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Atomic decompositions of weak Orlicz-Lorentz martingale spaces

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Abstract
In this paper we establish atomic decompositions of some weak Orlicz-Lorentz martingale spaces which are generalization of Orlicz martingale spaces and of Lorentz martingale spaces. With the help of atomic decompositions, the boundedness of sublinear operators is obtained.

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1 Introduction and preliminaries

The idea of atomic decomposition in martingale theory is derived from harmonic analysis [1]. Just as it does in harmonic analysis, the method is a key ingredient in dealing with many problems including martingale inequalities, duality, interpolation, and so on, especially for small-index martingale and multi-parameter martingales. As is well known, Weisz [2] gave some atomic decompositions on martingale Hardy spaces and proved many important theorems by atomic decompositions; Weisz [3] made a further study of atomic decompositions for weak Hardy spaces consisting of Vilenkin martingales, and he proved a weak version of the Hardy-Littlewood inequality; Liu and Hou [4] investigated the atomic decompositions for vector-valued martingales and some geometry properties of Banach spaces were characterized; Hou and Ren [5] considered the vector-valued weak atomic decompositions and weak martingale inequalities. Orlicz Hardy martingale spaces are also studied by some authors such as Miyamoto, Nakai, Sadasue and Jiao [6–8]. At the same time, the Lorentz spaces are discussed (see [9–13]). For example, the atomic decompositions of Lorentz martingales are first studied by Jiao *et al.* in [10], and in 2013 Ho investigated the atomic decomposition of Lorentz-Karamata martingale spaces similarly to the idea of [10]. As the generalization of Orlicz and Lorentz spaces, the Orlicz-Lorentz spaces attract more attention. Montgomery-Smith [14] discuss the comparison of Orlicz-Lorentz spaces. Rajeev and Romesh [15] studied composition operators on Orlicz-Lorentz spaces. Echandia [16] discussed the interpolation of Orlicz-Lorentz spaces.

Let (Ω, Σ, μ) be a measure space, $L^0(\mu)$ be a space of all Σ-measurable functions. Let there be given an Orlicz function $\varphi : [0,\infty) \to [0,\infty)$ (*i.e.*, it is a convex function and takes value zero only at zero) and a weight function $\omega : (0,\infty) \to (0,\infty)$ (*i.e.*, it is a non-increasing function and locally integrable and $\int_0^\infty \omega(x)\,dx = \infty$). The Orlicz-Lorentz space

$\Lambda_{\varphi,\omega}$ on (Ω, Σ, μ) is the set of all functions $f(x) \in L^0(\mu)$ such that

$$\int_0^\infty \varphi(\lambda f^*(x))\omega(x)\,dx < \infty \tag{1.1}$$

for some $\lambda > 0$, where f^* is the non-increasing rearrangement of f defined by

$$f^*(t) = \inf\{s > 0 : d_f(s) \leq t\}.$$

We shall not work with this definition of the Orlicz-Lorentz space, however, but with a different, equivalent definition. A Young function F is an even continuous and non-negative function in R^1, increasing on $(0,\infty)$, such that $\lim_{t\to 0^+} F(t) = 0$, $\lim_{t\to\infty} F(t) = \infty$, $F(t) = 0$ iff $t = 0$. A Young function F is said to satisfy the global $\triangle_2$-condition if there is $c > 0$ such that $F(2t) \leq cF(t)$ for all $t \in R^1$. We define $\tilde{F}(t)$ to be $1/F(1/t)$ if $t > 0$ and 0 if $t = 0$.

We define the Orlicz-Lorentz space $L_{F,G}$ as the set of all measurable f's on Ω for which the Orlicz-Lorentz functional

$$\|f\|_{F,G} = \|f^* \circ \tilde{F} \circ \tilde{G}^{-1}\|_G = \inf\left\{\lambda : \int_0^\infty G\left(\frac{f^*(\tilde{F}(\tilde{G}^{-1}(t)))}{\lambda}\right) dt \leq 1\right\}$$

is finite.

Similarly, by means of the weak Orlicz-Lorentz functional

$$\|f\|_{F,\infty} = \sup_{t\geq 0} \tilde{F}^{-1}(t) f^*(t),$$

we define the Orlicz-Lorentz space $L_{F,\infty}$.

