Lou *Journal of Inequalities and Applications* 2014, **2014**:523
http://www.journalofinequalitiesandapplications.com/content/2014/1/523

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Sharp constants for inequalities of Poincaré type: an application of optimal control theory

Hongwei Lou*

*Correspondence:
hwlou@fudan.edu.cn
School of Mathematical Sciences, and LMNS, Fudan University, Shanghai, 200433, China

Abstract
Sharp constants for an inequality of Poincaré type are studied. The problem is solved by using optimal control theory.

MSC: 26D10; 46E35; 49K15

Keywords: sharp constant; inequality of Poincaré type; optimal control

1 Introduction

Denote by $W^{1,2}(-1,1)$ the Sobolev space of all real-valued functions $f(\cdot)$ that are absolutely continuous on the closed interval $[-1,1]$ and such that $f'(\cdot) \in L^2(-1,1)$. Let $m \geq 1$ be an integer. Denote by $W(1,2,m)$ the space

$$W(1;2,m) \triangleq \left\{ y(\cdot) \in W^{1,2}(-1,1) \,\Big|\, \int_{-1}^{1} t^k y(t)\,dt = 0 \ (0 \leq k \leq m-1) \right\}. \tag{1.1}$$

Kalyabin considered in [1] the following problem.

Problem (B_x) Fix $x \in [-1,1]$, find the best constant $B_m(x)$ such that the following inequality holds:

$$|y(x)| \leq B_m(x) \left(\int_{-1}^{1} |y'(t)|^2\,dt \right)^{\frac{1}{2}}, \quad \forall y(\cdot) \in W(1,2,m). \tag{1.2}$$

It is proved in [1] that

$$B_m^2(\pm 1) = \frac{2}{m(m+2)}, \qquad B_1^2(x) = \frac{1+3x^2}{6}, \qquad B_2^2(0) = \frac{1}{6} \tag{1.3}$$

and the extremal functions for the case $x = 1$ are

$$C\big((m+2)P_m(t) + mP_{m+1}(t)\big), \tag{1.4}$$

where C is a constant and

$$P_k(t) = \frac{1}{2^k k!} \frac{d^k}{dt^k} (t^2 - 1)^k, \quad k = 0,1,2,\ldots, \tag{1.5}$$

are the classical Legendre polynomials. For notational simplicity, we denote

$$p_k(t) = \frac{1}{2^k k!}\frac{d^{k-2}}{dt^{k-2}}\left(t^2-1\right)^k, \quad k = 2, 3, \ldots \tag{1.6}$$

and

$$q_k(x) = \frac{\int_{-1}^1 P_k(s)|s-x|\,ds}{\int_{-1}^1 |P_k(s)|^2\,ds} = \begin{cases} \frac{x^2+1}{2}, & \text{if } k = 0, \\ \frac{x^3-3x}{2}, & \text{if } k = 1, \\ (2k+1)p_k(x), & \text{if } k \geq 2. \end{cases} \tag{1.7}$$

Many problems similar to Problem (B_x) were studied; see, for example, [2, 3], and [4].

In this paper, we will solve Problem (B_x) completely with the help of optimal control theory. Since the cases $x = \pm 1$ were solved in [1], we mainly consider the cases of $x \in (-1, 1)$. We have the following.

Theorem 1.1 *Assume $m \geq 1$ and $x \in (-1, 1)$. Then, $y(\cdot) \in W(1, 2, m)$ is an extremal function to Problem (B_x) if and only if*

$$y(t) = C\left[c(x)\left(Q_{m+1}(t) - |t-x|\right) + 1\right], \quad t \in [-1, 1], \tag{1.8}$$

where C is a constant,

$$Q_m(t) = A(x) + \sum_{1\leq k\leq m-1} q_k(x)P_k(t) + \alpha(x)P_m(t) + \beta(x)P_{m+1}(t), \quad t \in [-1, 1], \tag{1.9}$$

$$c(x) = \frac{2}{x^2 + 1 - 2A(x)}, \tag{1.10}$$

and $A(x)$, $\alpha(x)$, $\beta(x)$ are characterized by

$$Q_m'(-1) = -1, \qquad Q_m'(1) = 1 \tag{1.11}$$

and

$$Q_m(x) = 0. \tag{1.12}$$

The sharp constant of the inequality (1.2) *is*

$$B_m(x) = \frac{1}{\sqrt{2c(x)}}, \quad x \in (-1, 1). \tag{1.13}$$

More precisely, we have the following.

