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Some approximation properties of a kind of q-Gamma-Stancu operators

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Abstract

In this paper we propose the Stancu type generalization of a kind of q-Gamma operators. We estimate the moments of these operators and establish two direct and local approximation theorems of the operators. We also obtain the estimates of the rate of convergence and the weighted approximation of the operators. Furthermore, we present a Voronovskaya type asymptotic formula.

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1 Introduction

In 2007, Karsli [1] introduced a kind of new Gamma type operators defined as

$$L_n(f;x) = \frac{(2n+3)!x^{n+3}}{n!(n+2)!} \int_0^\infty \frac{t^n}{(x+t)^{2n+4}} f(t)\,dt, \quad x > 0. \tag{1}$$

He also estimated the rate of convergence of the operators (1) for functions with derivatives of bounded variation on $(0, \infty)$.

In 2009, Karsli *et al.* [2] gave an estimate of the rate of pointwise convergence of the operators (1) on a Lebesgue point of a bounded variation function f defined on the interval $(0, \infty)$.

In recent years Kim used q calculus to study several results on number theory and the related areas [3, 4]. Now we mention certain definitions based on q-integers, details can be found in [5–7]. For any fixed real number $q > 0$ and each nonnegative integer n, we denote q-integers by $[n]_q$, where

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & \text{if } q \neq 1; \\ n, & \text{if } q = 1. \end{cases} \tag{2}$$

Also q-factorial and q-binomial coefficients are defined as follows:

$$[n]_q! = \begin{cases} [1]_q[2]_q \cdots [n]_q, & \text{if } n = 1, 2, \ldots; \\ 1, & \text{if } n = 0; \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}, \quad k = 0, 1, \ldots, n.$$

It is obvious that q-binomial coefficients will reduce to the ordinary case when $q = 1$. The q-improper integrals are defined as (see [8, 9])

$$\int_0^a f(x)\,\mathrm{d}_q x = (1 - q)a \sum_{n=0}^{\infty} f(aq^n)q^n, \quad a \in \mathbb{R}$$

and

$$\int_0^{\infty/A} f(x)\,\mathrm{d}_q x = (1 - q) \sum_{-\infty}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A}, \quad A > 0, \tag{3}$$

provided the sums converge absolutely.

The q-Beta integral is defined as

$$B_q(t; s) = K(A; t) \int_0^{\infty/A} \frac{x^{t-1}}{(1 + x)_q^{t+s}}\,\mathrm{d}_q x, \tag{4}$$

where $K(x; t) = \frac{1}{x+1}x^t(1 + \frac{1}{x})_q^t(1 + x)_q^{1-t}$ and $(a + b)_q^s = \prod_{j=0}^{s-1}(a + q^j b)$, $s \in \mathbb{Z}^+$.

In particular, for any positive integers m, n

$$K(x; n) = q^{\frac{n(n-1)}{2}}, \qquad K(x; 0) = 1 \quad \text{and} \quad B_q(m; n) = \frac{\Gamma_q(m)\Gamma_q(n)}{\Gamma_q(m + n)}, \tag{5}$$

where $\Gamma_q(t)$ is the q-Gamma function.

For $m > 0$, the q-Gamma function is defined by

$$\Gamma_q(m) = \int_0^{\frac{1}{1-q}} x^{m-1} E_q(-qx)\,\mathrm{d}_q x, \tag{6}$$

where $E_q(x) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!}$. Obviously, it satisfies the following functional equations:

$$\Gamma_q(t + 1) = [t]_q \Gamma_q(t), \qquad \Gamma_q(1) = 1.$$

More details of the q-Gamma function and the q-Beta function can be found in [10].

Very recently, Cai and Zeng [11] proposed a kind of q-generalization of Gamma operators and studied their approximation properties. These operators are defined as follows:

$$G_{n,q}(f; x) = \frac{[2n + 3]_q!(q^{n+\frac{3}{2}}x)^{n+3}q^{\frac{n(n+1)}{2}}}{[n]_q![n + 2]_q!} \int_0^{\infty/A} \frac{t^n}{(q^{n+\frac{3}{2}}x + t)_q^{2n+4}}f(t)\,\mathrm{d}_q t, \quad x > 0. \tag{7}$$

In approximation theory the Stancu type modification of operators is an interesting research topic. In this paper, we propose a kind of q-Gamma-Stancu operators $G_{n,q}^{\alpha,\beta}(f; x)$ as follows.

Definition 1 For $f \in C(0, \infty)$, $q \in (0, 1)$, $0 \leq \alpha \leq \beta$ and $n \in \mathbb{N}$, we can define q-Gamma-Stancu operators $G_{n,q}^{\alpha,\beta}(f; x)$ as

$$G_{n,q}^{\alpha,\beta}(f; x) = \frac{[2n + 3]_q!(q^{n+\frac{3}{2}}x)^{n+3}q^{\frac{n(n+1)}{2}}}{[n]_q![n + 2]_q!} \int_0^{\infty/A} \frac{t^n}{(q^{n+\frac{3}{2}}x + t)_q^{2n+4}}f\left(\frac{[n]_q t + \alpha}{[n]_q + \beta}\right)\,\mathrm{d}_q t. \tag{8}$$

The rest of the paper is organized as follows. In Section 2 we present the moments of the operators $G_{n,q}^{\alpha,\beta}(f;x)$. In Section 3 we present two direct and local approximation results for the operators $G_{n,q}^{\alpha,\beta}(f;x)$ by means of the first- and second-order modulus of continuity and the second-order central moment of the operators. In Section 4 we study the rate of convergence of the operators $G_{n,q}^{\alpha,\beta}(f;x)$. In Section 5 we discuss the weighted approximation theorem and Voronovskaya type asymptotic formula of the operators.