Remark 1.1 By the fact $\sup_{t>0} t^s f^*(t) = \sup_{t>0} t(d_f(t))^s$, we have $\|f\|_{F,\infty} = \sup_{t\geq 0} t\tilde{F}^{-1}(d_f(t))$. We see that $L_{F,F} = L_F$, where L_F is Orlicz space, and that if $F(t) = t^p$ and $G(t) = t^q$, then $L_{F,G} = L_{p,q}$. If A is any measurable set, then $\|\chi_A\|_{F,G} = \|\chi_A\|_{F,\infty} = \|\chi_A\|_F = \tilde{F}^{-1}(\mu(A))$.

Let (Ω, Σ, P) be a complete probability space, and $(\Sigma_n)_{n\geq 0}$ a non-decreasing sequence of sub-σ-algebras of Σ with $\Sigma = \sigma(\bigcup_{n\geq 0} \Sigma_n)$. We denote by E and E_n the expectation and conditional expectation with respect to Σ and Σ_n, respectively. For a martingale $f = (f_n)_{n\geq 0}$ with martingale differences $df_n = f_n - f_{n-1}$, $n \geq 0, f_{-1} \equiv 0$, denote

$$M_n f = \sup_n |f_n|, \qquad S_n(f) = \left(\sum_{k=0}^n |df_k|^2\right)^{1/2}, \qquad \sigma_n(f) = \left(\sum_{k=0}^n E_{k-1}|df_k|^2\right)^{1/2},$$
$$Mf = \lim_{n\to\infty} M_n f, \qquad S(f) = \lim_{n\to\infty} S_n(f), \qquad \sigma(f) = \lim_{n\to\infty} \sigma_n(f).$$

Denote by Λ the collection of all sequences $(\lambda_n)_{n\geq 0}$ of non-decreasing, non-negative and adapted functions and set $\lambda_\infty = \lim_{n\to\infty} \lambda_n$. Thus we can define some weak Orlicz-Lorentz martingale spaces as follows:

$$H^\sigma_{F,\infty} = \{f = (f_n) : \|\sigma(f)\|_{F,\infty} < \infty\},$$
$$\mathcal{Q}_{F,\infty} = \{f = (f_n) : \exists(\lambda_n)_{n\geq 0} \in \Lambda,\ \text{s.t. } S_n(f) \leq \lambda_{n-1}, \lambda_\infty \in L_{F,\infty}\},$$

$$\|f\|_{\mathcal{Q}_{F,\infty}} = \inf_{(\lambda_n)\in\Lambda} \|\lambda_\infty\|_{L_{F,\infty}},$$

$$\mathcal{D}_{F,\infty} = \{f = (f_n) : \exists(\lambda_n)_{n\geq 0} \in \Lambda,\ \text{s.t. } |f_n| \leq \lambda_{n-1}, \lambda_\infty \in L_{F,\infty}\},$$

$$\|f\|_{\mathcal{D}_{F,\infty}} = \inf_{(\lambda_n)\in\Lambda} \|\lambda_\infty\|_{L_{F,\infty}}.$$

Definition 1.2 A measurable function a is called a weak atom of the first category (or of the second category, of the third category, respectively) if there exists a stopping time ν (ν is called the stopping time associated with a) such that

(i) $a_n = E_n a = 0$ if $\nu \geq n$;

(ii) $\|\sigma(a)\|_\infty < \infty$ (or (ii) $\|S(a)\|_\infty < \infty$, (ii) $\|Ma\|_\infty < \infty$, respectively).

These three category weak atoms are briefly called w-1-atom, w-2-atom, and w-3-atom, respectively.

Throughout this article, we denote the set of integers and the set of non-negative integers by $\mathbb{Z}$ and $\mathbb{N}$, respectively. We use c to denote constants and may denote different constants at different occurrences.

2 Weak atomic decompositions

Weak atomic decompositions of some weak martingale Hardy spaces were studied in [5, 7]. In this section, we will consider weak atomic decompositions of some weak Orlicz-Lorentz martingale spaces.

