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Some inequalities and applications on Borel direction and exceptional values of meromorphic functions

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Abstract

In view of the Nevanlinna theory in the angular domain, we study the exceptional values of meromorphic functions in the Borel direction and also establish some inequalities on the exceptional values of meromorphic functions in the Borel direction. Based on these inequalities, we also give two theorems and some corollaries as regards exceptional values of meromorphic functions in the Borel direction.

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1 Introduction and main results

To begin with, we assume that the reader is familiar with the basic results and the standard notations of the Nevanlinna theory of meromorphic functions (see [1–3]). We denote by $\mathbb{C}$ the open complex plane, by $\widehat{\mathbb{C}}$ ($= \mathbb{C} \cup \{\infty\}$) the extended complex plane, and by Ω ($\subset \mathbb{C}$) an angular domain. In addition, the order of the meromorphic function f is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

and the exponent of convergence of distinct a-points of f is defined by

$$\overline{\rho}(a,f) = \limsup_{r\to\infty} \frac{\log^+ \overline{N}(r,a,f)}{\log r}.$$

For f a meromorphic function of order ρ ($0 < \rho < \infty$), we say that a is an exceptional value in the sense of Borel (evB for short) for f for the distinct zeros if $\overline{\rho}(a,f) < \rho$. Thus, by the second fundamental theorem in the whole complex plane, we know that a meromorphic function f of order ρ ($0 < \rho < \infty$) at most has two evB for the distinct zeros.

It is well known that exceptional values of meromorphic functions are strictly relative with singular directions. For instance, Picard exceptional value relating with Julia direction and Borel exceptional value relating with Borel direction, and so on (see [4–8]). Moreover, the characteristics of meromorphic functions in the angular domain played an important role in studying on singular directions and exceptional values of meromorphic functions (see [9–12]). Now, we firstly introduce the characteristics of meromorphic functions in the angular domain as follows [13, 14].

Let f be a meromorphic function on the angular domain $\Omega(\alpha,\beta) = \{z : \alpha \leq \arg z \leq \beta\}$ and $0 < \beta - \alpha \leq 2\pi$. Define

$$A_{\alpha,\beta}(r,f) = \frac{\omega}{\pi}\int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\right)\left\{\log^+\left|f\left(te^{i\alpha}\right)\right| + \log^+\left|f\left(te^{i\beta}\right)\right|\right\}\frac{dt}{t},$$

$$B_{\alpha,\beta}(r,f) = \frac{2\omega}{\pi r^\omega}\int_\alpha^\beta \log^+\left|f\left(re^{i\theta}\right)\right| \sin\omega(\theta-\alpha)\,d\theta,$$

$$C_{\alpha,\beta}(r,f) = 2\sum_{1<|b_\mu|<r}\left(\frac{1}{|b_\mu|^\omega} - \frac{|b_\mu|^\omega}{r^{2\omega}}\right)\sin\omega(\theta_\mu - \alpha),$$

$$S_{\alpha,\beta}(r,f) = A_{\alpha,\beta}(r,f) + B_{\alpha,\beta}(r,f) + C_{\alpha,\beta}(r,f),$$

where $\omega = \frac{\pi}{\beta-\alpha}$ and $b_\mu = |b_\mu|e^{i\theta_\mu}$ ($\mu = 1,2,\ldots$) are the poles of f on $\Omega(\alpha,\beta)$ counted according to their multiplicities. $S_{\alpha,\beta}(r,f)$ is called Nevanlinna's angular characteristic, and $C_{\alpha,\beta}(r,f)$ is called the angular counting function of the poles of f on $\Omega(\alpha,\beta)$, and $\overline{C}_{\alpha,\beta}(r,f)$ is the reduced function of $C_{\alpha,\beta}(r,f)$. Similarly, the order of the meromorphic function f on $\Omega(\alpha,\beta)$ is defined by

$$\rho_{\alpha,\beta}(f) = \limsup_{r\to\infty}\frac{\log S_{\alpha,\beta}(r,f)}{\log r},$$

and the exponent of convergence of distinct a-points of f on $\Omega(\alpha,\beta)$ is defined by

$$\overline{\rho}_{\alpha,\beta}(a,f) = \limsup_{r\to\infty}\frac{\log^+\overline{C}_{\alpha,\beta}(r,a,f)}{\log r}.$$

For f is a meromorphic function of order $\rho_{\alpha,\beta}(f)$ ($0 < \rho_{\alpha,\beta}(f) < \infty$), then we say that a is an exceptional value on the angular domain in the sense of Borel (evaB for short) for f for the distinct zeros if $\overline{\rho}_{\alpha,\beta}(a,f) < \rho_{\alpha,\beta}(f)$.