Corollary 1.2 *Assume $m = 1$ and $x \in (-1, 1)$. Then*

$$B_1(x) = \sqrt{\frac{3x^2+1}{6}} \tag{1.14}$$

and $y(\cdot) \in W(1,2,1)$ is an extremal function to Problem (B_x) if and only if

$$y(t) = C\left[\frac{3}{3x^2+1}\left(\frac{t^2-x^2}{2} - |t-x|\right) + 1\right], \quad t \in [-1,1]. \tag{1.15}$$

Corollary 1.3 *Assume $m = 2$ and $x \in (-1,1)$. Then*

$$B_2(x) = \sqrt{\frac{8 - 21x^2 + 30x^4 - 5x^6}{48}} \tag{1.16}$$

and $y(\cdot) \in W(1,2,2)$ is an extremal function to Problem (B_x) if and only if

$$y(t) = C\left[\frac{4 + 33x^2 - 30x^4 + 5x^6 + 8P_2(t) - 2(x^3-3x)(P_3(t) - 6P_1(t)) - 24|t-x|}{8 - 21x^2 + 30x^4 - 5x^6} + 1\right]. \tag{1.17}$$

Corollary 1.4 *Assume $m = 2n+1$, $n \geq 1$ and $x \in (-1,1)$. Then*

$$\begin{aligned} B_{2n+1}(x) = \Biggl\{ & \frac{(x^2-1)^2}{4} + \frac{1}{2}\sum_{k=2}^{2n}(2k+1)p_k(x)P_k(x) \\ & - \frac{1}{2(n+1)(2n+1)}\left[\frac{x^3-3x}{2} + \sum_{1\leq k\leq n-1}(k+1)(2k+1)(4k+3)p_{2k+1}(x)\right]P_{2n+1}(x) \\ & + \frac{1}{2(n+1)(2n+3)}\left[1 - \sum_{k=1}^{n}k(2k+1)(4k+1)p_{2k}(x)\right]P_{2n+2}(x)\Biggr\}^{\frac{1}{2}}. \end{aligned} \tag{1.18}$$

In particular,

$$B_3(x) = \frac{\sqrt{297 + 1{,}260x^2 - 5{,}370x^4 + 5{,}900x^6 - 1{,}575x^8}}{16\sqrt{15}}, \tag{1.19}$$

$$B_5(x) = \frac{\sqrt{1{,}375 + 8{,}400x^2 - 95{,}025x^4 + 357{,}560x^6 - 597{,}555x^8 + 448{,}056x^{10} - 121{,}275x^{12}}}{16\sqrt{105}}. \tag{1.20}$$

Corollary 1.5 *Assume $m = 2n+2$, $n \geq 1$ and $x \in (-1,1)$. Then*

$$\begin{aligned} B_{2n+2}(x) = \Biggl\{ & \frac{(x^2-1)^2}{4} + \frac{1}{2}\sum_{k=2}^{2n+1}(2k+1)p_k(x)P_k(x) \\ & + \frac{1}{2(n+1)(2n+3)}\left[1 - \sum_{k=1}^{n}k(2k+1)(4k+1)p_{2k}(x)\right]P_{2n+2}(x) \\ & - \frac{1}{2(n+2)(2n+3)}\left[\frac{x^3-3x}{2} + \sum_{k=1}^{n}(k+1)(2k+1)(4k+3)p_{2k+1}(x)\right] \\ & \times P_{2n+3}(x)\Biggr\}^{\frac{1}{2}}. \end{aligned} \tag{1.21}$$

In particular,

$$B_4(x) = \frac{\sqrt{297 - 1{,}440x^2 + 9{,}030x^4 - 20{,}860x^6 + 18{,}585x^8 - 5{,}292x^{10}}}{16\sqrt{15}}, \tag{1.22}$$

$$B_6(x) = \frac{1}{32\sqrt{42}}\big(2{,}200 - 15{,}225x^2 + 211{,}050x^4 - 1{,}162{,}455x^6 + 3{,}017{,}700x^8 - 3{,}977{,}127x^{10} + 2{,}562{,}714x^{12} - 637{,}065x^{14}\big)^{\frac{1}{2}}. \tag{1.23}$$

When $x = 0$, we have the following.

