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The ergodic shadowing property and homoclinic classes

Manseob Lee*

*Correspondence:
lmsds@mokwon.ac.kr
Department of Mathematics,
Mokwon University, Daejeon,
302-729, Korea

Abstract
In this paper, we show that if a diffeomorphism satisfies a local star condition and it has the ergodic shadowing property then it is hyperbolic.

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1 Introduction

The notion of structural stability was introduced be Andronov and Pontrjagin [1]. This means that under small perturbations the dynamics are topologically equivalent. The system is Ω-stable; then it is called Axiom A, that is, the non-wandering set is the closure of the set of periodic points and it is hyperbolic. It turned out to be one of the most problems in the differentiable dynamical systems to find if a structurally stable system satisfies the Axiom A property. Let M be a closed C^∞ manifold. Mañé defined a set $\mathcal{F}(M)$ of diffeomorphisms having a C^1-neighborhood $\mathcal{U}$ such that every diffeomorphism inside of $\mathcal{U}$ has all periodic orbits of hyperbolic. In [2], Mañé proved that every surface diffeomorphism of $\mathcal{F}(M)$ satisfies Axiom A. Hayashi has shown in [3] that every diffeomorphism of $\mathcal{F}(M)$ satisfies Axiom A. Robinson has proven in [4] that a dynamical system is structurally stable when the system has the shadowing property. Also, in [5] Sakai showed that if a dynamical system belongs to the C^1-interior of the set of all systems having the shadowing property then it is a structurally stable diffeomorphism. Lee has shown in [6] that if a dynamical system belongs to the C^1-interior of the set of all systems having the ergodic shadowing property then it is a structurally stable diffeomorphism. Carvalho proved in [7] that the C^1-interior of the set of all systems having the two-side limit shadowing property is equal to the set of transitive Anosov diffeomorphisms, Pilyugin has shown in [8] that the C^1-interior of the set of all systems having the limit shadowing property is equal to the set Ω-stable diffeomorphisms. Recently, in [9] Sakai proved that for C^1-generically if a diffeomorphism has the s-limit shadowing property on the chain recurrent set then it is a Ω-stable diffeomorphism. From that, we know that the shadowing property is very close to the stability theory (see [10]). In [2], Mañé introduced the family of periodic sequences of linear isomorphisms of $\mathbb{R}^{\dim M}$, and from that we can define the local star condition (see [2, Proposition II.1]).

In this paper, we introduce the notion of the local star condition, and study under the local star condition the some shadowing property.

Let M be a closed C^∞ manifold, and denote by d the distance on M induced by a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Denote by $\mathrm{Diff}(M)$ the space of diffeomorphisms of M endowed with the C^1-topology. Let $f \in \mathrm{Diff}(M)$. We say that f has the *shadowing property* if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i=a}^b$ of f ($-\infty \le a < b \le \infty$), there is a point $y \in M$ such that $d(f^i(y), x_i) < \epsilon$ for all $a \le i \le b-1$. The notion of the ergodic shadowing property was introduced by Fakhari and Ghane in [11]. For any $\delta > 0$, a sequence $\xi = \{x_i\}_{i\in\mathbb{Z}}$ is *δ-ergodic pseudo orbit* of f if for $Np_n^+(\xi, f, \delta) = \{i : d(f(x_i), x_{i+1}) \ge \delta\} \cap \{0, 1, \ldots, n-1\}$, and $Np_n^-(\xi, f, \delta) = \{-i : d(f^{-1}(x_{-i}), x_{-i-1}) \ge \delta\} \cap \{-n+1, \ldots, -1, 0\}$

$$\lim_{n\to\infty} \frac{\#Np_n^+(\xi, f, \delta)}{n} = 0 \quad \text{and} \quad \lim_{n\to-\infty} \frac{\#Np_n^-(\xi, f, \delta)}{n} = 0.$$