2 Moment estimates

In order to obtain the approximation properties of the operators $G_{n,q}^{\alpha,\beta}(f;x)$, we need the following lemmas.

Lemma 1 ([11]) *For any $k \in \mathbb{N}$, $k \leq n+2$ and $q \in (0,1)$, we have*

$$G_{n,q}(t^k;x) = \frac{[n+k]_q![n-k+2]_q!}{[n]_q![n+2]_q!}q^{\frac{2k-k^2}{2}}x^k. \tag{9}$$

Lemma 2 *If we define the moments as*

$$
\begin{aligned}
T_{n,m}^{\alpha,\beta}(x) &= G_{n,q}^{\alpha,\beta}(t^m;x) \\
&= \frac{[2n+3]!(q^{n+\frac{3}{2}}x)^{n+3}q^{\frac{n(n+1)}{2}}}{[n]_q![n+2]_q!} \int_0^{\infty/A} \frac{t^n}{(q^{n+\frac{3}{2}}x+t)_q^{2n+4}}\left(\frac{[n]_q t + \alpha}{[n]_q + \beta}\right)^m d_q t,
\end{aligned} \tag{10}
$$

then we have

(i) $T_{n,0}^{\alpha,\beta}(x) = G_{n,q}^{\alpha,\beta}(1;x) = 1$,

(ii) $T_{n,1}^{\alpha,\beta}(x) = G_{n,q}^{\alpha,\beta}(t;x) = \frac{\sqrt{q}[n]_q[n+1]_q}{([n]_q+\beta)[n+2]_q}x + \frac{\alpha}{[n]_q+\beta}$, *for $n > 2$,*

(iii) $T_{n,2}^{\alpha,\beta}(x) = G_{n,q}^{\alpha,\beta}(t^2;x) = (\frac{[n]_q}{[n]_q+\beta})^2 x^2 + \frac{2\alpha\sqrt{q}[n]_q[n+1]_q}{([n]_q+\beta)^2[n+2]_q}x + (\frac{\alpha}{[n]_q+\beta})^2$, *for $n > 3$.*

Proof From Lemma 1, we have $T_{n,0}^{\alpha,\beta}(x) = G_{n,q}^{\alpha,\beta}(1;x) = 1$. Thus

$$
\begin{aligned}
T_{n,1}^{\alpha,\beta}(x) &:= \frac{[2n+3]_q!(q^{n+\frac{3}{2}}x)^{n+3}q^{\frac{n(n+1)}{2}}}{[n]_q![n+2]_q!} \int_0^{\infty/A} \frac{t^n}{(q^{n+\frac{3}{2}}x+t)_q^{2n+4}}\left(\frac{[n]_q t + \alpha}{[n]_q + \beta}\right) d_q t \\
&= \frac{[n]_q}{[n]_q+\beta}G_{n,q}(t;x) + \frac{\alpha}{[n]_q+\beta}G_{n,q}(1;x) \\
&= \frac{\sqrt{q}[n]_q[n+1]_q}{([n]_q+\beta)[n+2]_q}x + \frac{\alpha}{[n]_q+\beta}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
T_{n,2}^{\alpha,\beta}(x) &:= \frac{[2n+3]-q!(q^{n+\frac{3}{2}}x)^{n+3}q^{\frac{n(n+1)}{2}}}{[n]_q![n+2]_q!} \int_0^{\infty/A} \frac{t^n}{(q^{n+\frac{3}{2}}x+t)_q^{2n+4}}\left(\frac{[n]_q t + \alpha}{[n]_q + \beta}\right)^2 d_q t \\
&= \left(\frac{[n]_q}{[n]_q+\beta}\right)^2 G_{n,q}(t^2;x) + \frac{2[n]_q\alpha}{([n]_q+\beta)^2}G_{n,q}(t;x) + \left(\frac{\alpha}{[n]_q+\beta}\right)^2 G_{n,q}(1;x) \\
&= \left(\frac{[n]_q}{[n]_q+\beta}\right)^2 x^2 + \frac{2\alpha\sqrt{q}[n]_q[n+1]_q}{([n]_q+\beta)^2[n+2]_q}x + \left(\frac{\alpha}{[n]_q+\beta}\right)^2.
\end{aligned}
$$
□

Remark 1 If we put $q = 1$ and $\alpha = \beta = 0$, we get the moments of the Gamma operators (see [1]):

$$L_n(t; x) = \frac{n+1}{n+2} x, \quad n > 2,$$

$$L_n(t^2; x) = x^2, \quad n > 3.$$

Remark 2 Let $n > 2$ be a given natural number. For every $q \in (0, 1)$ we have

$$A_{n,q}^{\alpha,\beta}(x) = G_{n,q}^{\alpha,\beta}(t - x; x) = \left\{ \frac{\sqrt{q}[n]_q[n+1]_q}{([n]_q + \beta)[n+2]_q} - 1 \right\} x + \frac{\alpha}{[n]_q + \beta}, \tag{11}$$