An interesting subject arises naturally: *Does a meromorphic function f with order $\rho_{\alpha,\beta}$ ($0 < \rho_{\alpha,\beta} < \infty$) on $\Omega(\alpha,\beta)$ at most have two evaB for the distinct zeros*? By Lemma 2.2, Lemma 2.3, and Remark 2.1, we can give a negative answer to this question since $Q_{\alpha,\beta}(r,f) = O\{\log(rS_{\alpha,\beta}(r,f))\}$ is not valid, as $r \to \infty$ ($r \notin E$) and E is the set with finite linear measure. *Thus, it is an interesting topic in studying the exceptional value of meromorphic functions on the angular domain.*

The main purpose of this paper is to investigate the exceptional values of the meromorphic function with infinite order in its Borel direction. Valiron [15] proved that every meromorphic function of finite order $\rho > 0$ has at least one Borel direction of order ρ. Chuang [16, 17] investigated the existence of Borel directions of the meromorphic function of infinite order. Before stating Chuang's results, we will introduce the definition as follows.

Definition 1.1 [16] Let f be a meromorphic function of infinite order, $\rho(r)$ be a real function satisfying the following conditions:

(i) $\rho(r)$ is continuous, non-decreasing for $r \geq r_0$ and $\rho(r) \to \infty$ as $r \to \infty$;

(ii)

$$\lim_{r\to\infty} \frac{\log U(R)}{\log U(r)} = 1, \quad R = r + \frac{r}{\log U(r)},$$

where $U(r) = r^{\rho(r)}$ $(r \geq r_0)$;

(iii)

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{\log U(r)} = 1.$$

Then $\rho(r)$ is said to be of infinite order for the meromorphic function f. This definition was given by Xiong (see [16]).

We will give the definition of the Borel direction of the meromorphic functions f of infinite order $\rho(r)$ as follows.

Definition 1.2 [16] Let f be a meromorphic function of infinite order $\rho(r)$. If for any ε $(0 < \varepsilon < \pi)$, the equality

$$\limsup_{r\to\infty} \frac{\log n(\Omega(\theta-\varepsilon,\theta+\varepsilon,r), f = a)}{\rho(r)\log r} = 1$$

holds for any complex number $a \in \widehat{\mathbb{C}}$, at most except two exception, where $n(\Omega(\theta-\varepsilon,\theta+\varepsilon,r), f = a)$ is the counting function of zero of the function $f - a$ in the angular domain $\Omega(\theta-\varepsilon,\theta+\varepsilon)$, counting multiplicities. Then the ray $\arg z = \theta$ is called a Borel direction of $\rho(r)$ order of the meromorphic function f.

Remark 1.1 Chuang [16] proved that every meromorphic function f with infinite order $\rho(r)$ has as least one Borel direction of infinite order $\rho(r)$.

Now, the main theorem of this paper is listed as follows.

Theorem 1.1 *Let f be a transcendental meromorphic function of infinite order $\rho(r)$ on the whole complex plane, $\arg z = \theta$ $(0 \leq \theta < 2\pi)$ be one Borel direction of $\rho(r)$ order of the function f and $\Omega := \Omega(\theta-\varepsilon,\theta+\varepsilon)$ for any ε $(0 < \varepsilon < \pi)$. If there exist $a_1^1,\ldots, a_{p_1}^1, a_1^2,\ldots,a_{p_2}^2,\ldots,a_1^s,\ldots,a_{p_s}^s \in \widehat{\mathbb{C}}$ such that $a_1^i, a_2^i,\ldots,a_{p_i}^i$ are evBB for f for distinct zeros of multiplicity $\leq k_i$, $i = 1,2,\ldots,s$, where $s,p_1,\ldots,p_s \in \mathbb{N}_+$ and $k_1,k_2,\ldots,k_s$ are positive integers or infinity, then*

$$\Xi := \sum_{i=1}^{s} \frac{p_i k_i}{1+k_i} + \sum_{i=1}^{s}\left(\frac{1}{1+k_i}\sum_{j=1}^{p_i}\delta_{k_i}\left(a_j^i,f\right)\right) \leq 2. \tag{1}$$

Definition 1.3 Let $\arg z = \theta$ $(0 \leq \theta < 2\pi)$ be one Borel direction of $\rho(r)$ order of function f and k be a positive integer, we say that a is

(i) an exceptional value in the sense of Borel for f in the Borel direction (evBB for short) for distinct zeros of multiplicity $\leq k$, if $\overline{\rho}_{\theta}^{k}(a,f) < 1$;

(ii) an exceptional value in the sense of Borel for f in the Borel direction (evBB for short) for distinct zeros, if $\overline{\rho}_\theta(a,f) < 1$; where

$$\overline{\rho}_\theta^k(a,f) = \limsup_{r\to\infty} \frac{\log^+ \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f|\le k)}{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)},$$
$$\overline{\rho}_\theta(a,f) = \limsup_{r\to\infty} \frac{\log^+ \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f)}{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)},$$

and $\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f|\le k)$ is the counting function of distinct a-points of f on Ω whose multiplicities do not exceed k.