We say that f has the *ergodic shadowing property* if for any $\epsilon > 0$, there is a $\delta > 0$ such that every δ-ergodic pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}}$ of f is ϵ-shadowed in ergodic sense for some point $z \in M$, that is, for $Ns_n^+(\xi, f, z, \epsilon) = \{i : d(f^i(z), x_i) \ge \epsilon\} \cap \{0, 1, \ldots, n-1\}$, and $Ns_n^-(\xi, f, z, \epsilon) = \{-i : d(f^{-i}(z), x_{-i}) \ge \epsilon\} \cap \{-n+1, \ldots, -1, 0\}$,

$$\lim_{n\to\infty} \frac{\#Ns_n^+(\xi, f, z, \epsilon)}{N} = 0 \quad \text{and} \quad \lim_{n\to-\infty} \frac{\#Ns_n^-(\xi, f, z, \epsilon)}{N} = 0.$$

Let Λ be a closed f-invariant set. We say that f has the ergodic shadowing property in Λ if for any ϵ there is $\delta > 0$ such that for any δ-ergodic pseudo orbit $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ of f is ϵ-shadowed in ergodic sense for some point $z \in \Lambda$.

By the result of [11], if a diffeomorphism has the ergodic shadowing property then it is chain transitive, moreover, it is topologically mixing. Thus the diffeomorphism does not contain a sink and sources. We know that a Morse-Smale diffeomorphism has the shadowing property. But the diffeomorphism contains sinks and sources. Thus it does not have the ergodic shadowing property. We say that f is *topologically mixing* if for any nonempty open sets U and V, there is $N > 0$ such that $f^n(U) \cap V \neq \emptyset$ for $n \ge N$. In [11, Theorem A], Fakhari and Ghane proved that f has the ergodic shadowing property if and only if f has the shadowing property and it is topologically mixing. Let $P(f)$ be the set of periodic points of f. Denote by $\mathrm{Orb}(p)$ the periodic f-orbit of $p \in P(f)$. Let $p \in P(f)$ be a hyperbolic saddle with period $\pi(p) > 0$, then there are a local stable manifold $W_\epsilon^s(p)$ and a local unstable manifold $W_{\epsilon(p)}^u(p)$ for some $\epsilon = \epsilon(p) > 0$. Then we see that if $x \in W_\epsilon^s(p)$, then $d(f^i(x), f^i(p)) \le \epsilon$, for $i \ge 0$ and if $x \in W_\epsilon^u(p)$ then $d(f^{-i}(x), f^{-i}(p)) \le \epsilon$ for $i \ge 0$. The stable manifold $W^s(p)$ and the unstable manifold $W^u(p)$ of p are defined as usual. The dimension of the stable manifold $W^s(p)$ is called the *index of p*, and we denote it by $\mathrm{index}(p)$.

A point $x \in W^s(p) \cap W^u(p)$ is called a *transversal homoclinic point* of f if the above intersection is transversal at x; *i.e.*, $x \in W^s(p) \pitchfork W^u(p)$. The closure of the set of transversal homoclinic points of f associated to p is called the *transversal homoclinic class* of f associated to p, and it is denoted by $H_f(p)$. It is clear that $H_f(p)$ is compact, invariant, and transitive.

Let $\Lambda \subset M$ be an f-invariant closed set. We say that Λ is a *hyperbloc* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and

$0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s(x)}\| \le C\lambda^n \quad \text{and} \quad \|D_x f^{-n}|_{E^u(x)}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$.

For $\delta > 0$, a sequence $\{x_i\}_{\in\mathbb{Z}}$ is called a δ-average pseudo orbit of f if there is $N = N(\delta) > 0$ such that for all $n \ge N$ and $k \in \mathbb{Z}$,

$$\frac{1}{n}\sum_{i=0}^{n-1} d\big(f(x_{i+k}), x_{i+k+1}\big) < \delta.$$

We say that f has the *average shadowing property* if for any $\epsilon > 0$, there is $\delta > 0$ such that for any δ-average pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$, there is $z \in M$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d\big(f^i(z), x_i\big) < \epsilon.$$

In [12], Lee showed that if $f \in \mathcal{F}(H_f(p))$ and f has the average shadowing property in $H_f(p)$ then $H_f(p)$ is hyperbolic. For that, we show the following.