$$\begin{aligned}
B_{n,q}^{\alpha,\beta}(x) &= G_{n,q}^{\alpha,\beta}\big((t - x)^2; x\big) \\
&= \left(\frac{[n]_q}{[n]_q + \beta} \right)^2 x^2 + \frac{2\alpha\sqrt{q}[n]_q[n+1]_q}{([n]_q + \beta)^2[n+2]_q} x + \left(\frac{\alpha}{[n]_q + \beta} \right)^2 \\
&\quad - 2x \left\{ \frac{\sqrt{q}[n]_q[n+1]_q}{([n]_q + \beta)[n+2]_q} x + \frac{\alpha}{[n]_q + \beta} \right\} + x^2 \\
&= \left\{ \left(\frac{[n]_q}{[n]_q + \beta} \right)^2 - \frac{2\sqrt{q}[n]_q[n+1]_q}{([n]_q + \beta)[n+2]_q} + 1 \right\} x^2 \\
&\quad + 2\alpha \left\{ \frac{\sqrt{q}[n]_q[n+1]_q}{([n]_q + \beta)^2[n+2]_q} - \frac{1}{[n]_q + \beta} \right\} x + \left(\frac{\alpha}{[n]_q + \beta} \right)^2, \tag{12}
\end{aligned}$$

$$\begin{aligned}
C_{n,q}^{\alpha,\beta}(x) &= G_{n,q}^{\alpha,\beta}\big((t - x)^4; x\big) \\
&= \left\{ \frac{([n]_q)^3[n+3]_q[n+4]_q}{[n-1]_q([n]_q + \beta)^4 q^4} - \frac{4\sqrt{q}([n]_q)^2[n+3]_q}{q^2([n]_q + \beta)^3} \right. \\
&\quad \left. + \frac{6([n]_q)^2}{([n]_q + \beta)^2} - \frac{4\sqrt{q}[n]_q[n+1]_q}{[n+2]_q([n]_q + \beta)} + 1 \right\} x^4 \\
&\quad + \frac{4\alpha}{[n]_q + \beta} \left\{ \frac{\sqrt{q}([n]_q)^2[n+3]_q}{q^2([n]_q + \beta)^3} - \frac{3([n]_q)^2}{([n]_q + \beta)^2} + \frac{3\sqrt{q}[n+1]_q[n]_q}{[n]_q([n]_q + \beta)} - 1 \right\} x^3 \\
&\quad + \frac{6\alpha^2}{([n]_q + \beta)^2} \left\{ \frac{([n]_q)^2}{([n]_q + \beta)^2} - \frac{2\sqrt{q}[n]_q[n+1]_q}{[n+2]_q([n]_q + \beta)} + 1 \right\} x^2 \\
&\quad + \frac{4\alpha^3}{([n]_q + \beta)^3} \left\{ \frac{\sqrt{q}[n]_q[n+1]_q}{[n+2]_q([n]_q + \beta)} - 1 \right\} x + \frac{\alpha^4}{([n]_q + \beta)^4}. \tag{13}
\end{aligned}$$

3 Local approximation

In this section we establish two direct and the local approximation theorems of the operators $G_{n,q}^{\alpha,\beta}(f; x)$.

Let $C_B[0, +\infty)$ denote the space of all real valued continuous bounded functions f defined on the interval $[0, +\infty)$. The norm $\| \cdot \|$ on the space $C_B[0, +\infty)$ is given by $\|f\| = \sup\{|f(x)| : x \in [0, +\infty)\}$.

Further let us consider Peetre's K-functional:

$$K_2(f; \delta) = \inf_{g \in W^2} \left\{ \|f - g\| + \delta \|g''\| \right\},$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0, +\infty) : g', g'' \in C_B[0, +\infty)\}$.

For $f \in C_B[0, +\infty)$, the modulus of continuity of second order is defined by

$$\omega_2(f; \delta) := \sup_{0 < h \leq \delta} \sup_{x \in [0, +\infty)} \left| f(x + 2h) - 2f(x + h) + f(x) \right|.$$

From [12, 13], there exists an absolute constant $M_0 > 0$ such that

$$K_2(f; \delta) \leq M_0 \omega_2(f; \sqrt{\delta}), \quad \delta > 0. \tag{14}$$

Also we set

$$\omega(f; \delta) := \sup_{0 < h \leq \delta} \sup_{x \in [0, +\infty)} \left| f(x + h) - f(x) \right|.$$

In order to prove the theorems of this section, we need the following lemma.

Lemma 3 *Let $q \in (0, 1)$, $x \in (0, +\infty)$, $f \in C_B[0, +\infty)$. Then, for all $g \in C_B^2[0, +\infty)$, we have*

$$\left| \hat{G}_{n,q}^{\alpha,\beta}(g; x) - g(x) \right| \leq \left(B_{n,q}^{\alpha,\beta}(x) + D_{n,q}^2(x) \right) \left\| g'' \right\|,$$

where

$$\hat{G}_{n,q}^{\alpha,\beta}(f; x) = G_{n,q}^{\alpha,\beta}(f; x) + f(x) - f\left(T_{n,1}^{\alpha,\beta}(x) \right), \tag{15}$$

$$D_{n,q}^{\alpha,\beta}(x) = T_{n,1}^{\alpha,\beta}(x) - x.$$

Proof From (15) and Lemma 2, we have

$$\hat{G}_{n,q}^{\alpha,\beta}(t - x; x) = 0.$$

For $x \in (0, +\infty)$ and $g \in C_B^2[0, +\infty)$, using Taylor's formula,

$$g(t) - g(x) = (t - x)g'(x) + \int_x^t (t - u)g''(u)\, du,$$

we have

$$\hat{G}_{n,q}^{\alpha,\beta}(g; x) - g(x) = \hat{G}_{n,q}^{\alpha,\beta}\left((t - x)g'(x); x \right) + \hat{G}_{n,q}^{\alpha,\beta}\left(\int_x^t (t - u)g''(u)\, du; x \right)$$

$$= g'(x)\hat{G}_{n,q}^{\alpha,\beta}\left((t - x); x \right) + G_{n,q}^{\alpha,\beta}\left(\int_x^t (t - u)g''(u)\, du; x \right)$$

$$- \int_x^{T_{n,1}^{\alpha,\beta}(x)} \left(T_{n,1}^{\alpha,\beta}(x) - u \right)g''(u)\, du$$

$$= G_{n,q}^{\alpha,\beta}\left(\int_x^t (t - u)g''(u)\, du; x \right) - \int_x^{T_{n,1}^{\alpha,\beta}(x)} \left(T_{n,1}^{\alpha,\beta}(x) - u \right)g''(u)\, du.$$