Corollary 1.6 *The following holds*:

$$B_1^2(0) = B_2^2(0) = \frac{1}{6}, \qquad B_3^2(0) = B_4^2(0) = \frac{99}{1{,}280} \tag{1.24}$$

and

$$B_{2n+1}^2(0) = B_{2n+2}^2(0) = \frac{3}{32} - \sum_{k=2}^{n} \frac{(4k+1)}{2^{4k+2}(k+1)(2k-1)}\binom{2k}{k}^2 + (-1)^n \frac{\binom{2n+2}{n+1}}{2^{2n+2}\binom{2n+3}{2}}\left[\frac{7}{16} - \sum_{k=2}^{n} \frac{(-1)^k k(4k+1)}{2^{2k+3}(2k-1)}\binom{2k+2}{k+1}\right], \quad n \geq 2. \tag{1.25}$$

In particular,

$$B_5^2(0) = B_6^2(0) = \frac{275}{5{,}376}, \qquad B_7^2(0) = B_8^2(0) = \frac{45{,}325}{1{,}179{,}648}. \tag{1.26}$$

2 Transmitting Problem (B_x) to optimal control problem

We introduce the equivalent optimal control problem to Problem (B_x). Let $\mathcal{U} = L^2(-1,1)$. We define the following control system:

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ w(t) \\ z_0(t) \\ z_1(t) \\ \vdots \\ z_{m-1}(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ u(t)\chi_{(-1,x)}(t) \\ y(t) \\ ty(t) \\ \vdots \\ t^{m-1}y(t) \end{pmatrix}, \quad t \in (-1,1) \tag{2.1}$$

and the state constraints

$$y(-1) = w(-1), \qquad w(1) = 1, \qquad z_k(\pm 1) = 0 \quad (k = 0,1,\ldots,m-1). \tag{2.2}$$

Let

$$\mathcal{P}_{ad} = \big\{\big(Y(\cdot),u(\cdot)\big) \in \big(W^{1,2}(-1,1)\big)^{m+2} \times \mathcal{U} \,|\, \big(Y(\cdot),u(\cdot)\big) \text{ satisfying (2.1)-(2.2)}\big\} \tag{2.3}$$

and

$$\mathcal{U}_{ad} = \{u(\cdot) | (Y(\cdot), u(\cdot)) \in \mathcal{P}_{ad}\}, \tag{2.4}$$

where

$$Y(\cdot) = \begin{pmatrix} y(\cdot) \\ w(\cdot) \\ z_0(\cdot) \\ z_1(\cdot) \\ \vdots \\ z_{m-1}(\cdot) \end{pmatrix}. \tag{2.5}$$

Our optimal control problem corresponding to Problem (B_x) is as follows.

Problem (C_x) Let $x \in (-1, 1)$. Find $(\overline{Y}(\cdot), \bar{u}(\cdot)) \in \mathcal{P}_{ad}$ such that

$$\int_{-1}^{1} \bar{u}^2(t)\, dt = \inf_{(Y(\cdot),u(\cdot))\in\mathcal{P}_{ad}} \int_{-1}^{1} u^2(t)\, dt. \tag{2.6}$$

It is obvious that $(Y(\cdot), u(\cdot)) \mapsto y(\cdot)$ is a bijection from $\mathcal{P}_{ad}$ to $\{f(\cdot) \in W(1, 2, m) | f(x) = 1\}$. Then one can easily see that

$$B_m(x) = \left(\inf_{(Y(\cdot),u(\cdot))\in\mathcal{P}_{ad}} \int_{-1}^{1} u^2(t)\, dt \right)^{-\frac{1}{2}}. \tag{2.7}$$

Therefore, we can solve Problem (B_x) by solving Problem (C_x).

3 Pontryagin's maximum principle

We state Pontryagin's maximum principle for optimal control problems. Symbols in this section will have similar but probably different meanings from other sections. Thus we set this part as a separate section. We will state a result given in [5]. For simplicity, we only state it in a simple way. In other words, Lemma 3.1 below is a special case of Theorem 3.1 and Corollary 3.1 in Chapter V of [5].

Now, let $T > t_0$ and $U \subseteq \mathbb{R}^m$. A measurable function $u(\cdot)$ defined on $[t_0, T]$ with range in U is said to be a control.