Theorem 1.1 *Let $H_f(p)$ be the homoclinic class associated to the hyperbolic periodic point p. Assume $H_f(p)$ satisfies the following properties*, (i) *and* (ii):

(i) *f satisfies the local star condition, and*

(ii) *f has the ergodic shadowing property in $H_f(p)$.*

Then $H_f(p)$ is hyperbolic.

The average shadowing property is not the ergodic shadowing property. Indeed, the map $f : [0,1] \to [0,1]$ is defined by $f(x) = 2x$ if $0 \le x < 1/2$, and $f(x) = -2x + 2$ if $1/2 \le x \le 1$. Then the map has two fixed points. In [11, Example], the map has the ergodic shadowing property. However, in [13, Theorem 3.1], Park and Zhang proved that if the number of the fixed points is greater than two, then the map f does not have the average shadowing property.

2 Proof of Theorem 1.1

Let M be as before, and let $f \in \mathrm{Diff}(M)$. We say that a compact f-invariant set $\Lambda \subset M$ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n|_{E(x)}\| \cdot \|Df^{-n}|_{F(f^n(x))}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$. Note that the above dominated splitting can be rewritten as

$$\|Df|_{E(x)}\| / m(Df|_{F(x)}) < \lambda^2$$

for every $x \in \Lambda$, where $m(A) = \inf\{\|Av\| : \|v\| = 1\}$ denotes the *mininorm* of a linear map A. It always extends to a neighborhood which is called an *admissible neighborhood* of Λ. By Mañé (see [2]), the family of periodic sequences of linear isomorphisms of $\mathbb{R}^{\dim M}$ generated by Dg (g close to f) along the hyperbolic periodic point $q \in \Lambda_g(U) \cap P(g)$ is uniformly hyperbolic. This means that there is $\epsilon > 0$ such that for any g C^1-nearby f, $q \in \Lambda_g(U) \cap P(g)$ and any sequence of linear maps $A_i : T_{g^i(q)}M \to T_{g^{i+1}(q)}M$ with $\|A_i - D_{g^i(q)}g\| < \epsilon$ ($i = 1, 2, \ldots, \pi(q)$), $\prod_{i=0}^{\pi(q)-1} A_i$, is hyperbolic. Thus if $f \in \mathcal{F}(H_f(p))$ then we have the following.

Proposition 2.1 *Suppose that $f \in \mathcal{F}(H_f(p))$. Let $\mathcal{U}(f)$ and U be given by the definition of $\mathcal{F}(H_f(p))$. Then there are $m > 0$, $C > 0$ and $\lambda \in (0,1)$ such that*

(a) *$H_f(p)$ admits a dominated splitting $T_{H_f(p)}M = E \oplus F$ with $\dim E = \dim W^s(p)$.*

(b) *For any $q \in \mathcal{U}(f)$ if $q \in \Lambda_g(U) \cap P(g)$ has minimum period $\pi(q)$ then*

$$\prod_{i=0}^{k-1} \left\| D_{g^{im}(q)} g^m|_{E^s(g^{im}(q))} \right\| < C\lambda^k \quad \text{and} \quad \prod_{i=0}^{k-1} \left\| D_{g^{-im}(q)} g^{-m}|_{E^u(g^{-im}(q))} \right\| < C\lambda^k,$$

where $k = [\pi(q)/m]$.

By Proposition 2.1, we get the following, which was found by [14, Theorem 3.2].

Proposition 2.2 *Suppose that $f \in \mathcal{F}(H_f(p))$. Let $\mathcal{U}(f)$ and U be given by the definition of $\mathcal{F}(H_f(p))$. Then there are $m > 0$, $\lambda \in (0,1)$, and $L > 0$ such that we have the following.*