Theorem 2.1 *Let $F^{-1} \in \triangle_2$. Then $f = (f_n) \in H^\sigma_{F,\infty}$ if and only if there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of w-1-atoms and the corresponding stopping times $(\tau_k)_{k\in\mathbb{Z}}$ such that*

(1) $f_n = \sum_{k\in\mathbb{Z}} E_n a^k$;

(2) $\sigma(a^k) \leq A \cdot 2^k$, $\forall k \in \mathbb{Z}$ *for some constant* $A > 0$, *and* $\sup_{k\in\mathbb{Z}} 2^k \tilde{F}^{-1}(P(\tau_k < \infty)) < \infty$.

Moreover the following equivalence of norms holds:

$$\|f\|_{H^\sigma_{F,\infty}} \sim \inf \sup_{k\in\mathbb{Z}} 2^k \tilde{F}^{-1}\bigl(P(\tau_k < \infty)\bigr), \tag{2.1}$$

where the infimum is taken over all the preceding decompositions of f.

Proof Assume $f = (f_n, n \in \mathbb{N}) \in H^\sigma_{F,\infty}$. Let us consider the following stopping time for all $k \in \mathbb{Z}$:

$$\tau_k = \inf\{n \in \mathbb{N}, \sigma_{n+1}(f) > 2^k\}, \quad \inf(\emptyset) = \infty.$$

Then the sequence of these stopping times is non-decreasing and $\tau_k \uparrow \infty$. Let $f^{(\tau_k)} = (f_{(\tau_k \wedge n)}, n \in \mathbb{N})$ be the stopping martingale. It is easy to see that

$$f_n = \sum_{k\in\mathbb{Z}} \bigl(f_n^{(\tau_{k+1})} - f_n^{(\tau_k)}\bigr). \tag{2.2}$$

Now let $a_n^k = f_n^{(\tau_{k+1})} - f_n^{(\tau_k)}$. Then for a fixed $k \in \mathbb{N}$ (a_n^k, $n \in \mathbb{N}$) is a martingale and

$$\sigma\bigl(a^k\bigr) = \left(\sum_{i=0}^\infty E_{i-1}\bigl|da_i^k\bigr|^2\right)^{1/2} \leq \bigl(\sigma\bigl(f^{\tau_{k+1}}\bigr) + \sigma\bigl(f^{\tau_k}\bigr)\bigr) \leq 3 \cdot 2^k, \quad \forall n. \tag{2.3}$$

Thus $\|M(a^k)\|_2 \leq c\|\sigma(a^k)\|_2 < \infty$ and $(a_n^k)_{n\geq 0}$ is L_2 bounded. So there exists an integrable function a^k such that $a_n^k = E_n a^k$. If $n \leq \tau_k$, then $E_n a^k = 0$, so we get a^k is really a w-1-atom. Moreover, we have

$$f_n = \sum_{k\in Z}\left(f_n^{(\tau_{k+1})} - f_n^{(\tau_k)}\right) = \sum_{k\in Z} a_n^k = \sum_{k\in Z} E_n a^k. \tag{2.4}$$

Hence we get (1). As $\{\tau_k < \infty\} = \{\sigma(f) > 2^k\}$ for any $k \in \mathbb{Z}$, we have

$$\begin{aligned} 2^k \tilde{F}^{-1}\bigl(P(\tau_k < \infty)\bigr) &= 2^k \tilde{F}^{-1}\bigl(P\bigl(\sigma(f) > 2^k\bigr)\bigr) \\ &\leq \sup_{t>0} t\tilde{F}^{-1}\bigl(P\bigl(\sigma(f) > t\bigr)\bigr) = \|f\|_{H^\sigma_{F,\infty}}, \end{aligned} \tag{2.5}$$

which implies $\sup_{k\in\mathbb{Z}} 2^k \tilde{F}^{-1}(P(\tau_k < \infty)) \leq \|f\|_{H^\sigma_{F,\infty}} < \infty$.

Conversely, assume that $f = (f_n)_{n\geq 0}$ has a decomposition of the form (1). Let $M = \sup_{k\in\mathbb{Z}} 2^k \tilde{F}^{-1}(P(\tau_k < \infty))$. For any fixed $y > 0$ choose $j \in \mathbb{Z}$ such that $2^j \leq y < 2^{j+1}$.