In particular, we say that a is an evBB for f for simple zeros if $k = 1$, a is an evBB for f for simple and double zeros if $k = 2$.

Definition 1.4 For positive integers k, μ, we define

$$\delta_k^\theta(a,f) = 1 - \limsup_{r\to\infty} \frac{C_{\theta-\varepsilon,\theta+\varepsilon}^k(r,a,f)}{S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)},$$
$$\Theta^\theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f)}{S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)},$$

where $\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}^k(r,a,f)$ the counting function of a-points of f on Ω where an a-point of multiplicity μ is counted μ times if $\mu \le k$ and $1 + k$ times if $\mu > k$. In particular, if $k = \infty$, we denote

$$\delta^\theta(a,f) = \delta_\infty^\theta(a,f) = 1 - \limsup_{r\to\infty} \frac{C_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f)}{S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}.$$

Theorem 1.2 *Let f be a transcendental meromorphic function of infinite order $\rho(r)$ on the whole complex plane, $\arg z = \theta$ $(0 \le \theta < 2\pi)$ be one Borel direction of $\rho(r)$ order of the function f and $\Omega := \Omega(\theta - \varepsilon, \theta + \varepsilon)$ for any ε $(0 < \varepsilon < \pi)$. If there exist $a \in \widehat{\mathbb{C}}$ and two positive integers k and p such that*

$$(1 + k)\Theta^\theta(a,f) + \sum_{b\neq a} \delta^\theta(b,f) > 2 - k(p - 1),$$

then there exist at most p elements $\widehat{\mathbb{C}}\backslash\{a\}$ which are evBB for f for distinct zeros of multiplicity not exceeding k.

2 Some lemmas

To prove our results, we need the following lemmas.

Lemma 2.1 (see [18, 19]) *Let f be a non-constant meromorphic function on $\Omega(\alpha, \beta)$. Then for arbitrary complex number a, we have*

$$S_{\alpha,\beta}\left(r, \frac{1}{f - a}\right) = S_{\alpha,\beta}(r,f) + \varepsilon(r,a),$$

where $\varepsilon(r,a) = O(1)$ as $r \to \infty$.

Lemma 2.2 (see [13, 14]) *Suppose that f is a non-constant meromorphic function in one angular domain $\Omega(\alpha,\beta)$ with $0<\beta-\alpha\le 2\pi$, then for arbitrary q distinct $a_j\in\widehat{\mathbb{C}}$ $(1\le j\le q)$, we have*

$$(q-2)S_{\alpha,\beta}(r,f)\le\sum_{j=1}^{q}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)+Q_{\alpha,\beta}(r,f),$$

where the term $\overline{C}_{\alpha,\beta}(r,\frac{1}{f-a_j})$ will be replaced by $\overline{C}_{\alpha,\beta}(r,f)$ when some $a_j=\infty$ and

$$\begin{aligned}Q_{\alpha,\beta}(r,f)&=A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)\\&\quad+\sum_{j=1}^{q}\left\{A_{\alpha,\beta}\left(r,\frac{f'}{f-a_j}\right)+B_{\alpha,\beta}\left(r,\frac{f'}{f-a_j}\right)\right\}+O(1).\end{aligned}\tag{2}$$

Lemma 2.3 (see [18, p.138]) *Let f be a non-constant meromorphic function in the whole complex plane $\mathbb{C}$. Let one angular domain be given on $\Omega(\alpha,\beta)$. Then for any $1\le r<R$, we have*

$$A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)\le K\left\{\left(\frac{R}{r}\right)^{\omega}\int_1^R\frac{\log^+T(r,f)}{t^{1+\omega}}\,dt+\log^+\frac{r}{R-r}+\log\frac{R}{r}+1\right\}$$

and

$$B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)\le\frac{4\omega}{r^{\omega}}m\left(r,\frac{f'}{f}\right),$$

where $\omega=\frac{\pi}{\beta-\alpha}$ and K is a positive constant not depending on r and R.

Remark 2.1 Nevanlinna conjectured that

$$D_{\alpha,\beta}\left(r,\frac{f'}{f}\right)=A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)=o\big(S_{\alpha,\beta}(r,f)\big)\tag{3}$$

when r tends to $+\infty$ outside an exceptional set of finite linear measure, and he proved that $A_{\alpha,\beta}(r,\frac{f'}{f})+B_{\alpha,\beta}(r,\frac{f'}{f})=O(1)$ when the function f is meromorphic in $\mathbb{C}$ and has finite order. In 1974, Gol'dberg [13] constructed a counter-example to show that (3) is not valid.