On the other hand, from

$$\left| \int_x^t (t - u)g''(u)\, du \right| \leq \left| \int_x^t |t - u| \left| g''(u) \right|\, du \right| \leq \left\| g'' \right\| \left| \int_x^t |t - u|\, du \right| \leq (t - x)^2 \left\| g'' \right\|$$

and

$$\left| \int_{x}^{T_{n,1}^{\alpha,\beta}(x)} \left(T_{n,1}^{\alpha,\beta}(x) - u \right) g''(u) \, du \right| \leq \left(T_{n,1}^{\alpha,\beta}(x) - x \right)^2 \left\| g'' \right\| = \left(D_{n,q}^{\alpha,\beta}(x) \right)^2 \left\| g'' \right\|,$$

we conclude that

$$\left| \hat{G}_{n,q}^{\alpha,\beta}(g;x) - g(x) \right| = \left| G_{n,q}^{\alpha,\beta}\left(\int_{x}^{t} (t-u)g''(u) \, du; x \right) - \int_{x}^{T_{n,1}^{\alpha,\beta}(x)} \left(T_{n,1}^{\alpha,\beta}(x) - u \right) g''(u) \, du \right|$$

$$\leq G_{n,q}^{\alpha,\beta}\left((t-x)^2 \left\| g'' \right\|; x \right) + \left(D_{n,q}^{\alpha,\beta}(x) \right)^2 \left\| g'' \right\|$$

$$\leq \left(B_{n,q}^{\alpha,\beta}(x) + \left(D_{n,q}^{\alpha,\beta}(x) \right)^2 \right) \left\| g'' \right\|.$$

This completes the proof. □

Theorem 1 *Let $q \in (0,1)$, $f \in C_B[0,+\infty)$. Then, for every $x \in (0,+\infty)$, there exists a constant $M_1 > 0$ such that*

$$\left| G_{n,q}^{\alpha,\beta}(f;x) - f(x) \right| \leq M_1 \omega_2\left(f; \sqrt{B_{n,q}^{\alpha,\beta}(x) + \left(D_{n,q}^{\alpha,\beta}(x) \right)^2} \right) + \omega\left(f; \left| D_{n,q}^{\alpha,\beta}(x) \right| \right).$$

Proof By (15), we have

$$\left| \hat{G}_{n,q}^{\alpha,\beta}(f;x) \right| \leq \left| G_{n,q}^{\alpha,\beta}(f;x) \right| + 2\|f\| \leq 3\|f\|. \tag{16}$$

Using Lemma 3, for every $g \in C_B^2[0,+\infty)$, we obtain

$$\left| G_{n,q}^{\alpha,\beta}(f;x) - f(x) \right| \leq \left| \hat{G}_{n,q}^{\alpha,\beta}(f;x) - f(x) \right| + \left| f(x) - f\left(T_{n,1}^{\alpha,\beta}(x) \right) \right|$$

$$\leq \left| \hat{G}_{n,q}^{\alpha,\beta}(f-g;x) - (f-g)(x) \right| + \left| f(x) - f\left(T_{n,1}^{\alpha,\beta}(x) \right) \right|$$

$$\quad + \left| \hat{G}_{n,q}^{\alpha,\beta}(g;x) - g(x) \right|$$

$$\leq 4\|f-g\| + \left| f(x) - f\left(T_{n,1}^{\alpha,\beta}(x) \right) \right| + \left(B_{n,q}^{\alpha,\beta}(x) + \left(D_{n,q}^{\alpha,\beta}(x) \right)^2 \right) \left\| g'' \right\|$$

$$\leq 4\|f-g\| + \omega\left(f; \left| D_{n,q}^{\alpha,\beta}(x) \right| \right) + \left(B_{n,q}^{\alpha,\beta}(x) + \left(D_{n,q}^{\alpha,\beta}(x) \right)^2 \right) \left\| g'' \right\|.$$

Now, by taking infimum on the right-hand side for all $g \in C_B^2[0,\infty)$ and using (14), we get the following result:

$$\left| G_{n,q}^{\alpha,\beta}(f;x) - f(x) \right| \leq 4 K_2\left(f; B_{n,q}^{\alpha,\beta}(x) + \left(D_{n,q}^{\alpha,\beta}(x) \right)^2 \right) + \omega\left(f; \left| D_{n,q}^{\alpha,\beta}(x) \right| \right)$$

$$\leq 4 M_0 \omega_2\left(f; \sqrt{B_{n,q}^{\alpha,\beta}(x) + \left(D_{n,q}^{\alpha,\beta}(x) \right)^2} \right) + \omega\left(f; \left| D_{n,q}^{\alpha,\beta}(x) \right| \right)$$

$$= M_1 \omega_2\left(f; \sqrt{B_{n,q}^{\alpha,\beta}(x) + \left(D_{n,q}^{\alpha,\beta}(x) \right)^2} \right) + \omega\left(f; \left| D_{n,q}^{\alpha,\beta}(x) \right| \right).$$