Let

$$f_n = \sum_{k\in\mathbb{Z}} E_n a^k = \sum_{k=\infty}^{j-1} E_n a^k + \sum_{k=j}^{\infty} E_n a^k =: g_n + h_n, \quad \forall n \in \mathbb{N}.$$

Thus by the fact that $\sigma(f) \leq \sigma(g) + \sigma(h)$, we have

$$P\bigl(\sigma(f) > 2Ay\bigr) \leq P\bigl(\sigma(g) > Ay\bigr) + P\bigl(\sigma(h) > Ay\bigr).$$

Since $\sigma(a^k) \leq A \cdot 2^k$, $\forall k \in \mathbb{Z}$, we have

$$\sigma(g) \leq \sum_{k=\infty}^{j-1} \sigma\left(a^k\right) \leq A \sum_{k=\infty}^{j-1} 2^k \leq A2^j. \tag{2.6}$$

Then $P(\sigma(g) > Ay) \leq P(\sigma(g) > A2^j) = 0$.

On the other hand, since $a_n^k = E_n(a^k) = 0$ if $n \leq \tau_k$, thus $\sigma(a^k) = 0$ on the set $\{\tau_k = \infty\}$. Moreover $\sigma(h) \leq \sum_{k=j}^{\infty} \sigma(a^k)$ and $\{\sigma(h) > 0\} \subset \bigcup_{k=j}^{\infty}\{\tau_k < \infty\}$. Since $F^{-1} \in \triangle_2$, then $\tilde{F}^{-1} = \widetilde{F^{-1}} \in \triangle_2$. Moreover $\tilde{F}^{-1}$ is c-subadditive, *i.e.*,

$$\tilde{F}^{-1}(t_1 + t_2) \leq c\bigl(\tilde{F}^{-1}(t_1) + \tilde{F}^{-1}(t_2)\bigr), \quad \forall t_1, t_2 > 0. \tag{2.7}$$

Consequently,

$$\begin{aligned} \tilde{F}^{-1}\bigl(P\bigl(\sigma(f) > 2Ay\bigr)\bigr) &\leq \tilde{F}^{-1}\bigl(P\bigl(\sigma(h) > Ay\bigr)\bigr) \leq \tilde{F}^{-1}\bigl(P\bigl(\sigma(h) > 0\bigr)\bigr) \\ &\leq \sum_{k=j}^{\infty} c\tilde{F}^{-1}\bigl(P(\tau_k < \infty)\bigr) \leq \sum_{k=j}^{\infty} cM2^{-k} \leq cM2^{-j-1} \leq cMy^{-1}, \end{aligned} \tag{2.8}$$

which implies

$$\|f\|_{H^\sigma_{F,\infty}} = \sup_{y>0} 2Ay\tilde{F}^{-1}\bigl(P\bigl(\sigma(f) > 2Ay\bigr)\bigr) \leq c \sup_{k\in\mathbb{Z}} 2^k \tilde{F}^{-1}\bigl(P(\tau_k < \infty)\bigr). \tag{2.9}$$

We combine (2.5) and (2.9) to obtain (2.1). Thus we prove Theorem 2.1. □

Theorems similar to Theorem 2.1 hold for the spaces $\mathcal{Q}_{F,\infty}$ and $\mathcal{D}_{F,\infty}$.

Theorem 2.2 *Let $F^{-1} \in \triangle_2$. Then $f = (f_n) \in \mathcal{Q}_{F,\infty}$ if and only if there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of w-2-atoms and the corresponding stopping times $(\tau_k)_{k\in\mathbb{Z}}$ such that*

(1) $f_n = \sum_{k\in\mathbb{Z}} E_n a^k$;

(2) $S(a^k) \leq A \cdot 2^k$, $\forall k \in \mathbb{Z}$ *for some constant* $A > 0$, *and* $\sup_{k\in\mathbb{Z}} 2^k \tilde{F}^{-1}(P(\tau_k < \infty)) < \infty$.

Moreover the following equivalence of norms holds:

$$\|f\|_{\mathcal{Q}_{F,\infty}} \sim \inf \sup_{k\in\mathbb{Z}} 2^k \tilde{F}^{-1}\bigl(P(\tau_k < \infty)\bigr), \tag{2.10}$$

where the infimum is taken over all the preceding decompositions of f.