Lemma 2.4 (see [20, Lemma 4]) *Let f be a meromorphic function in $\mathbb{C}$, $\Omega(\alpha,\beta)$ $(0<\beta-\alpha\le 2\pi)$ be a closed angular domain, then*

$$Q_{\alpha,\beta}(r,f)=\begin{cases}O(1), & f \text{ is of finite order,}\\ O(\log U(r)), & f \text{ is of infinite order,}\end{cases}$$

where $Q_{\alpha,\beta}(r,f)$ is stated as in (2), $U(r)=r^{\rho(r)}$, $\rho(r)$ is the precise order of $T(r,f)$ when f is of infinite order, E is a set of finite linear measure.

Lemma 2.5 (see [20, Lemma 5]) *Let f be a meromorphic function on a closed angular domain $\Omega(\alpha,\beta)$ and $\omega = \frac{\pi}{\beta-\alpha}$, then for any $a \in \widehat{\mathbb{C}}$ and for any $\varepsilon \in (0, \frac{\beta-\alpha}{2})$,*

$$C_{\alpha,\beta}(r,a,f) \geq 2\omega\sin(\omega\varepsilon)\int_1^r \frac{n(t,\Omega_\varepsilon,f=a)}{t^{\omega+1}}\,dt + O(1),$$

$$C_{\alpha,\beta}(r,a,f) \geq \frac{4\omega\sin(\omega\varepsilon)}{r^\omega}N(r,\Omega_\varepsilon,f=a) + o(1),$$

$$C_{\alpha,\beta}(r,a,f) \leq 4\omega\int_1^r \frac{n(t,\Omega,f=a)}{t^{\omega+1}}\,dt,$$

$$C_{\alpha,\beta}(r,a,f) \leq 2n(r,\Omega,f=a),$$

where $\Omega_\varepsilon = (\alpha+\varepsilon,\beta-\varepsilon)$.

Remark 2.2 For the reduced case, that is, each multiple zero of $f-a$ in $\Omega(\alpha,\beta)$ is counted only once (ignoring multiplicities), Lemma 2.5 still holds, and its proof is similar to the case of counting multiplicities.

Lemma 2.6 (see [17]) *Let f be a meromorphic function of infinite order $\rho(r)$. Then the ray $\arg z = \theta$ is one Borel direction of $\rho(r)$ order of the meromorphic function f if and only if f satisfies the equality*

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}{\rho(r)\log r} = 1$$

for any ε $(0<\varepsilon<\frac{\pi}{2})$.

Lemma 2.7 *Let f be a transcendental meromorphic function of infinite order $\rho(r)$ on the whole complex plane, $\arg z = \theta$ $(0 \leq \theta < 2\pi)$ be one Borel direction of $\rho(r)$ order of the function f and $\Omega := \Omega(\theta-\varepsilon,\theta+\varepsilon)$ for any ε $(0<\varepsilon<\pi)$. Then*

$$\limsup_{r\to\infty}\frac{C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f')}{S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)} \leq 2-\Theta^\theta(\infty,f)-\sum_{b\in\mathbb{C}}\delta^\theta(b,f).$$

Proof Suppose that $b_1,b_2,\ldots,b_t \in \mathbb{C}$ are t distinct complex constants. Since $\arg z = \theta$ $(0 \leq \theta < 2\pi)$ is one Borel direction of $\rho(r)$ order of the function f, then we have

$$\begin{aligned}\sum_{i=1}^t D_{\theta-\varepsilon,\theta+\varepsilon}(r,b_i,f) &\leq D_{\theta-\varepsilon,\theta+\varepsilon}\bigl(r,0,f'\bigr) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\\ &\leq S_{\theta-\varepsilon,\theta+\varepsilon}\bigl(r,f'\bigr) - C_{\theta-\varepsilon,\theta+\varepsilon}\bigl(r,0,f'\bigr) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f),\end{aligned}$$

where $D_{\theta-\varepsilon,\theta+\varepsilon}(r,b_i,f) := A_{\theta-\varepsilon,\theta+\varepsilon}(r,b_i,f) + B_{\theta-\varepsilon,\theta+\varepsilon}(r,b_i,f)$. From $S_{\theta-\varepsilon,\theta+\varepsilon}(r,f') \leq S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f)$, then we have

$$\sum_{i=1}^t D_{\theta-\varepsilon,\theta+\varepsilon}(r,b_i,f) + C_{\theta-\varepsilon,\theta+\varepsilon}\bigl(r,0,f'\bigr) \leq S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f),$$

it follows by Lemmas 2.4-2.6 that

$$\sum_{i=1}^{t}\delta^{\theta}(b_i,f)+\limsup_{r\to\infty}\frac{C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f')}{S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}\leq 2-\Theta^{\theta}(\infty,f). \tag{4}$$