Let the function $\hat{f} = (f^0, f) = (f^0, f^1, \ldots, f^n)$ be an $\mathbb{R}^{n+1}$-valued vector function on $[t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m$. Assume that $\hat{f}$ is Borel measurable on $t \in [t_0, T]$, continuous on $(y, u) \in \mathbb{R}^n \times \mathbb{R}^m$ and continuously differentiable on $y \in \mathbb{R}^n$.

If $y(\cdot)$ is an absolutely function on $[t_0, T]$ with range in $\mathbb{R}^n$ such that

$$\frac{dy(t)}{dt} = f(t, y(t), u(t)), \quad \text{a.e. on } [t_0, T],$$

then $y(\cdot)$ is called a state/trajectory corresponding to $u(\cdot)$.

Let $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be a C^1 manifold of dimensional k, where $0 \le k \le 2n$. We say $(y(\cdot), u(\cdot))$ is an admissible pair if

(i) $u(\cdot)$ is a control,

(ii) $y(\cdot)$ is a state corresponding to $u(\cdot)$,

(iii) $(y(t_0), y(T)) \in \Omega$.

Denote by $\mathcal{P}_{ad}$ the set of all admissible pairs. The set $\mathcal{U}_{ad} = \{u(\cdot)|(Y(\cdot), u(\cdot)) \in \mathcal{P}_{ad}\}$ is called the set of admissible controls.

If an admissible pair $(\bar{y}(\cdot), \bar{u}(\cdot))$ satisfies

$$J(\bar{y}(\cdot), \bar{u}(\cdot)) = \inf_{(y(\cdot),u(\cdot))\in\mathcal{P}_{ad}} J(y(\cdot), u(\cdot)),$$

then it is called an optimal pair, where

$$J(y(\cdot), u(\cdot)) := \int_0^T f^0(t, y(t), u(t))\, dt.$$

Assume that for each compact $Y \subset \mathbb{R}^n$ and admissible control $u(\cdot)$, there exists a function $\mu(\cdot) = \mu(\cdot; Y, u(\cdot))$ such that for almost all $t \in [t_0, T]$ and all $y \in Y$,

$$\left|\hat{f}(t, y, u(t))\right| \le \mu(t), \qquad \left|\hat{f}_y(t, y, u(t))\right| \le \mu(t). \tag{3.1}$$

We have the following.

Lemma 3.1 *Let assumptions listed in this section hold. Let $(\bar{y}(\cdot), \bar{u}(\cdot))$ be an optimal pair. Then there exists a constant $\lambda^0 \le 0$ and an absolutely continuous vector function $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ defined on $[t_0, T]$ such that the following hold*:

(i) *The vector $(\lambda^0, \lambda(t))$ is never zero on $[t_0, T]$.*

(ii) *For a.e. $t \in [t_0, T]$,*

$$\lambda'(t) = -f_y(t, \bar{y}(t), \bar{u}(t))\lambda(t) - \lambda^0 f_y^0 t, \bar{y}(t), \bar{u}(t)),$$

where

$$f_y = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_2}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_1} \\ \frac{\partial f_1}{\partial y_2} & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_n}{\partial y_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial y_n} & \frac{\partial f_2}{\partial y_n} & \cdots & \frac{\partial f_n}{\partial y_n} \end{pmatrix}.$$

(iii) *The pointwise maximum condition holds*: *for almost all $t \in [t_0, T]$ and all $u \in U$,*

$$\lambda^0 f^0(t, \bar{y}(t), \bar{u}(t)) + \langle \lambda(t), f(t, \bar{y}(t), \bar{u}(t)) \rangle \ge \lambda^0 f^0(t, \bar{y}(t), \bar{u}(t)) + \langle \lambda(t), f(t, \bar{y}(t), u) \rangle.$$

(iv) *The transversality condition holds*: *if the mapping $t \mapsto \hat{f}(t, \bar{y}(t), \bar{u}(t))$ is continuous at $t = t_0$ and $t = T$, then $(-\lambda(t_0), \lambda(T))$ is orthogonal to Ω.*

4 Proof of Theorem 1.1

We give the following lemma first.