Theorem 2.3 *Let $F^{-1} \in \triangle_2$. Then $f = (f_n) \in \mathcal{D}_{F,\infty}$ if and only if there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of w-3-atoms and the corresponding stopping times $(\tau_k)_{k\in\mathbb{Z}}$ such that*

(1) $f_n = \sum_{k\in\mathbb{Z}} E_n a^k$;

(2) $M(a^k) \leq A \cdot 2^k$, $\forall k \in \mathbb{Z}$ *for some constant* $A > 0$, *and* $\sup_{k\in\mathbb{Z}} 2^k \tilde{F}^{-1}(P(\tau_k < \infty)) < \infty$.

Moreover the following equivalence of norms holds:

$$\|f\|_{\mathcal{D}_{F,\infty}} \sim \inf \sup_{k\in\mathbb{Z}} 2^k \tilde{F}^{-1}\bigl(P(\tau_k < \infty)\bigr), \tag{2.11}$$

where the infimum is taken over all the preceding decompositions of f.

We sketch the proofs of Theorem 2.2 and Theorem 2.3 and omit the details since they are similar to that of Theorem 2.1. Let $\tau_k = \inf\{n \in \mathbb{N} : \lambda_n > 2^k\}$ in these cases where $(\lambda_n)_{n\geq 0}$ is the sequence in the definitions of $\mathcal{Q}_{F,\infty}$ and $\mathcal{D}_{F,\infty}$, respectively. Let a^k be defined as in the proof of Theorem 2.1. Equation (1) and the analogs of (2.5) can be proved in the same way as in Theorem 2.1. For the converse parts of the proof of Theorem 2.2 assume that $f = (f_n)_{n\geq 0}$ has a decomposition of the form (1) and let $\lambda_n = \sum_{k\in\mathbb{Z}} \chi_{\{\tau_k \leq n\}} \|S(a^k)\|_\infty$. Then $(\lambda_n)_{n\geq 0}$ is a non-negative, non-decreasing and adapted sequence with $S_{n+1}(f) \leq \lambda_n$. For any fixed $y > 0$ choose $j \in \mathbb{Z}$ such that $2^j \leq y < 2^{j+1}$, then $\lambda_\infty = \lambda_\infty^{(1)} + \lambda_\infty^{(2)}$ with

$$\lambda_\infty^{(1)} = \sum_{k=-\infty}^{j-1} \chi_{\{\tau_k<\infty\}} \bigl\|S(a^k)\bigr\|_\infty, \qquad \lambda_\infty^{(2)} = \sum_{k=j}^{\infty} \chi_{\{\tau_k<\infty\}} \bigl\|S(a^k)\bigr\|_\infty.$$

Similarly to the argument of (2.8) (replacing $\sigma(g)$ and $\sigma(h)$ by $\lambda_\infty^{(1)}$ and $\lambda_\infty^{(2)}$, respectively) we have

$$\begin{aligned}\tilde{F}^{-1}\bigl(P(\lambda_\infty > 2Ay)\bigr) &\leq \tilde{F}^{-1}\bigl(P\bigl(\lambda_\infty^{(1)} > Ay\bigr)\bigr) \leq \tilde{F}^{-1}\bigl(P\bigl(\lambda_\infty^{(2)} > 0\bigr)\bigr) \\ &\leq \sum_{k=j}^{\infty} c\tilde{F}^{-1}\bigl(P(\tau_k < \infty)\bigr) \leq \sum_{k=j}^{\infty} cM2^{-k} \leq cM2^{-j-1} \leq cMy^{-1}.\end{aligned} \tag{2.12}$$

It follows that $\|f\|_{\mathcal{Q}_{F,\infty}} \leq c \sup_{k\in\mathbb{Z}} 2^k \tilde{F}^{-1}(P(\tau_k < \infty))$, which shows that $f \in \mathcal{Q}_{F,\infty}$ and (2.10) holds. As for the converse parts of the proof of Theorem 2.3 we let

$$\lambda_n = \sum_{k\in\mathbb{Z}} \chi_{\{\tau_k\leq n\}} \bigl\|M\bigl(a^k\bigr)\bigr\|_\infty.$$

3 Sublinear operators on weak Orlicz-Lorentz martingale spaces

As one of the applications of the atomic decompositions, we shall obtain a sufficient condition for a sublinear operator to be bounded from weak Orlicz-Lorentz martingale spaces to weak Orlicz-Lorentz function spaces.

An operator $T : X \to Y$ is called a sublinear operator if it satisfies $|T(f+g)| \leq |Tf| + |Tg|$, $|T(\alpha f)| \leq |\alpha||Tf|$, where X is a martingale space, Y is a measurable function space. In this paper, we will add some restrictions to the function F.