Since t is arbitrary, from (4) we can easily complete the proof of Lemma 2.7. □

3 Proof of Theorem 1.1

Proof Since f is a meromorphic function of infinite order $\rho(r)$ and $\arg z=\theta$ $(0\leq\theta<2\pi)$ is one Borel direction of $\rho(r)$ order of the meromorphic function f, by Lemma 2.6, we can get for any ε $(0<\varepsilon<\pi)$

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}{\rho(r)\log r}=1. \tag{5}$$

For any positive integer k or ∞ and $a\in\widehat{\mathbb{C}}$, we have

$$\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f)\leq\frac{k}{1+k}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f|\leq k)+\frac{1}{1+k}C^{k}_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f), \tag{6}$$

where $\frac{k}{k+1}=1$ and $\frac{1}{k+1}=0$ if $k=\infty$. Then, from (6) and Lemma 2.2, we have

$$\begin{aligned}\left(\sum_{i=1}^{s}p_i-2\right)S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\leq{}&\sum_{i=1}^{s}\frac{k_i}{1+k_i}\sum_{j=1}^{p_i}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}\bigl(r,a^i_j,f|\leq k_i\bigr)\\&+\sum_{i=1}^{s}\frac{1}{1+k_i}\sum_{j=1}^{p_i}C^{k_i}_{\theta-\varepsilon,\theta+\varepsilon}\bigl(r,a^i_j,f\bigr)+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f).\end{aligned} \tag{7}$$

From (5), Lemma 2.5 and the assumptions of Theorem 1.1, there exists a constant η $(0<\eta<1)$ such that for sufficiently large r,

$$\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}\bigl(r,a^i_j,f|\leq k_i\bigr)<\bigl(U(r)\bigr)^{\eta},\quad j=1,2,\dots,p_i;i=1,2,\dots,s. \tag{8}$$

Hence, from (5) and for sufficiently large r, we have

$$\begin{aligned}\left(\sum_{i=1}^{s}p_i-2\right)S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\leq{}&\sum_{i=1}^{s}\frac{1}{1+k_i}\sum_{j=1}^{p_i}C^{k_i}_{\theta-\varepsilon,\theta+\varepsilon}\bigl(r,a^i_j,f\bigr)\\&+O\bigl(\bigl(U(r)\bigr)^{\eta}\bigr)+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f).\end{aligned} \tag{9}$$

Thus, for sufficiently large r and arbitrary ε (>0), we can get from (8) and the definition of $\delta^{\theta}_{k}(a,f)$

$$\begin{aligned}\left(\sum_{i=1}^{s}p_i-2\right)S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\leq{}&\sum_{i=1}^{s}\frac{1}{1+k_i}\sum_{j=1}^{p_i}\bigl(1-\delta^{\theta}_{k_i}\bigl(a^i_j,f\bigr)+\varepsilon\bigr)S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\\&+O\bigl(\bigl(U(r)\bigr)^{\eta}\bigr)+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f),\end{aligned}$$

that is,

$$(\Xi - 2 - \varepsilon)S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \leq O\big(\big(U(r)\big)^{\eta}\big) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \tag{10}$$

If $\Xi > 2$, we can choose an arbitrary ε (> 0) satisfying $\eta + \varepsilon < 1$ and $\Xi - 2 - \varepsilon > 0$. Thus, from (5), (10) and for sufficiently large r, we easily get a contradiction.

Therefore, we get the conclusion of Theorem 1.1. □

4 Proof of Theorem 1.2

Proof Without loss of generality, we assume that $a = \infty$. Next, we use reduction to absurdity to prove the conclusion of Theorem 1.2. Suppose that there exist $p + 1$ elements $a_1, a_2, \ldots, a_{p+1} \in \mathbb{C}$ which are evBB for f for distinct zeros of multiplicity $\leq k$. Since $\arg z = \theta$ $(0 \leq \theta < 2\pi)$ is one Borel direction of $\rho(r)$ order of the meromorphic function f and $\Omega := \Omega(\theta - \varepsilon, \theta + \varepsilon)$ for any ε $(0 < \varepsilon < \pi)$, and if z_0 is a zero of $f - b$ on Ω of multiplicity d (> 1) for $b \in \mathbb{C}$, then z_0 is a zero of f' on Ω of multiplicity $d - 1$, and it follows that

$$\sum_{i=1}^{p+1} \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, a_i, f) \leq \sum_{i=1}^{p+1} \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, a_i, f| \leq k) + \frac{1}{k} C_{\theta-\varepsilon,\theta+\varepsilon}\big(r, 0, f'\big).$$