Lemma 4.1 *Let $n \ge 1$, $c \in \mathbb{R}$, $x \in (-1, 1)$, $Q(\cdot)$ is an $(n+1)$th degree polynomial satisfying*

$$Q(x) = 0, \qquad Q'(1) = 1, \qquad Q'(-1) = -1 \tag{4.1}$$

and

$$\int_{-1}^{1} t^k\big[c\big(Q(t) - |t-x|\big) + 1\big]\,dt = 0, \quad \forall k = 0,1,\ldots,n-1. \tag{4.2}$$

Then

$$\int_{-1}^{1}\left[\frac{\partial}{\partial t}\big(Q(t) - |t-x|\big)\right]^2 dt = \frac{2}{c}. \tag{4.3}$$

Proof Noting that $Q''(\cdot)$ is an $(n-1)$th degree polynomial, by (4.2), we have

$$\int_{-1}^{1} Q''(t)\big[c\big(Q(t) - |t-x|\big) + 1\big]\,dt = 0. \tag{4.4}$$

Therefore,

$$\begin{aligned}
&\int_{-1}^{1}\left[\frac{\partial}{\partial t}\big(Q(t) - |t-x|\big)\right]^2 dt \\
&\quad = \int_{-1}^{x}\big(Q'(t)+1\big)^2\,dt + \int_{x}^{1}\big(Q'(t)-1\big)^2\,dt \\
&\quad = \int_{-1}^{1}\big(Q'(t)\big)^2\,dt + 2 + 2\int_{-1}^{x} Q'(t)\,dt - 2\int_{x}^{1} Q'(t)\,dt \\
&\quad = Q(1)Q'(1) - Q(-1)Q'(-1) - \int_{-1}^{1} Q''(t)Q(t)\,dt + 2 + 4Q(x) - 2Q(-1) - 2Q(1) \\
&\quad = \int_{-1}^{1} Q''(t)\left(\frac{1}{c} - |t-x|\right)dt + 2 - Q(-1) - Q(1) \\
&\quad = \frac{2}{c} + \int_{-1}^{x} Q''(t)(t-x)\,dt + \int_{x}^{1} Q''(t)(x-t)\,dt + 2 - Q(-1) - Q(1) \\
&\quad = \frac{2}{c} - (1+x) - \int_{-1}^{x} Q'(t)\,dt - (1-x) + \int_{x}^{1} Q'(t)\,dt + 2 - Q(-1) - Q(1) \\
&\quad = \frac{2}{c}.
\end{aligned} \tag{4.5}$$

□

Now, we list some properties of Legendre polynomials. We can easily get

$$P_k(0) = \begin{cases} 0, & \text{if } k = 2n+1, n \ge 0, \\ 1, & \text{if } k = 0, \\ (-1)^n \frac{(2n-1)!!}{(2n)!!}, & \text{if } k = 2n, n \ge 1, \end{cases} \tag{4.6}$$

$$p_k(0) = \begin{cases} 0, & \text{if } k = 2n+1, n \ge 1, \\ \frac{1}{8}, & \text{if } k = 2, \\ (-1)^{n-1} \frac{(2n-3)!!}{(2n+2)!!}, & \text{if } k = 2n, n \ge 2, \end{cases} \tag{4.7}$$

$$P_k(-1) = (-1)^k, \qquad P_k(1) = 1, \tag{4.8}$$

$$P_k'(-1) = (-1)^{k-1}\frac{k(k+1)}{2}, \qquad P_k'(1) = \frac{k(k+1)}{2}, \tag{4.9}$$

$$\int_{-1}^{1} \left|P_k(t)\right|^2 dt = \frac{2}{2k+1}, \tag{4.10}$$

and

$$\int_{-1}^{1} \left(P_k'(t)\right)^2 dt = P_k'(1)P_k(1) - P_k'(-1)P_k(-1) = k(k+1), \tag{4.11}$$

$$\int_{-1}^{1} P_k'(t)P_{k+1}'(t)\, dt = P_k'(1)P_{k+1}(1) - P_k'(-1)P_{k+1}(-1) = 0, \tag{4.12}$$

$$\int_{-1}^{1} P_k(t)|t-x|\, dt = \begin{cases} x^2+1, & \text{if } k=0, \\ \frac{1}{3}x^3 - x, & \text{if } k=1, \\ 2p_k(x), & \text{if } k \geq 2. \end{cases} \tag{4.13}$$

We turn to the proof of Theorem 1.1.