(a) *$H_f(p)$ admits a dominated splitting $T_{H_f(p)}M = E \oplus F$ with $\dim E = \dim W^s(p)$ such that for every $x \in H_f(p)$,*

$$\left\| Df^m|_{E(x)} \right\| / m\left(Df^m|_{F(x)}\right) < \lambda^2.$$

(b) *For any $q \in \mathcal{U}(f)$ if $q \in \Lambda_g(U) \cap P(g)$ then $\mathrm{index}(q) = \mathrm{index}(p_g)$, and if $\pi(q) > L$ then*

$$\prod_{i=0}^{\pi(q)-1} \left\| D_{g^{im}(q)} g^m|_{E^s(g^{im}(q))} \right\| < \lambda^{\pi(q)} \quad \text{and} \quad \prod_{i=0}^{\pi(q)-1} \left\| D_{g^{-im}(q)} g^{-m}|_{E^u(g^{-im}(q))} \right\| < \lambda^{\pi(q)}.$$

Theorem 2.3 [14, Proposition 2.3] *Let $\lambda \in (0,1)$ and let Λ be a closed f-invariant set with a continuous Df-invariant splitting $T_\Lambda M = E \oplus F$ such that*

$$\|Df|_{E(x)}\| / m(Df|_{F(x)}) < \lambda^2$$

for any $x \in \Lambda$. Assume that there is a point $x \in \Lambda$ such that

$$\log \lambda < \log \lambda_1 = \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\left(\|Df|_{E(f^i(x))}\|\right) < 0$$

and

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\left(\|Df|_{E(f^i(x))}\|\right) < \log \lambda_1.$$

Then for any λ_2 and λ_3 with $\lambda < \lambda_2 < \lambda_1 < \lambda_3 < 1$, and any neighborhood U of Λ, there exists a hyperbolic periodic point q if $\mathrm{index}(q) = \dim E$ such that its orbit $\mathrm{Orb}(q)$ is entirely *contained in U and the derivatives along* $\mathrm{Orb}(q)$ *satisfy*

$$\prod_{i=0}^{k-1} \|Df|_{E^s(f^i(q))}\| < \lambda_3^k \quad \text{and} \quad \prod_{i=k-1}^{\pi(q)-1} \|Df|_{E^s(f^i(q))}\| > \lambda_2^{\pi(q)-k+1}$$

for all $k = 1, 2, \dots, \pi(q)$. Moreover, q can be chosen such that $\pi(q)$ is arbitrarily large.

Lemma 2.4 *Let $\{x_i\}_{i\in\mathbb{Z}}$ be a δ-ergodic pseudo orbit of f in Λ. If $\{x_i\}_{i\in\mathbb{Z}}$ is ϵ shadowed in ergodic by some point $z \in \Lambda$ then*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} d\bigl(f^i(z), x_i\bigr) < \epsilon.$$

Proof Suppose that f has the ergodic shadowing property in Λ. Since $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ is a δ-ergodic pseudo orbit of f, there is $z \in \Lambda$ such that

$$\lim_{n\to\infty} \frac{\#\{i \in \{0,1,\dots,n\} : d(f^i(z), x_i) \geq \epsilon\}}{n+1} = 0.$$

Set $k = \#\{i \in \{0,1,\dots,n\} : d(f^i(z), x_i) \geq \epsilon\}$, and $\operatorname{diam}\Lambda = l$. Then

$$\sum_{i=0}^{n} d\bigl(f^i(z), x_i\bigr) < kl + (n+1-k)\epsilon.$$

Thus

$$\frac{1}{n+1} \sum_{i=0}^{n} d\bigl(f^i(z), x_i\bigr) < \frac{1}{n+1}\bigl(kl + (n+1-k)\epsilon\bigr) = \frac{kl}{n+1} + \frac{n\epsilon}{n+1} - \frac{k\epsilon}{n+1}.$$

Therefore, if $n \to \infty$ then

$$\frac{1}{n+1} \sum_{i=0}^{n} d\bigl(f^i(z), x_i\bigr) < \frac{kl}{n+1} + \frac{n\epsilon}{n+1} - \frac{k\epsilon}{n+1} \to \epsilon.$$

Thus

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} d\bigl(f^i(z), x_i\bigr) < \epsilon.$$

□

Let Λ be a closed f-invariant set.