This completes the proof. □

Theorem 2 *Let $0 < \gamma \leq 1$ and E be any bounded subset of the interval $[0,+\infty)$. If $f \in C_B[0,+\infty) \cap \mathrm{Lip}_{M_3}(\gamma)$, then we have*

$$\left| G_{n,q}^{\alpha,\beta}(f;x) - f(x) \right| \leq M_3\left\{ \left(B_{n,q}^{\alpha,\beta}(x) \right)^{\frac{\gamma}{2}} + 2\left(d(x;E) \right)^{\gamma} \right\},$$

where M_3 is a constant depending only on α, $d(x; E)$ is the distance between x and E defined as

$$d(x; E) = \inf\{|t - x| : t \in E \text{ and } x \in (0, +\infty)\}.$$

Proof From the properties of the infimum, there is at least one point t_0 in the closure of E such that

$$d(x; E) = |t_0 - x|.$$

By the triangle inequality, we have

$$\big|f(t) - f(x)\big| \le \big|f(t) - f(t_0)\big| + \big|f(t_0) - f(x)\big|.$$

Thus

$$
\begin{aligned}
\big|G_{n,q}^{\alpha,\beta}(f; x) - f(x)\big| &\le G_{n,q}^{\alpha,\beta}\big(\big|f(t) - f(x)\big|; x\big) \\
&\le G_{n,q}^{\alpha,\beta}\big(\big|f(t) - f(t_0)\big|; x\big) + G_{n,q}^{\alpha,\beta}\big(\big|f(t_0) - f(x)\big|; x\big) \\
&\le M_3\big\{G_{n,q}^{\alpha,\beta}\big(|t - t_0|^\gamma; x\big) + |t_0 - x|^\gamma\big\} \\
&\le M_3\big\{G_{n,q}^{\alpha,\beta}\big(|t - x|^\gamma; x\big) + 2|t_0 - x|^\gamma\big\}
\end{aligned}
$$

holds. Now we choose $p_1 = \frac{2}{\gamma}$ and $p_2 = \frac{2}{2-\gamma}$ such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$, then by Hölder inequality we have

$$
\begin{aligned}
\big|G_{n,q}^{\alpha,\beta}(f; x) - f(x)\big| &\le M_3\big\{\big[G_{n,q}^{\alpha,\beta}\big(|t - x|^{\gamma p_1}; x\big)\big]^{\frac{1}{p_1}}\big[G_{n,q}^{\alpha,\beta}\big(1^{p_2}; x\big)\big]^{\frac{1}{p_2}} + 2|t_0 - x|^\gamma\big\} \\
&= M_3\big\{G_{n,q}^{\alpha,\beta}\big(|t - x|^2; x\big)^{\frac{\gamma}{2}} + 2|t_0 - x|^\gamma\big\} \\
&= M_3\big\{\big(B_{n,q}^{\alpha,\beta}(x)\big)^{\frac{\gamma}{2}} + 2\big(d(x; E)\big)^\gamma\big\}.
\end{aligned}
$$

This completes the proof. □

4 Rate of convergence

Let $B_{x^2}[0, +\infty)$ be the set of all functions f defined on $[0, +\infty)$ satisfying the condition $|f(x)| \le M_f(1 + x^2)$, where M_f is a constant depending only on f. Let $C_{x^2}[0, +\infty)$ denote the subset of all continuous functions belonging to $B_{x^2}[0, +\infty)$. If $f \in C_{x^2}[0, +\infty)$ and $\lim_{x \to +\infty} \frac{f(x)}{1+x^2}$ exists, we write $f \in C_{x^2}^*[0, +\infty)$. The norm on $C_{x^2}^*[0, +\infty)$ is given by $\|f\|_{x^2} = \sup_{x \in [0, +\infty)} \frac{|f(x)|}{1+x^2}$. The modulus of continuity of f on the closed interval $[0, a]$ is defined by

$$\omega_a(f; \delta) = \sup_{|t-x| < \delta} \sup_{x, t \in [0,a]} \big|f(t) - f(x)\big|.$$

We know that for a function $f \in C_{x^2}[0, +\infty)$, the modulus of continuity $\omega_a(f; \delta)$ tends to zero as $\delta \to 0$.

Now we give a rate of convergence theorem for the operators $G_{n,q}^{\alpha,\beta}(f; x)$.

Theorem 3 *Let $f \in C_{x^2}[0, +\infty)$, $q \in (0, 1)$ and let $\omega_{a+1}(f; \delta)$ be modulus of the continuity of f on the finite interval $[0, a + 1] \subset [0, +\infty)$, where $a > 0$. Then for $n > 3$,*

$$\left| G_{n,q}^{\alpha,\beta}(f; x) - f(x) \right| \leq 5M_f\left(1 + a^2\right)B_{n,q}^{\alpha,\beta}(x) + \omega_{a+1}(f; \delta)\left(1 + \frac{1}{\delta}\left[B_{n,q}^{\alpha,\beta}(x)\right]^{\frac{1}{2}}\right). \quad (17)$$

Proof For $x \in (0, a]$ and $t > a + 1$, since $t - x > 1$, we have

$$\begin{aligned}
\left| f(t) - f(x) \right| &\leq M_f\left(2 + x^2 + t^2\right) \\
&\leq M_f\left(2 + x^2 + t^2 + (2x - t)^2\right) \\
&\leq M_f\left(2 + 3x^2 + 2(x - t)^2\right) \\
&\leq 5M_f\left(1 + a^2\right)(t - x)^2. \quad (18)
\end{aligned}$$

For $x \in (0, a]$ and $t \leq a + 1$, we have

$$\left| f(t) - f(x) \right| \leq \omega_{a+1}(f; |t - x|) \leq \left(1 + \frac{|t - x|}{\delta}\right)\omega_{a+1}(f; |t - x|) \quad (19)$$

with $\delta > 0$.