Therefore, by Lemma 2.2 we have

$$\begin{aligned} pS_{\theta-\varepsilon,\theta+\varepsilon}(r,f) &\leq \sum_{i=1}^{p+1} \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, a_i, f) + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \\ &\leq \sum_{i=1}^{p+1} \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, a_i, f| \leq k) + \frac{1}{k} C_{\theta-\varepsilon,\theta+\varepsilon}\big(r, 0, f'\big) \\ &\quad + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \end{aligned} \tag{11}$$

From (5), Lemma 2.5 and the assumptions of Theorem 1.2, there exists a number η $(0 < \eta < 1)$ such that for sufficiently large r,

$$\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r, a_i, f| \leq k) < \big(U(r)\big)^{\eta}, \quad i = 1, 2, \ldots, p + 1. \tag{12}$$

From (11), (12) and Lemma 2.6, for sufficiently large r, it follows that

$$\begin{aligned} &\left(p - 1 - \frac{1}{k}\left(2 - \Theta^{\theta}(\infty,f) - \sum_{b \neq \infty} \delta^{\theta}(b,f)\right) + \Theta^{\theta}(\infty,f)\right) S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \\ &\quad \leq O\big(\big(U(r)\big)^{\eta}\big) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \end{aligned} \tag{13}$$

Since $\eta < 1$, from (5) and (13) for sufficiently large r, we can get

$$(1 + k)\Theta^{\theta}(\infty,f) + \sum_{b \neq \infty} \delta^{\theta}(b,f) > 2 - k(p - 1),$$

which is a contradiction with the assumption of Theorem 1.2.

Thus, this completes the proof of Theorem 1.2. □

5 Some consequences of Theorems 1.1 and 1.2

In this section, we will give some consequences of Theorem 1.1. Before giving these results, some definitions will be introduced below.

Definition 5.1 Let $\arg z = \theta$ $(0 \leq \theta < 2\pi)$ be one Borel direction of the function f and we have any ε $(0 < \varepsilon < \pi)$, for $a \in \widehat{\mathbb{C}}$. Then

(i) a is called an exceptional value in the sense of Nevanlinna in the Borel direction (evNB for short), if $\delta^{\theta}(a,f) > 0$;
(ii) a is called a normal value in the sense of Nevanlinna in the Borel direction (nvNB for short), if $\delta^{\theta}(a,f) = 0$.

In addition, similar to the Picard exceptional value in the whole complex plane, by definition a is called an exceptional value in the sense of Picard in the Borel direction of f (evPB for short), if f has at most a finite number of a-points in the Borel direction.

Consequence 5.1 Under the assumptions of Theorem 1.1, if $k_1 = 1$, from Theorem 1.1, we get

$$p_1 + 2\sum_{i=2}^{s} \frac{p_i k_i}{1+k_i} + \sum_{j=1}^{p_1} \delta_1^{\theta}\bigl(a_j^1,f\bigr) + \sum_{i=2}^{s} \frac{2}{1+k_i} \sum_{j=1}^{p_i} \delta_1^{\theta}\bigl(a_j^i,f\bigr) \leq 4.$$

Since $p_i \geq 0$ and $\delta_1^{\theta}(a_j^i,f) \geq 0$ for $i = 1, \ldots, s$, it follows that

(i) if f has an evBB for simple zeros which is also an evNB for f, then f has at most three evBB for simple zeros;
(ii) if a_1, a_2 are two evPB for f then no other element is an evBB for f for simple zeros;
(iii) there exist at most four elements which are evBB for f simple zeros since $\delta_1(a_j^1,f) \geq 0$, moreover, all these four values are nvNB for f.

Consequence 5.2 Under the assumptions of Theorem 1.1, if $k_1 = 1$, $k_2 = 2$, $p_1 = 1$, we have

$$\frac{2p_2}{3} + \sum_{i=3}^{s} \frac{p_i k_i}{1+k_i} + \frac{1}{2}\delta_1^{\theta}\bigl(a_1^1,f\bigr) + \frac{1}{3}\sum_{j=1}^{p_2} \delta_2^{\theta}\bigl(a_j^2,f\bigr) + \sum_{i=3}^{s} \frac{1}{1+k_i} \sum_{j=1}^{p_i} \delta_{k_i}^{\theta}\bigl(a_j^i,f\bigr) \leq \frac{3}{2}. \qquad (14)$$

Since $p_i \geq 0$ and $\delta_{k_i}^{\theta}(a_j^i,f) \geq 0$, it follows from (14) that $p_2 < 3$. Thus, if a_1^1 is an evBB for f for simple zeros, that is, $\delta_1^{\theta}(a_1^1,f) > 0$, then there exist at most two other elements which are evBB for f for distinct simple zeros and double zeros. Furthermore,

(i) if $\delta_1^{\theta}(a_1^1,f) = \frac{1}{3}$, then two other elements evBB are also evNB for f;
(ii) if any one of the two other elements a_1^2, a_2^2, say a_1^2, satisfies $\delta_1^{\theta}(a_1^2,f) = \frac{1}{2}$, then a_1^1, a_2^2 are also evNB for f.