Proof of Theorem 1.1

I. Existence of optimal pair. One can prove directly that the sharp constant $B_m(x)$ is attainable, *i.e.*, there is a nontrivial $\bar{y}(\cdot) \in W(1,2,m)$ such that

$$\left|\bar{y}(x)\right| = B_m(x)\left(\int_{-1}^{1} \left|y'(t)\right|^2 dt\right)^{-\frac{1}{2}}. \tag{4.14}$$

Now, we give an optimal control version of this fact.

Let $(Y_j(\cdot), u_j(\cdot)) \in \mathcal{P}_{ad}$ be a minimizing sequence of Problem (C_x). That is

$$\lim_{j\to+\infty} \int_{-1}^{1} u_j^2(t)\, dt = \inf_{(Y(\cdot),u(\cdot))\in\mathcal{P}_{ad}} \int_{-1}^{1} u^2(t)\, dt. \tag{4.15}$$

Then $u_j(\cdot)$ is bounded in $L^2(-1,1)$. That is

$$\left\|u_j(\cdot)\right\|_{L^2(-1,1)} \leq M$$

for some constant $M > 0$. Denote $Y_j(\cdot) \equiv (y^j(\cdot), w^j(\cdot), z_0^j(\cdot), \ldots, z_{m-1}^j(\cdot))$. Then, by the state equation (2.1) and the constraints (2.2), we have

$$\int_{-1}^{1} y^j(t)\, dt = z_0^j(1) - z_0^j(-1) = 0.$$

Then (2.1)-(2.2), and Poincaré's inequality imply

$$\left\|y^j(\cdot)\right\|_{W^{1,2}(-1,1)} \leq C_1, \qquad \left\|w^j(\cdot)\right\|_{W^{1,2}(-1,1)} \leq C_1, \quad \forall j \geq 1, \tag{4.16}$$

and consequently

$$\left\|z_k^j(\cdot)\right\|_{W^{1,2}(-1,1)} \leq C_2, \quad \forall k = 0,1,\ldots,m-1; j \geq 1 \tag{4.17}$$

for some constant C_1, C_2. That is, $Y_j(\cdot)$ is bounded in $(W^{1,2}(-1,1))^{m+2}$. Then, by Sobolev's imbedding theorem, $Y_j(\cdot)$ is bounded and equicontinuous in $(C[-1,1])^{m+2}$.

Thus, Eberlein-Shmulyan's theorem and Arzelá-Ascoli's theorem (see Chapter V, Appendix 4 and Chapter III, Section 3 in [6], for example), we can suppose that

$$u_j(\cdot) \to \bar{u}(\cdot), \quad \text{weakly in } L^2(-1,1) \tag{4.18}$$

and

$$Y_j(\cdot) \to \overline{Y}(\cdot), \quad \text{weakly in } \big(W^{1,2}(-1,1)\big)^{m+2}, \text{ strongly in } \big(C[-1,1]\big)^{m+2} \tag{4.19}$$

for some $\bar{u}(\cdot) \in L^2(-1,1)$ and $\overline{Y}(\cdot) \in (W^{1,2}(-1,1))^{m+2}$. One can easily see that $(\overline{Y}(\cdot), \bar{u}(\cdot))$ satisfies (2.1) and (2.2). Thus, $(\overline{Y}(\cdot), \bar{u}(\cdot)) \in \mathcal{P}_{ad}$. Moreover,

$$\int_{-1}^{1} \bar{u}^2(t)\,dt \le \lim_{j\to+\infty} \int_{-1}^{1} u_j^2(t)\,dt. \tag{4.20}$$

Therefore $(\overline{Y}(\cdot), \bar{u}(\cdot))$ is a solution to Problem (C_x). We call it an optimal pair of Problem (C_x).

II. Pontryagin's maximum principle for the optimal pair. We now apply Lemma 4.1 - Pontryagin's maximum principle to Problem (C_x). We can easily verify that all the conditions posed in Section 3 hold. For example, conditions on state constraints and the local existence of a dominating integrable function (see (3.1)) hold. More precisely, let

$$\Omega = \big\{(s,s) | s \in \mathbb{R}\big\} \times \{0\}^m \times \mathbb{R} \times \{1\} \times \{0\}^m.$$

Then Ω is a C^1 manifold of dimensional 2. While the state constraints (2.2) is equivalent to $(Y(-1), Y(1)) \in \Omega$.

