_2, \ldots\} &= \operatorname{span}\bigl\{f_0, P(-\triangle_k)f_0, P(-\triangle_k)^2 f_0, \ldots\bigr\}\\ &= \operatorname{span}\bigl\{f_0, P(-\triangle_k)f_0, \ldots, P^n(-\triangle_k)f_0\bigr\}.\end{aligned}$$

We shall now show that we can take $N = 0$ in (6.9). If not then $(P(-\triangle_k) - 1)f_0 \neq 0$. Let p be the largest positive integer so that $(P(-\triangle_k) - 1)^p f_0 \neq 0$. Clearly $p \leq N$. Thus

$$f := \bigl(P(-\triangle_k) - 1\bigr)^{p-1} f_0 \in \operatorname{span}\{f_0, f_1, \ldots, f_N\}$$

will satisfy

$$\bigl(P(-\triangle_k) - 1\bigr)^2 f = 0 \quad \text{and} \quad \bigl(P(-\triangle_k) - 1\bigr)f \neq 0. \tag{6.10}$$

Write

$$f = a_0 f_0 + \cdots + a_N f_N,$$

for constants $a_0, \ldots, a_N$. Then

$$P^j(-\triangle_k)f = a_0 f_j + \cdots + a_N f_{N+j}.$$

If

$$C_j = |a_0|M_j + \cdots + |a_N|M_{j+N},$$

then this and (6.1) imply

$$\bigl|P^j(-\triangle_k)f(x)\bigr| \leq C_j\bigl(1 + \|x\|\bigr)^a. \tag{6.11}$$

By (6.2) these satisfy the sublinear growth condition,

$$\lim_{j\to\infty} \frac{C_j}{j} = 0. \tag{6.12}$$

An induction using (6.10) implies for $j \geq 2$ that

$$P^j(-\triangle_k)f = jP(-\triangle_k)f - (j-1)f = j\bigl(P(-\triangle_k) - 1\bigr)f + f.$$

Thus

$$\left|\big(P(-\triangle_k)-1\big)f(x)\right| \le \frac{1}{j}\left|P^j(-\triangle_k)f(x)\right| + \frac{|f(x)|}{j} \le \frac{C_j}{j}\big(1+\|x\|\big)^a + \frac{|f(x)|}{j}.$$

Letting $j \to \infty$ and using (6.12) implies $(P(-\triangle_k)-1)f = 0$. But this contradicts (6.10). Consequently, $N = 0$ in (6.9). This completes the proof in the case that -1 is not in the range of P.

In the case that 1 is not in the range of P we apply the same argument to $-P(-\triangle_k)$ to conclude $P(-\triangle_k)f_0 = -f_0$.

In the general case, let $\mathfrak{L}_k = P(-\triangle_k)^2$. Then $\mathcal{H}^W(\mathfrak{L}_k f)(\xi) = P(\|\xi\|^2)^2\mathcal{H}^W(f)(\xi)$. We have $\mathfrak{L}_k f_{2p} = f_{2(p+1)}$ and $P(\|\xi\|^2)^2 \neq -1$. Thus we can (as before) conclude for the sequence $(f_{2p})_{p\in\mathbb{Z}}$ that

$$\mathfrak{L}_k f_0 = P(-\triangle_k)^2 f_0 = f_0.$$

Set $f_+ = \frac{1}{2}(f_0 + P(-\triangle_k)f_0)$ and $f_- = \frac{1}{2}(f_0 - P(-\triangle_k)f_0)$. Then $f = f_+ + f_-$, $P(-\triangle_k)f_+ = f_+$ and $P(-\triangle_k)f_- = -f_-$. This completes the proof of Theorem 11. □

Remark 6 (i) If we take $P(y) = -y$, then $P(-\triangle_k) = \triangle_k$ and Theorem 11 gives $\triangle_k f_0 = -f_0$. This characterizes the eigenfunctions f of the generalized Laplace operator $\triangle_k$ with polynomial growth in terms of the size of the powers $\triangle_k^j f$, $-\infty < j < \infty$.

(ii) We note that the analogue of Theorem 11 and Lemma 1, studied in [14, 15], is missing it the term $e^{-\varrho|x|}$ (as a typing error).

(iii) We note that Barhoumi and Mili in [19] have studied the range of the generalized Fourier transform associated with a Cherednick type operator on the real line, and have generalized the Roe's theorem for this operator.

As an application of the above theorem we have the following.

Theorem 12 *If, in Theorem* 11, *we replace* (6.2) *with*

$$\lim_{j\to\infty} \frac{M_{|j|}}{(1+\varepsilon)^j} = 0, \tag{6.13}$$

for all $j > 0$, *then the span of* $(f_j)_j$ *is finite dimensional. Moreover,* $f_0 = f_+ + f_-$, *where, for some integer* N, $(P(-\triangle_k)-1)^N f_+ = 0$ *and* $(P(-\triangle_k)+1)^N f_- = 0$. *Thus* f_+ (*or* f_-) *is a generalized eigenfunction of* $P(-\triangle_k)$ *with eigenvalue* 1 (*or* -1).

Before we demonstrate this theorem we needs the following lemma.

Lemma 3 ([18]) *Let* X *be a finite dimensional complex vector space, and let* $T : X \to X$ *be a linear map with eigenvalues* $\lambda_1, \dots, \lambda_p$. *Then* $X = X_1 \oplus \cdots \oplus X_p$, *where* $X_j = \ker((T-\lambda_j)^N)$ *and* $\dim X = N$.

Proof We prove Theorem 12.

We first prove the result under the assumption that $P(\|\xi\|^2) \neq -1$. Using the growth condition (6.13) and Lemma 3, we may still conclude that

$$\operatorname{support}\big(\mathcal{H}^W(f_0)\big) \subset S = \big\{\xi : P\big(\|\xi\|^2\big) = 1\big\}.$$

But then, as before, we can conclude that (6.9) holds. But this is enough to complete the proof in this case. A similar argument shows that if $P(\|\xi\|^2) \neq 1$, then $(P(-\triangle_k) + 1)^N f_0 = 0$.

In the general case we again let $\mathfrak{L}_k = P(-\triangle_k)^2$ and $P_0 = P^2$. Then $P_0(\|\xi\|^2) \neq -1$ and the span of $(f_{2j})_j$ is finite dimensional. The map $P(-\triangle_k)$ takes the span of $(f_{2j})_j$ onto the span of $(f_{2j+1})_j$. Thus X is finite dimensional. Any $f \in X$ will have support(f) inside the set defined by $P(\|\xi\|^2) = \pm 1$. From this the only possible eigenvalues of $P(-\triangle_k)$ restricted to X are 1 and −1, as it is not hard to show. The result now follows from the last lemma. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors jointly worked, read and approved the final version of the paper.

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