Consequence 5.3 Under the assumptions of Theorem 1.1, if $k_1 = 1$, $k_2 = 3$, $p_1 = 1$, then we have

$$\frac{3p_2}{4} + \sum_{i=3}^{s} \frac{p_i k_i}{1+k_i} + \frac{1}{2}\delta_1^{\theta}(a_1,f) + \frac{1}{4}\sum_{j=1}^{p_2} \delta_3^{\theta}\bigl(a_j^2,f\bigr) + \sum_{i=3}^{s} \frac{1}{1+k_i} \sum_{j=1}^{p_i} \delta_{k_i}^{\theta}\bigl(a_j^i,f\bigr) \leq \frac{3}{2}.$$

From the above inequality and $\delta^{\theta}(a,f) \geq 0$ for $a \in \mathbb{C} \cup \{\infty\}$, we see that $p_1 = 1$, $p_2 \leq 2$ and $p_1 = 1$, $p_2 = 2$, $p_i = 0$ $(i = 3, \ldots, s)$. Thus, if f has an evBB for simple zeros, then there exist at most two other elements which are evBB for f distinct zeros of multiplicity ≤ 3, moreover, all these exceptional values are nvNB for f.

Consequence 5.4 Under the assumptions of Theorem 1.1, if $k_1 = 2$, we have

$$\frac{2p_1}{3} + \sum_{i=2}^{s} \frac{p_i k_i}{1 + k_i} + \frac{1}{3} \sum_{j=1}^{p_1} \delta_2^{\theta}\bigl(a_j^1, f\bigr) + \sum_{i=2}^{s} \frac{1}{1 + k_i} \sum_{j=1}^{p_i} \delta_{k_i}^{\theta}\bigl(a_j^i, f\bigr) \leq 2. \tag{15}$$

Since $\sum_{i=2}^{s} \frac{p_i k_i}{1+k_i} + \sum_{i=2}^{s} \frac{1}{1+k_i} \sum_{j=1}^{p_i} \delta_{k_i}^{\theta}(a_j^i, f) \geq 0$, from (15) we have

$$\frac{2p_1}{3} + \frac{1}{3} \sum_{j=1}^{p_1} \delta_2^{\theta}\bigl(a_j^1, f\bigr) \leq 2.$$

Thus, it follows that $p_1 \leq 3$ and $p_1 = 3$, $p_i = 0$, $i = 2, \ldots, s$. Hence, we see that f has at most three evBB for distinct simple and double zeros, moreover, all three evBB for distinct simple and double zeros are nvNB for f.

Consequence 5.5 Under the assumptions of Theorem 1.1, if $k_1 = 2$, $p_1 = 1$, $k_2 = 1$, then we have

$$p_2 + 2 \sum_{i=3}^{s} \frac{p_i k_i}{1 + k_i} + \frac{2}{3} \delta_2^{\theta}\bigl(a_1^1, f\bigr) + \sum_{j=1}^{p_2} \delta_1^{\theta}\bigl(a_j^2, f\bigr) + \sum_{i=3}^{s} \frac{2}{1 + k_i} \sum_{j=1}^{p_i} \delta_{k_i}^{\theta}\bigl(a_j^i, f\bigr) \leq \frac{8}{3}.$$

Thus, it follows that $p_2 \leq 2$. So, if there exists an evBB for f for distinct and double zeros, say a_1^1, then there exist at most two other evBB for f for simple zeros, say a_1^2, a_2^2. Furthermore, if $p_1 = 1$, $p_2 = 2$, it follows that

$$2\delta_2^{\theta}\bigl(a_1^1, f\bigr) + 3\delta_1^{\theta}\bigl(a_1^2, f\bigr) + \delta_1^{\theta}\bigl(a_2^2, f\bigr) \leq 2.$$

Thus, we can see that any one of a_1^2, a_2^2 may not be an evPB for f, furthermore, if a_1^1 is an evPB for f, then a_1^2, a_2^2 are nvNB for f.

Now, some consequences of Theorem 1.2 are listed.