From (18) and (19) we get

$$\left| f(t) - f(x) \right| \leq 5M_f\left(1 + a^2\right)(t - x)^2 + \left(1 + \frac{|t - x|}{\delta}\right)\omega_{a+1}(f; \delta), \quad (20)$$

for $x \in (0, a]$ and $t > 0$. Thus

$$\begin{aligned}
\left| G_{n,q}^{\alpha,\beta}(f; x) - f(x) \right| &\leq G_{n,q}^{\alpha,\beta}\left(\left| f(t) - f(x) \right|; x\right) \\
&\leq 5M_f\left(1 + a^2\right)G_{n,q}^{\alpha,\beta}\left((t - x)^2; x\right) \\
&\quad + \omega_{a+1}(f; \delta)\left(1 + \frac{1}{\delta}\left[G_{n,q}^{\alpha,\beta}\left((t - x)^2; x\right)\right]^{\frac{1}{2}}\right) \\
&\leq 5M_f\left(1 + a^2\right)B_{n,q}^{\alpha,\beta}(x) + \omega_{a+1}(f; \delta)\left(1 + \frac{1}{\delta}\left[B_{n,q}^{\alpha,\beta}(x)\right]^{\frac{1}{2}}\right).
\end{aligned}$$

The proof is completed. □

As is well known, if f is not uniformly continuous on the interval $[0, \infty)$, then the usual first modulus of continuity $\omega(f; \delta)$ does not tend to zero as $\delta \to 0$. For every $f \in C_{x^2}^*[0, \infty)$, we would like to take a weighted modulus of continuity $\Omega(f; \delta)$ which tends to zero as $\delta \to 0$.

Let

$$\Omega(f; \delta) = \sup_{0 < h \leq \delta, x \geq 0} \frac{|f(x + h) - f(x)|}{1 + (x + h)^2}, \quad \text{for every } f \in C_{x^2}^*[0, \infty).$$

The weighted modulus of continuity $\Omega(f; \delta)$ was defined by Yuksel and Ispir in [14]. It is well known that $\Omega(f; \delta)$ has the following properties.

Lemma 4 ([14]) *Let $f \in C^*_{x^2}[0,\infty)$, then*:

  (i) $\Omega(f;\delta)$ *is a monotone increasing function of δ.*

  (ii) *For each $f \in C^*_{x^2}[0,\infty)$, $\lim_{\delta\to 0^+} \Omega(f;\delta) = 0$.*

  (iii) *For each $m \in \mathbb{N} \setminus \{0\}$, $\Omega(f;m\delta) \leq m\Omega(f;\delta)$.*

  (iv) *For each $\lambda \in \mathbb{R}^+$, $\Omega(f;\lambda\delta) \leq (1+\lambda)\Omega(f;\delta)$.*

Theorem 4 *Let $f \in C^*_{x^2}[0,\infty)$ and $q = q_n \in (0,1)$ such that $q_n \to 1$ and $[n]_{q_n} \to \infty$ as $n \to \infty$, then there exists a positive constant A such that the inequality*

$$\sup_{x\in(0,\infty)} \frac{|G^{\alpha,\beta}_{n,q_n}(f;x) - f(x)|}{(1+x^2)^{\frac{5}{2}}} \leq A\Omega\left(f; \frac{1}{\sqrt{[n+2]_{q_n}}}\right) \tag{21}$$

holds.

Proof For $t > 0$, $x \in (0,\infty)$ and $\delta > 0$, by the definition of $\Omega(f;\delta)$ and Lemma 4, we get

$$\big|f(t) - f(x)\big| \leq \big(1 + (x + |x - t|)\big)^2 \Omega\big(f; |t - x|\big)$$

$$\leq 2\big(1 + x^2\big)\big(1 + (t - x)^2\big)\left(1 + \frac{|t - x|}{\delta}\right)\Omega(f;\delta).$$

Since $G^{\alpha,\beta}_{n,q_n}$ is linear and positive, we have

$$\big|G^{\alpha,\beta}_{n,q_n}(f;x) - f(x)\big|$$

$$\leq 2\big(1 + x^2\big)\Omega(f;\delta)\left\{1 + G^{\alpha,\beta}_{n,q_n}\big[(t - x)^2; x\big] + G^{\alpha,\beta}_{n,q_n}\left[\big(1 + (t - x)^2\big)\frac{|t - x|}{\delta}; x\right]\right\}. \tag{22}$$

From Remark 2, we have

$$G^{\alpha,\beta}_{n,q_n}\big((t - x)^2; x\big) \leq A_1 \frac{1 + x^2}{[n + 2]_{q_n}}, \tag{23}$$

for some positive constant A_1. To estimate the second term of (22), applying the Cauchy-Schwartz inequality, we have

$$G^{\alpha,\beta}_{n,q_n}\left(\big(1 + (t - x)^2\big)\frac{|t - x|}{\delta}; x\right) \leq 2\big(G^{\alpha,\beta}_{n,q_n}\big(1 + (t - x)^4; x\big)\big)^{\frac{1}{2}}\left(G^{\alpha,\beta}_{n,q_n}\left(\frac{(t - x)^2}{\delta^2}; x\right)\right)^{\frac{1}{2}}. \tag{24}$$

By Remark 2 and (23), there exist two positive constants A_2, A_3 such that

$$\big(G^{\alpha,\beta}_{n,q_n}\big(1 + (t - x)^4; x\big)\big)^{\frac{1}{2}} \leq A_2\big(1 + x^2\big) \tag{25}$$

and

$$\left(G^{\alpha,\beta}_{n,q_n}\left(\frac{(t - x)^2}{\delta^2}; x\right)\right)^{\frac{1}{2}} \leq \frac{A_3}{\delta}\sqrt{\frac{1 + x^2}{[n + 2]_{q_n}}}. \tag{26}$$

Now we take $A = 1 + 2A_1 + 2A_2A_3$ and $\delta = \frac{1}{\sqrt{[n+2]_{q_n}}}$, and combining the above estimates, we obtain the inequality (21). $\qquad\square$

5 Weighted approximation and Voronovskaya type asymptotic formula

In this section we will discuss the weighted approximation theorem and Voronovskaya type asymptotic formula.