On the other hand, for any $u(\cdot) \in L^2(-1,1)$ and $|Y| \le R$, if we choose $\mu(t) = 2|u(t)| + mR$, then the condition (4.1) corresponding to (2.1) holds.

Now, by Lemma 4.1, the optimal pair $(\overline{Y}(\cdot), \bar{u}(\cdot))$ satisfies the following Pontryagin maximum principle: there exists a $\varphi^0 \le 0$ and a solution to the following adjoint equation:

$$\frac{d}{dt}\begin{pmatrix} \varphi(t) \\ \zeta(t) \\ \psi_0(t) \\ \psi_1(t) \\ \vdots \\ \psi_{m-1}(t) \end{pmatrix} = \begin{pmatrix} -\sum_{j=0}^{m-1} t^j \psi_j(t) \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad t \in (-1,1) \tag{4.21}$$

such that the following conditions hold:

(i) we have the following non-trivial condition:

$$\big(\varphi^0, \varphi(\cdot), \zeta(\cdot), \psi_0(\cdot), \dots, \psi_{m-1}(\cdot)\big) \neq 0, \tag{4.22}$$

(ii) the maximum condition:

$$\begin{aligned} &\varphi^0 \bar{u}^2(t) + \big(\varphi(t) + \zeta(t)\chi_{(-1,x)}(t)\big)\bar{u}(t) \\ &\quad = \max_{u\in\mathbb{R}} \big[\varphi^0 u^2 + \big(\varphi(t) + \zeta(t)\chi_{(-1,x)}(t)\big)u\big], \quad \text{a.e. } t \in (-1,1), \end{aligned} \tag{4.23}$$

(iii) the transversality condition

$$\varphi(-1)+\zeta(-1)=0,\qquad \varphi(1)=0. \tag{4.24}$$

III. Simplification. By (4.21), $\zeta(\cdot)\equiv\zeta$, $\psi_0(\cdot)\equiv\psi_0,\dots,\psi_{m-1}(\cdot)\equiv\psi_{m-1}$ are constants and

$$\varphi'(t)=-\sum_{j=0}^{m-1}\psi_j t^j,\quad t\in(-1,1). \tag{4.25}$$

Then $\varphi(\cdot)$ is an mth degree polynomial.

By (4.23), we have

$$2\varphi^0\bar{u}(t)+\varphi(t)+\zeta\chi_{(-1,x)}=0,\quad \text{a.e. } t\in(-1,1). \tag{4.26}$$

If $\varphi^0=0$, we get

$$\varphi(t)=\begin{cases}-\zeta, & t\in(-1,x),\\ 0, & t\in[x,1),\end{cases}\qquad \text{a.e. } t\in(-1,1).$$

Therefore, since $\varphi(\cdot)$ is a polynomial, we have the following:

$$\varphi(\cdot)\equiv 0,\qquad \zeta=0. \tag{4.27}$$

Consequently, by (4.25),

$$\psi_0=\psi_1=\cdots=\psi_{m-1}=0. \tag{4.28}$$

This contradicts the non-trivial condition (4.22). Therefore, we must have $\varphi^0<0$. Without loss of generality, we can suppose that $\varphi^0=-\frac{1}{2}$. Then it follows from (4.26) that

$$\bar{u}(t)=\varphi(t)+\zeta\chi_{(-1,x)}(t),\quad t\in(-1,1). \tag{4.29}$$

Combining the above with (4.24), we see that the corresponding function $\bar{y}(\cdot)$ is continuously differentiable on $[-1,x)\cup(x,1]$ and

$$\bar{y}'(\pm1)=\bar{u}(\pm1)=0. \tag{4.30}$$

Moreover, $\bar{y}(\cdot)$ can be expressed as

$$\bar{y}(t)=-c(x)|t-x|+\widetilde{Q}_m(t),\quad t\in[-1,1], \tag{4.31}$$

where $c\equiv c(x)=\frac{\zeta}{2}$ and $\widetilde{Q}_m(\cdot)$ is an $(m+1)$th degree polynomial.