Lemma 2.5 [12, Lemma 2.2] *Let $\varphi(x)$ be a continuous function defined on Λ. For any $\epsilon > 0$ there is a $\delta > 0$ such that for any two sequences $\{x_i\}_{i\in\mathbb{Z}}$, $\{y_i\}_{i\in\mathbb{Z}}$ if*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(x_i, y_i) < \delta,$$

then

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}\big\|\varphi(x_i)-\varphi(y_i)\big\| < \epsilon.$$

Proposition 2.6 *Let p be a hyperbolic periodic point and let $\lambda \in (0,1)$ and $L \geq 1$ be given. Assume that the homoclinic class $H_f(p)$ satisfies the following properties*:

(a) *$H_f(p)$ admits a dominated splitting $T_{H_f(p)}M = E \oplus F$ with $\dim E = \dim W^s(p)$ such that for every $x \in H_f(p)$,*

$$\|Df|_{E(x)}\|/m(Df|_{F(x)}) < \lambda^2.$$

(b) *For any $q \in H_f(p) \cap P(f)$ if q is hyperbolic and $\pi(q) > L$, then $\operatorname{index}(q) = \operatorname{index}(p)$, and*

$$\prod_{i=0}^{\pi(q)-1}\|Df|_{E^s(f^i(q))}\| < \lambda^{\pi(q)} \quad \textit{and} \quad \prod_{i=0}^{\pi(q)-1}\|Df|_{E^u(f^{-i}(q))}\| < \lambda^{\pi(q)}.$$

(c) *f has the ergodic shadowing property in $H_f(p)$.*

Then $H_f(p)$ is hyperbolic for f.

Let $E \subset T_\Lambda M$ be a subbundle. We say that $E \subset T_\Lambda M$ is *contracting* if there exist $C > 0$ and $0 < \lambda < 1$ such that $\|Df^n|_{E(x)}\| < C\lambda^n$ for every $x \in \Lambda$ and every $n \in \mathbb{N}$. We will say that E is *expanding* if E is contracting respecting f^{-1}.

Lemma 2.7 *Let $H_f(p)$ satisfy* (a)-(c) *of Proposition* 2.6. *Suppose that E is not contracting. Then for any $\lambda < \gamma_1 < \gamma_2 < 1$, there is $z \in H_f(p)$ such that*

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\big(\|Df|_{E(f^i(z))}\|\big) < \log\gamma_1 < \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\big(\|Df|_{E(f^i(z))}\big) < \log\gamma_2.$$

Proof Suppose that E is not contracting. Then there is $y \in H_f(p)$ such that

$$\prod_{i=0}^{n-1}\|Df|_{E(f^i(y))}\| \geq 1$$

for all $n \in \mathbb{N}$. For any $x \in H_f(p)$ and $i \in \mathbb{N}$, we define $\varphi(x) = \log\|Df|_{E(f^i(x))}\|$. By Lemma 2.5, for any $\xi > 0$ there is $\epsilon > 0$ such that for any sequences $\{x_i\}_{i\in\mathbb{Z}}, \{y_i\}_{i\in\mathbb{Z}} \subset H_f(p)$ if

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d(x_i,y_i) < \epsilon,$$

then

$$\limsup \frac{1}{n}\sum_{i=0}^{n-1}\big\|\varphi(x_i)-\varphi(y_i)\big\| < \xi.$$

Fix $0 < \xi < \min\{(\log\gamma_2 - \log\gamma_1)/2, (\log\gamma_1 - \log\lambda)/2\}$. Since f has the ergodic shadowing property in $H_f(p)$, there is $\delta > 0$ such that any δ-ergodic pseudo orbit in $H_f(p)$ can be ergodic shadowed by some point in $H_f(p)$. Since $H_f(p) = \overline{\{q \in P(f) : q \sim p\}}$, there is a hyperbolic periodic orbit $\mathrm{Orb}(q) \subset H_f(p)$ with $\pi(q) > L$ such that for any $y \in H_f(p)$, there is $q \in \mathrm{Orb}(q)$ such that $d(y,q) < \delta$. Then we can construct a δ-ergodic pseudo orbit $\{x_i\}_{i\in\mathbb{Z}} \subset H_f(p)$ as in the proof of [12, Lemma 2.3]. We obtain the sequence

$$\{x_i\}_{i\in\mathbb{Z}} = \bigl\{\ldots, f(q), q, y, f(y), \ldots, f^{l_1}(y), q, \ldots, q, f^{l_1+1}(y), \ldots, f^{l_1+l_2}(y), q, \ldots\bigr\}.$$