Definition 3.1 A strict concave function F is said to obey the $\triangle$-condition written often as $F \in \triangle$, if there exists a positive constant b such that $F(xy) \leq bF(x)F(y)$ for arbitrary $x, y \in R^+$; and it obeys the ∇-condition denoted symbolically as $f \in \nabla$, if there exists a positive constant B such that $F(x)F(y) \leq F(Bxy)$ for arbitrary $x, y \in R^+$, where $B \geq 1$ (see [17]).

Here we should notice that:

(1) Any strict concave function $F \in \triangle_2$ since $F(2x) \leq 2F(x)$, $\forall x > 0$;

(2) Not only the power function $F \in \nabla$, for example $F(x) = x/\ln(1+e^{x+1})$.

Proof By the definition of $F(x)$, we have $F'(x) = \frac{\ln(1+e^{x+1}) - x\frac{e^{x+1}}{1+e^{x+1}}}{(\ln(1+e^{x+1}))^2}$. Thus we have

$$0 < \sup_{x>0} \frac{xF'(x)}{F(x)} = \sup_{x>0} 1 - \frac{x}{\ln(1+e^{x+1})} \frac{e^{x+1}}{1+e^{x+1}} < 1,$$

which means $F(x)$ is a strict concave function. Now we will prove that $F(xy) \geq F(x)F(y)$, $\forall x, y > 0$. Since

$$1 + e^{xy+1} \leq \left(1 + e^{x+1}\right)^{y+1} < \left(1 + e^{x+1}\right)^{\ln(1+e^{y+1})},$$

we have $\ln(1+e^{xy+1}) < \ln(1+e^{x+1})\ln(1+e^{y+1})$. Then $F(xy) \geq F(x)F(y)$, $\forall x, y > 0$. Thus we complete the proof of (2). □

Theorem 3.2 *Let concave function $F^{-1} \in \triangle \cap \nabla$ and $T : L_2(X) \to L_2(Y)$ be a bounded sublinear operator. If*

$$P\left(|Ta| > 0\right) \leq cP(\tau < \infty) \tag{3.1}$$

for all w-1-atom a, where τ is the stopping time associated with a, then

$$\|Tf\|_{L_{F,\infty}} \leq c\|f\|_{H^{\sigma}_{F,\infty}}.$$

Proof By Theorem 2.1, f can be decomposed into the sum of a sequence of w-1-atoms and $\sigma(a^k) \leq A2^k$, $\forall k \in \mathbb{Z}$ for some constant A. For any fixed $y > 0$ choose $j \in \mathbb{Z}$ such that $2^{j-1} \leq y < 2^j$ and let

$$f_n = \sum_{k\in\mathbb{Z}} E_n a^k = \sum_{k=-\infty}^{j-1} E_n a^k + \sum_{k=j}^{\infty} E_n a^k := g_n + h_n.$$

Recall that $\sigma(a^k) = 0$ on the set $\{\tau_k = \infty\}$, we have

$$\begin{aligned}
\|g\|_2 &\le \sum_{k=-\infty}^{j-1} \|a^k\|_2 \le c \sum_{k=-\infty}^{j-1} \|\sigma(a)\|_2 \\
&= c \sum_{k=-\infty}^{j-1} \left(\int_{\{\tau_k<\infty\}} \sigma(a^k)^2 \, dP \right)^{1/2} \\
&\le c \sum_{k=-\infty}^{j-1} \|\sigma(a^k)\|_\infty P(\tau_k < \infty)^{1/2} \\
&\le c \sum_{k=-\infty}^{j-1} 2^k P(\tau_k < \infty)^{1/2}.
\end{aligned}$$

Since $F^{-1} \in \triangle \cap \bigtriangledown$, we have for any $x, y > 0$

$$\tilde{F}^{-1}(xy) = \tilde{F^{-1}}(xy) = \frac{1}{F^{-1}(\frac{1}{xy})} \ge \frac{1}{bF^{-1}(\frac{1}{x})F^{-1}(\frac{1}{y})} \ge \frac{1}{b}\tilde{F}^{-1}(x)\tilde{F}^{-1}(y), \tag{3.2}$$

$$\tilde{F}^{-1}(xy) = \tilde{F^{-1}}(xy) = \frac{1}{F^{-1}(\frac{1}{xy})} \le \frac{1}{F^{-1}(\frac{1}{Bx})F^{-1}(\frac{1}{y})} \le \tilde{F}^{-1}(Bx)\tilde{F}^{-1}(y) \le B\tilde{F}^{-1}(x)\tilde{F}^{-1}(y). \tag{3.3}$$