Theorem 5 *Let the sequence $q = \{q_n\}$ satisfy $0 < q_n < 1$, $q_n \to 1$ and $[n]_{q_n} \to \infty$ as $n \to \infty$. Then for $f \in C_{x^2}^*[0, \infty)$, we have*

$$\lim_{n \to \infty} \left\| G_{n,q_n}^{\alpha,\beta}(f) - f \right\|_{x^2} = 0. \tag{27}$$

Proof Using the Korovkin theorem in [15], we know that it is sufficient to verify the following three equations:

$$\lim_{n \to \infty} \left\| G_{n,q_n}^{\alpha,\beta}(t^k; x) - x^k \right\|_{x^2} = 0, \quad k = 0, 1, 2. \tag{28}$$

Since $G_{n,q_n}^{\alpha,\beta}(1; x) = 1$, (28) holds true for $k = 0$.

By Lemma 2, for $n > 1$, we have

$$
\begin{aligned}
\left\| G_{n,q_n}^{\alpha,\beta}(t; x) - x \right\|_{x^2} &= \sup_{x \in [0,\infty)} \left\{ \left| \left\{ \frac{\sqrt{q_n}[n]_{q_n}[n+1]_{q_n}}{([n]_{q_n} + \beta)[n+2]_{q_n}} - 1 \right\} x + \frac{\alpha}{[n]_{q_n} + \beta} \right| \times \frac{1}{1 + x^2} \right\} \\
&\leq \left| \frac{\sqrt{q_n}[n]_{q_n}[n+1]_{q_n}}{([n]_{q_n} + \beta)[n+2]_{q_n}} - 1 \right| \sup_{x \in [0,\infty)} \frac{x}{1 + x^2} + \frac{\alpha}{[n]_{q_n} + \beta} \sup_{x \in [0,\infty)} \frac{1}{1 + x^2} \\
&\leq \left| \frac{\sqrt{q_n}[n]_{q_n}[n+1]_{q_n}}{([n]_{q_n} + \beta)[n+2]_{q_n}} - 1 \right| + \frac{\alpha}{[n]_{q_n} + \beta} \\
&\leq \left| \frac{\sqrt{q_n}[n+1]_{q_n}}{[n+2]_{q_n}} - 1 \right| + \frac{\beta \sqrt{q_n}[n+1]_{q_n}}{([n]_{q_n} + \beta)[n+2]_{q_n}} + \frac{\alpha}{[n]_{q_n} + \beta} \\
&\leq \left| \frac{1}{\sqrt{q_n}} - 1 \right| + \frac{1}{\sqrt{q_n}[n+2]_{q_n}} + \frac{\alpha + \beta \sqrt{q_n}}{[n]_{q_n} + \beta}.
\end{aligned}
$$

Thus

$$\lim_{n \to \infty} \left\| G_{n,q_n}^{\alpha,\beta}(t; x) - x \right\|_{x^2} = 0.$$

Similarly, for $n > 2$, we have

$$
\begin{aligned}
\left\| G_{n,q_n}^{\alpha,\beta}(t^2; x) - x^2 \right\|_{x^2} &= \sup_{x \in [0,\infty)} \left\{ \left| \left(\frac{[n]_{q_n}}{[n]_{q_n} + \beta} \right)^2 x^2 + \frac{2\alpha \sqrt{q_n}[n]_{q_n}[n+1]_{q_n}}{([n]_{q_n} + \beta)^2[n+2]_{q_n}} x \right. \right. \\
&\quad \left. \left. + \left(\frac{\alpha}{[n]_{q_n} + \beta} \right)^2 - x^2 \right| \times \frac{1}{1 + x^2} \right\} \\
&\leq \frac{\beta^2}{([n]_{q_n} + \beta)^2} + \frac{2\beta}{[n]_{q_n} + \beta} + \frac{\alpha^2}{([n]_{q_n} + \beta)^2} + \frac{2\alpha \sqrt{q_n}[n]_{q_n}[n+1]_{q_n}}{([n]_{q_n} + \beta)^2[n+2]_{q_n}} \\
&\leq \frac{\alpha^2 + \beta^2}{([n]_{q_n} + \beta)^2} + \frac{2\alpha \sqrt{q_n} + 2\beta}{[n]_{q_n} + \beta},
\end{aligned}
$$

which implies that

$$\lim_{n \to \infty} \left\| G_{n,q_n}^{\alpha,\beta}(t^2; x) - x^2 \right\|_{x^2} = 0.$$

The proof is completed. □

Finally, we give a Voronovskaya type asymptotic formula for $G_{n,q_n}^{\alpha,\beta}(f;x)$ by means of the second and the fourth central moments.