It is a δ-pseudo orbit of f. Thus we know $\#\{i \in \{0,1,2,\ldots,n\} : d(f(x_i), x_{i+1}) \geq \delta\} = 0$ and so, it is a δ-ergodic pseudo orbit of f. As in the proof of [12, Lemma 2.3], we have

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\bigl(\|Df|_{E(x_i)}\|\bigr) = \frac{1}{2}(\log\gamma_1 + \log\gamma_2) \quad \text{and}$$

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\bigl(\|Df|_{E(x_i)}\|\bigr) \leq \frac{1}{2}(\log\gamma_1 + \log\lambda).$$

Since f has the ergodic shadowing property in $H_f(p)$, we can take $z \in H_f(p)$ such that z is the ergodic shadowing point of $\{x_i\}_{i\in\mathbb{Z}}$. Then we show that

(i)

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\bigl(\|Df|_{E(f^i(z))}\|\bigr) < \log\gamma_1,$$

(ii)

$$\log\gamma_1 < \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\bigl(\|Df|_{E(f^i(z))}\|\bigr), \quad \text{and}$$

(iii)

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\bigl(\|Df|_{E(f^i(z))}\|\bigr) < \log\gamma_2.$$

Proof of (i) Since $\xi < (\log\gamma_1 - \log\lambda)/2$, by Lemma 2.5

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi\bigl(f^i(z)\bigr) &< \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(x_i) + \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \bigl\|\varphi\bigl(f^i(z)\bigr) - \varphi(x_i)\bigr\| \\ &< \frac{1}{2}(\log\gamma_1 + \log\lambda) + \xi < \log\gamma_1.\end{aligned}$$

Proof of (ii) Since $\xi < (\log\gamma_2 - \log\gamma_1)/2$, by Lemma 2.5

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi\bigl(f^i(z)\bigr) &> \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(x_i) - \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \bigl\|\varphi\bigl(f^i(z)\bigr) - \varphi(x_i)\bigr\| \\ &> \frac{1}{2}(\log\gamma_1 + \log\gamma_2) - \xi > \log\gamma_1.\end{aligned}$$

Proof of (iii) Since $\xi < (\log\gamma_2 - \log\gamma_1)/2$, by Lemma 2.5

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi\bigl(f^i(z)\bigr) &< \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi(x_i) + \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\bigl\|\varphi\bigl(f^i(z)\bigr) - \varphi(x_i)\bigr\| \\ &< \frac{1}{2}(\log\gamma_1 + \log\gamma_2) + \xi < \log\gamma_2. \qquad\square\end{aligned}$$

Proof of Theorem 1.1 Since $f \in \mathcal{F}(H_f(p))$, $H_f(p)$ admits a dominated splitting. Then we have $T_{H_f(p)}M = E \oplus F$. To derive a contradiction, we may assume that E is not contracting. Then by Lemma 2.7, for any $\lambda < \gamma_1 < \gamma_2 < 1$ there is $z \in H_f(p)$ such that

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\bigl(\|Df|_{E(f^i(z))}\|\bigr) < \log\gamma_1 < \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\bigl(\|Df|_{E(f^i(z))}\bigr) < \log\gamma_2.$$

By Theorem 2.3, for any $\lambda < \lambda_2 < \lambda_3 < 1$, there is a periodic point q close to $H_f(p)$ such that

$$\begin{aligned}&\prod_{i=0}^{k-1}\|Df|_{E^s(f^i(q))}\| < \lambda_3^k \quad\text{and}\\ &\prod_{i=k}^{\pi(q)-1}\|Df|_{E^s(f^i(q))}\| > \lambda_2^{\pi(q)-k+1}.\end{aligned}\tag{1}$$

Since $H_f(p)$ is locally maximal, $\mathrm{Orb}(q) \subset H_f(p)$. Since $f \in \mathcal{F}(H_f(p))$, (1) is a contradiction by Proposition 2.6(b). This is the proof of Theorem 1.1. $\square$

3 Stably ergodic shadowing property in $H_f(p)$

Let M be as before, and let $f \in \mathrm{Diff}(M)$. We introduce the notion of the C^1-stably ergodic shadowing property.