It follows from the boundedness of T and that

$$\begin{aligned}
\tilde{F}^{-1}\big(P(|Tg| > y)\big) &\le \tilde{F}^{-1}\big(y^{-2}\|Tg\|_2^2\big) \le \tilde{F}^{-1}\big(y^{-2}c^2\|g\|_2^2\big) \\
&\le cB\big(\tilde{F}^{-1}\big(y^{-1}\|g\|_2\big)\big)^2 \le cB\left(\tilde{F}^{-1}\left(y^{-1} \sum_{k=\infty}^{j-1} 2^k P(\tau_k < \infty)^{1/2} \right)\right)^2 \\
&\le cB\left(\sum_{k=\infty}^{j-1} y^{-1} 2^k \tilde{F}^{-1}\big(P(\tau_k < \infty)^{1/2}\big) \right)^2 \\
&\le cB\left(\sum_{k=\infty}^{j-1} y^{-1} 2^{k/2} 2^{k/2} \tilde{F}^{-1}\big(P(\tau_k < \infty)^{1/2}\big) \right)^2 \\
&\le cB\big(y^{-1/2}\|f\|_{H^\sigma_{F,\infty}}^{1/2}\big)^2 \\
&\le cy^{-1}\|f\|_{H^\sigma_{F,\infty}},
\end{aligned}$$

which implies

$$\|Tg\|_{L_{F,\infty}} \le c\|f\|_{H^\sigma_{F,\infty}}. \tag{3.4}$$

On the other hand, by the assumption (3.1) we have

$$\begin{aligned}
y\tilde{F}^{-1}\big(P(|Th| > y)\big) &\le y\tilde{F}^{-1}\big(P(|Th| > 0)\big) \\
&\le y\tilde{F}^{-1}\left(\sum_{k=j}^{\infty} P\big(|Ta^k| > 0\big) \right) \le cy \sum_{k=j}^{\infty} 2^{-k} 2^k \tilde{F}^{-1}\big(P(\tau < \infty)\big) \\
&\le cy 2^{-j}\|f\|_{H^\sigma_{F,\infty}} \le c\|f\|_{H^\sigma_{F,\infty}},
\end{aligned}$$

which implies

$$\|Th\|_{L_{F,\infty}} \le c\|f\|_{H^{\sigma}_{F,\infty}}. \tag{3.5}$$

By (3.4) and (3.5),

$$\|Tf\|_{L_{F,\infty}} \le c\bigl(\|Tg\|_{L_{F,\infty}} + \|Th\|_{L_{F,\infty}}\bigr) \le c\|f\|_{H^{\sigma}_{F,\infty}}.$$

Thus we complete the proof. □

Similarly to the proof of Theorem 3.2, we can prove the following two theorems. In the proof we need Theorem 2.2 and Theorem 2.3 instead of Theorem 2.1, respectively. Here we only give the two theorems and omit the proofs

Theorem 3.3 *Let the concave function $F^{-1} \in \triangle \cap \triangledown$ and $T : L_2(X) \to L_2(Y)$ be a bounded sublinear operator. If*

$$P\bigl(|Ta| > 0\bigr) \le cP(\tau < \infty) \tag{3.6}$$

for all w-2-atom a, where τ is the stopping time associated with a, then

$$\|Tf\|_{L_{F,\infty}} \le c\|f\|_{\mathcal{Q}_{F,\infty}}.$$

Theorem 3.4 *Let concave function $F^{-1} \in \triangle \cap \triangledown$ and $T : L_2(X) \to L_2(Y)$ be a bounded sublinear operator. If*

$$P\bigl(|Ta| > 0\bigr) \le cP(\tau < \infty) \tag{3.7}$$

for all w-3-atom a, where τ is the stopping time associated with a, then

$$\|Tf\|_{L_{F,\infty}} \le c\|f\|_{\mathcal{D}_{F,\infty}}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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