Consequence 5.6 Under the assumptions of Theorem 1.2, if $k = 1$ and

$$2\Theta^{\theta}(a, f) + \sum_{b \neq a} \delta^{\theta}(b, f) > 3 - p,$$

we have

(i) if $p = 1$ and $2\Theta^{\theta}(a, f) + \sum_{b \neq a} \delta^{\theta}(b, f) > 2$, then there exists at most one element $b \neq a$ which is an evBB for f simple zeros; in particular, this holds if there exists an $a \in \widehat{\mathbb{C}}$ satisfying $\Theta^{\theta}(a, f) = 1$;

(ii) if $p = 2$ and $2\Theta^{\theta}(a, f) + \sum_{b \neq a} \delta^{\theta}(b, f) > 1$, then there exist at most two elements $b_1, b_2 \neq a$ which are evBB for f simple zeros; in particular, this holds if there exists an $a \in \widehat{\mathbb{C}}$ satisfying $\Theta^{\theta}(a, f) > \frac{1}{2}$;

(iii) if $p = 3$ and $2\Theta^{\theta}(a,f) + \sum_{b \neq a} \delta^{\theta}(b,f) > 0$, then there exists at most three elements $b_1, b_2, b_3 \neq a$ which are evBB for f simple zeros; in particular, this holds if there exists an $a \in \widehat{\mathbb{C}}$ satisfying $\Theta^{\theta}(a,f) > 0$.

Remark 5.1 Under the assumptions of Theorem 1.2, from Consequence 5.6, we see that if there exist four distinct elements $b_1, b_2, b_3, b_4 \in \widehat{\mathbb{C}}$ which are evBB for f for simple zeros, then $\Theta^{\theta}(b_i,f) \leq \frac{1}{2}$ and $\Theta^{\theta}(a,f) = 0$ for $a \neq b_i$ and $i = 1, 2, 3, 4$.

Consequence 5.7 Under the assumptions of Theorem 1.2, if $k = 2$ and

$$3\Theta^{\theta}(a,f) + \sum_{b \neq a} \delta^{\theta}(b,f) > 4 - 2p,$$

we have

(i) if $p = 1$ and $3\Theta^{\theta}(a,f) + \sum_{b \neq a} \delta^{\theta}(b,f) > 2$, then there exists at most one element $b \neq a$ which is an evBB for f distinct simple and double zeros; in particular, this holds if there exists an $a \in \widehat{\mathbb{C}}$ satisfying $\Theta^{\theta}(a,f) > \frac{2}{3}$;

(ii) if $p = 2$ and $3\Theta^{\theta}(a,f) + \sum_{b \neq a} \delta^{\theta}(b,f) > 0$, then there exist at most two elements $b_1, b_2 \neq a$ which are evBB for f distinct simple and double zeros; in particular, this holds if there exists an $a \in \widehat{\mathbb{C}}$ satisfying $\Theta^{\theta}(a,f) > 0$.

Remark 5.2 Under the assumptions of Theorem 1.2, from Consequence 5.7, we find that if there exist three distinct elements $b_1, b_2, b_3 \in \widehat{\mathbb{C}}$ which are evBB for f for distinct simple and double zeros, then $\Theta^{\theta}(b_i,f) \leq \frac{2}{3}$ and $\Theta^{\theta}(a,f) = 0$ for $a \neq b_i$ and $i = 1, 2, 3$.

6 Remarks

From Theorems 1.1 and 1.2, it is a natural question to ask: could we get the same conclusions of Theorems 1.1 and 1.2 when f is a transcendental meromorphic function with finite order ρ $(0 < \rho < \infty)$ on the whole complex plane? However, we cannot give a positive answer to the above question. Now we give a simple procedure to show that the conclusion of Theorem 1.1 cannot hold when f is a transcendental meromorphic function with finite order ρ $(0 < \rho < \infty)$ on the whole complex plane.

If f is of finite order ρ $(0 < \rho < \infty)$, that is, $\rho(r) = \rho$, then we say a is an exceptional value in the sense of Borel for f in the Borel direction (evBB for short) for distinct zeros of multiplicity $\leq k$, if $\overline{\rho}_{\theta}^{k}(a,f) < \rho$. Thus, by Lemma 2.5 and the definition of the Borel direction, (8) can be replaced by

$$\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}\bigl(r, a_j^i, f \mid \leq k_i\bigr) < r^{\eta'}, \quad j = 1, 2, \ldots, p_i; i = 1, 2, \ldots, s,$$

where $\eta' < \rho$ and r is sufficiently large, and (10) can be replaced by

$$(\Xi - 2 - \varepsilon) S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \leq O\bigl(r^{\eta'}\bigr) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \tag{16}$$

However, by Lemmas 2.1-2.5, we get

$$\rho - \frac{\pi}{2\varepsilon} \leq \eta'' := \limsup_{r \to \infty} \frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}{\log r} \leq \rho.$$

Moreover, from the above inequality, we cannot be sure whether η' is greater than η''. If $\eta'' \leq \eta' < \rho$, then from (16) we cannot easily get a contradiction. Therefore, Theorems 1.1 and 1.2 may not be true when f is of finite order.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HYX, ZJW and JT completed the main part of this article, HYX corrected the main theorems. The authors read and approved the final manuscript.

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