Definition 3.1 We say that f has the C^1-stably ergodic shadowing property in Λ if there are a compact neighborhood U of f and a C^1-neighborhood $\mathcal{U}(f)$ of f such that $\Lambda = \Lambda_f(U) = \bigcap_{n\in\mathbb{Z}} f^n(U)$ (locally maximal), and for any $g \in \mathcal{U}(f)$, g has the ergodic shadowing property in $\Lambda_g(U)$, where $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is the *continuation* of Λ.

For given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a δ-pseudo orbit $\{x_i\}_{i=a}^b$ ($-\infty \le a < b \le \infty$) of f such that $x_a = x$ and $x_b = y$. We write $x \leftrightsquigarrow y$ if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. The set of points $\{x \in M : x \rightsquigarrow x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{R}(f)$. It is well known that $\mathcal{R}(f)$ is a closed and f-invariant set. If we denote the set of periodic points of f by $P(f)$, then $P(f) \subset \Omega(f) \subset \mathcal{R}(f)$. Here $\Omega(f)$ is the non-wandering set of f. We write $x \leftrightsquigarrow y$ if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. The relation $\leftrightsquigarrow$ induces an equivalence relation on $\mathcal{R}(f)$, whose classes are called *chain components* of f. Denote by $C_f(p) = \{x \in M : x \rightsquigarrow p \text{ and } p \rightsquigarrow x\}$. Then we know that $H_f(p) \subset C_f(p)$ (see [15]). In [15], Lee *et al.* proved that if f has the C^1-stably shadowing property on $C_f(p)$ then $C_f(p)$ is a hyperbolic homoclinic class. They used Mañé's ergodic closing lemma. In this section we use Theorem 1.1. We say that Λ is *topologically transitive* if for any neighborhoods U, V in Λ there is $n > 0$ such that $f^n(U) \cap V \neq \emptyset$. Note that it can be rewritten as follows: there is $x \in \Lambda$ such that $\omega(x) = \Lambda$, where $\omega(x)$ is the omega limit set. Note that if Λ is topologically mixing then Λ is topologically transitive. In [16], Lee showed that if f has the C^1-stably ergodic shadowing property on a transitive set Λ then it admits a dominated splitting. In this section, we will show that if f has the C^1-stably ergodic shadowing property in $H_f(p)$ then it is hyperbolic. The following is the main theorem in this section.

Theorem 3.2 *Let Λ be a closed f-invariant set. Suppose that f has the C^1-stably ergodic shadowing property in Λ. Then $f \in \mathcal{F}(\Lambda)$.*

To prove Theorem 3.2, we need the following lemmas.

Lemma 3.3 [11, Theorem A] *f has the ergodic shadowing property if and only if f has the shadowing property and it is topologically mixing.*

Lemma 3.4 *Let $p, q \in \Lambda$ be hyperbolic periodic points. If f has the ergodic shadowing property in Λ then $W^s(p) \cap W^u(q) \neq \emptyset$ and $W^u(p) \cap W^s(q) \neq \emptyset$.*