Theorem 6 *Let f be a bounded and integrable function on the interval $(0,\infty)$ and $\{q_n\}_{n=1}^{\infty}$ be a sequence such that $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. Suppose that the second derivative $f''(x)$ exists at a point $x \in (0,\infty)$, then we have*

$$\lim_{n\to\infty}[n+2]_{q_n}\big(G_{n,q_n}^{\alpha,\beta}(f;x) - f(x)\big) = \big(\alpha - (1+\beta)x\big)f'(x) + x^2 f''(x). \tag{29}$$

Proof By the Taylor formula we have

$$f(t) - f(x) = (t-x)f'(x) + \frac{1}{2}f''(x)(t-x)^2 + r(t,x)(t-x)^2,$$

where $r(t,x)$ is bounded and $\lim_{t\to x} r(t,x) = 0$. By applying the operator $G_{n,q}^{\alpha,\beta}(f;x)$ to the above equation we obtain

$$\begin{aligned}
G_{n,q_n}^{\alpha,\beta}(f;x) - f(x) &= f'(x)G_{n,q_n}^{\alpha,\beta}\big((t-x);x\big) + \frac{1}{2}f''(x)G_{n,q_n}^{\alpha,\beta}\big((t-x)^2;x\big) \\
&\quad + G_{n,q_n}^{\alpha,\beta}\big(r(t,x)(t-x)^2;x\big) \\
&= f'(x)A_{n,q_n}^{\alpha,\beta}(x) + \frac{1}{2}f''(x)B_{n,q_n}^{\alpha,\beta}(x) + G_{n,q_n}^{\alpha,\beta}\big(r(t,x)(t-x)^2;x\big).
\end{aligned}$$

By direct calculation, we obtain

$$\begin{aligned}
[n+2]_{q_n}A_{n,q_n}^{\alpha,\beta}(x) &= \left\{\frac{\sqrt{q_n}[n]_{q_n}[n+1]_{q_n}}{([n]_{q_n}+\beta)} - [n+2]_{q_n}\right\}x + \frac{\alpha[n+2]_{q_n}}{[n]_{q_n}+\beta} \\
&= \left\{\sqrt{q_n}[n+1]_{q_n} - [n+2]_{q_n}\right\}x - \left\{\frac{\sqrt{q_n}\beta[n+1]_{q_n}}{[n]_{q_n}+\beta}\right\}x + \frac{\alpha[n+2]_{q_n}}{[n]_{q_n}+\beta} \\
&= (\sqrt{q_n} - q_n)[n+1]_{q_n}x - \left\{1 + \frac{\sqrt{q_n}\beta[n+1]_{q_n}}{[n]_{q_n}+\beta}\right\}x + \frac{\alpha[n+2]_{q_n}}{[n]_{q_n}+\beta} \\
&= \frac{\sqrt{q_n}(1-q_n^{n+1})}{1+\sqrt{q_n}}x - \left\{1 + \frac{\sqrt{q_n}\beta[n+1]_{q_n}}{[n]_{q_n}+\beta}\right\}x + \frac{\alpha[n+2]_{q_n}}{[n]_{q_n}+\beta} \\
&\to \alpha - (1+\beta)x \quad (n\to\infty).
\end{aligned}$$

Similarly,

$$\begin{aligned}
[n+2]_{q_n}&\left[\left(\frac{[n]_{q_n}}{[n]_{q_n}+\beta}\right)^2 - \frac{2\sqrt{q_n}[n]_{q_n}[n]_{q_n}}{([n]_{q_n}+\beta)([n+2]_{q_n})} + 1\right] \\
&= 2\left\{[n+2]_{q_n} - \frac{\sqrt{q_n}[n]_{q_n}[n+1]_{q_n}}{[n]_{q_n}+\beta} - \frac{\beta[n+2]_{q_n}}{[n]_{q_n}+\beta}\right\} \\
&= 2\left\{[n+2]_{q_n} - \sqrt{q_n}[n+1]_{q_n} + \frac{\sqrt{q_n}\beta[n+1]_{q_n}}{[n]_{q_n}+\beta} - \frac{\beta[n+2]_{q_n}}{[n]_{q_n}+\beta}\right\} \\
&= 2\left\{1 + (q_n - \sqrt{q_n})[n+1]_{q_n} + \frac{\sqrt{q_n}\beta[n+1]_{q_n}}{[n]_{q_n}+\beta} - \frac{\beta[n+2]_{q_n}}{[n]_{q_n}+\beta}\right\} \\
&\to 2 \quad (n\to\infty).
\end{aligned}$$

That means

$$[n+2]_{q_n} B_{n,q_n}^{\alpha,\beta}(x) \to 2x^2 \quad (n \to \infty). \tag{30}$$

On the other hand, by simple calculation we obtain

$$[n+2]_{q_n}\left(\frac{[n+3]_{q_n}[n+4]_{q_n}}{[n]_{q_n}[n-1]_{q_n}} - 1\right) \to 8 \quad (n \to \infty), \tag{31}$$

$$[n+2]_{q_n}\left(\frac{[n+3]_{q_n}}{[n]_{q_n}} - 1\right) \to 3 \quad (n \to \infty). \tag{32}$$

Thus from Remark 2, we have

$$[n+2]_{q_n} G_{n,q_n}^{\alpha,\beta}\left((t-x)^4;x\right) \to 0 \quad (n \to \infty). \tag{33}$$

Since $r(t,x)$ is bounded and $\lim_{t \to x} r(t,x) = 0$, then for any given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left|r(t,x)\right| \le \epsilon + \frac{M}{\delta^2}(t-x)^2 \quad \left(t,x \in (0,\infty), M \text{ is a positive constant}\right).$$

Thus

$$[n+2]_{q_n}\left|G_{n,q_n}^{\alpha,\beta}\left(r(t,x)(t-x)^2;x\right)\right|$$

$$\le \epsilon[n+2]_{q_n} G_{n,q_n}^{\alpha,\beta}\left((t-x)^2;x\right) + \frac{M}{\delta^2}[n+2]_{q_n} G_{n,q_n}^{\alpha,\beta}\left((t-x)^4;x\right) \to 0 \quad (n \to \infty).$$

The proof is completed. $\square$

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