Proof Since $p, q \in \Lambda$ are hyperbolic periodic points, there are $\epsilon(p) > 0$ and $\epsilon(q) > 0$ such that $W^{\sigma}_{\epsilon(p)}(p)$ and $W^{\sigma}_{\epsilon(q)}(q)$ are defined, where $\sigma = s, u$. Suppose that f has the ergodic shadowing property in Λ. By Lemma 3.3, f has the shadowing property in Λ and Λ is topologically mixing. Since f has the shadowing property in Λ, we can take $\epsilon = \min\{\epsilon(p), \epsilon(q)\}$. For that $\epsilon > 0$, take $\delta > 0$ be as in the definition of the shadowing property. Since Λ is topologically mixing, Λ is topologically transitive. Then there is a point $x \in \Lambda$ such that $\omega(x) = \Lambda$. For simplicity, we may assume that $f(p) = p$ and $f(q) = q$. Then there exist $l_1 > 0$ and $l_2 > 0$ such that $d(f^{l_1}(x), p) < \delta$ and $d(f^{l_2}(x), q) < \delta$. To construct a δ-pseudo orbit of f, we assume that $l_2 = l_1 + k$ for some $k > 0$. Put (i) $x_i = f^i(p)$ for $i \leq 0$ (ii) $x_i = f^{l_1+i}(x)$ for $0 < i < k$ and (iii) $x_{k+i} = f^{l_2+i}(x)$ for all $i \geq 0$. Then as in the proof of [17, Lemma 2.3], we get $W^u(p) \cap W^s(q) \neq \emptyset$. The other case is similar. □

We say that f is a *Kupka Smale diffeomorphism* if every periodic points are hyperbolic and if $p, q \in P(f)$, then $W^s(p)$ is transversal to $W^u(q)$. It is well known that if f is a Kupka Smale then f is residual in $\mathrm{Diff}(M)$. Denote by $\mathcal{KS}$ the set of all Kupka Smale diffeomorphisms.

Proof of Theorem 3.2 Since f has the C^1-stably ergodic shadowing property in Λ, there exist a C^1-neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, g has the ergodic shadowing property in $\Lambda_g(U)$. To derive a contradiction, we may assume that $f \notin \mathcal{F}(H_f(p))$. Then there are $g \in \mathcal{U}(f)$ and $q \in \Lambda_g(U) \cap P(g)$ such that q is not hyperbolic. Then there is $g_1 \in \mathcal{U}(f)$ close to g such that g_1 has two hyperbolic periodic points $\gamma_1, \gamma_2 \in \Lambda_{g_1}(U) \cap P(g_1)$ with different indices. Then we know $\dim W^s(\gamma_1) + \dim W^u(\gamma_2) < \dim M$ or $\dim W^u(\gamma_1) + \dim W^s(\gamma_2) < \dim M$. Without loss of generality, we assume that $\dim W^s(\gamma_1) + \dim W^u(\gamma_2) < \dim M$. Take $h \in \mathcal{U}(f) \cap \mathcal{KS}$. Since h is Kupka Smale, $W^s(\gamma_{1,h}) \cap W^u(\gamma_{2,h}) = \emptyset$, where $\gamma_{1,h}$ and $\gamma_{2,h}$ are continuations of γ_1 and γ_2, respectively. Since h has the ergodic shadowing property in $\Lambda_h(U)$, h has the shadowing property in $\Lambda_h(U)$ and $\Lambda_h(U)$ is topologically mixing. Since $\gamma_{1,h}, \gamma_{2,h} \in \Lambda_h(U) \cap P(h)$, by Lemma 3.4 we get a contradiction. □

We say that Λ is a *basic set* if Λ is transitive, and locally maximal. If the basic set Λ is hyperbolic then we can easily show that there is a periodic point such that the orbit of the periodic point is dense in Λ. We say that Λ is an *elementary set* if Λ is mixing, and locally maximal. Note that every elementary set is a basic set.

Theorem 3.5 *Let $H_f(p)$ be the homoclinic class. If f has the C^1-stably ergodic shadowing property in $H_f(p)$ then $H_f(p)$ is a hyperbolic elementary set.*

Proof of Theorem 3.5 Suppose that f has the C^1-stably ergodic shadowing property in $H_f(p)$. By Theorem 3.2, $f \in \mathcal{F}(H_f(p))$. Since f has the ergodic shadowing property in $H_f(p)$, by Lemma 3.3 $H_f(p)$ is topologically mixing. Thus by Theorem 1.1, $H_f(p)$ is a hyperbolic elementary set. □

Competing interests
The author declares that they have no competing interests.

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