We claim that $c \neq 0$. Otherwise, $c = 0$. Then it follows from (4.31) and

$$\int_{-1}^{1} t^k \bar{y}(t)\,dt = 0, \quad k = 0,1,2,\ldots,m-1, \tag{4.32}$$

that

$$\widetilde{Q}_m(t) = c_m P_m(t) + c_{m+1} P_{m+1}(t), \quad k = 0,1,2,\ldots,m-1 \tag{4.33}$$

for some constant c_m, c_{m+1}.

Then (4.30) and (4.9) imply $c_m = c_{m+1} = 0$. This contradicts the nontrivial condition. Therefore $c \neq 0$ and we can rewrite $\bar{y}(\cdot)$ as

$$\bar{y}(t) = c(x)\big(Q_m(t) - |t-x|\big) + 1, \quad t \in [-1,1], \tag{4.34}$$

where $Q_m(\cdot)$ is an $(m+1)$th degree polynomial such that

$$Q_m(x) = 0, \qquad Q_m'(-1) = -1, \qquad Q_m'(1) = 1. \tag{4.35}$$

IV. Conclusion. By (4.32),

$$\int_{-1}^{1} Q_m(t)P_k(t)\,dt = \int_{-1}^{1} P_k(t)|x-t|\,dt, \quad 1 \leq k \leq m-1. \tag{4.36}$$

Thus we see that

$$Q_m(t) = A(x) + \sum_{1\leq k\leq m-1} q_k(x)P_k(t) + \alpha(x)P_m(t) + \beta(x)P_{m+1}(t), \tag{4.37}$$

where $q_k(x)$ is defined by (1.7). Moreover, by (4.35), we can determine $A(x)$, $\alpha(x)$ and $\beta(x)$. Finally, using (4.32) again, we get

$$c(x) = \frac{2}{x^2 + 1 - 2A(x)}. \tag{4.38}$$

By Lemma 4.1, we have the following:

$$\int_{-1}^{1} \left|\bar{u}(t)\right|^2 dt = \int_{-1}^{1} \left|\bar{y}'(t)\right|^2 dt = 2c(x). \tag{4.39}$$

Then Theorem 1.1 follows from (2.7). □

Remark 4.1 If $x = -1$, instead of (4.34)-(4.35), we get

$$\bar{y}(t) = Q_m(t), \quad t \in [-1,1] \tag{4.40}$$

with

$$Q_m(-1) = 1, \qquad Q_m'(1) = 0 \tag{4.41}$$

and

$$Q_m(t) = \alpha P_m(t) + \beta P_{m+1}(t), \quad t \in [-1,1]. \tag{4.42}$$

The above equations imply the results in [1] for $x = \pm 1$.

On the other hand, since $B_m(x)$ is obviously continuous respect to $x \in [-1,1]$, we can certainly get $B_m(\pm 1)$ from Theorem 1.1.

5 Results for some special cases

We prove Corollaries 1.2-1.6 in this section.

Proof of Corollary 1.2 By (1.9), we have

$$Q_1(t) = A(x) + \alpha(x)P_1(t) + \beta(x)P_2(t). \tag{5.1}$$

Thus (1.11) and (4.9) imply $\alpha(x) = 0$, $\beta(x) = \frac{1}{3}$. Then (1.12) implies

$$A(x) = -\frac{1}{3}P_2(x) = \frac{1-3x^2}{6}. \tag{5.2}$$

Consequently,

$$Q_1(t) = \frac{1}{3}\bigl(P_2(t) - P_2(x)\bigr) = \frac{t^2 - x^2}{2} \tag{5.3}$$

and

$$c(x) \equiv \frac{2}{x^2 + 1 - 2A(x)} = \frac{3}{3x^2+1}. \tag{5.4}$$

Therefore, the extremal functions to Problem (B_x) are $C\bar{y}(\cdot)$ with

$$\bar{y}(x) = \frac{3}{3x^2+1}\left(\frac{t^2-x^2}{2} - |t-x|\right) + 1, \tag{5.5}$$

while

$$B_1(x) = \frac{1}{\sqrt{2c(x)}} = \sqrt{\frac{3x^2+1}{6}}. \tag{5.6}$$

□

Proof of Corollary 1.3 By (1.7) and (1.9), we have

$$Q_2(t) = A(x) + \frac{x^3-3x}{2}P_1(t) + \alpha(x)P_2(t) + \beta(x)P_3(t). \tag{5.7}$$

Then it follows easily from (1.11) that

$$\alpha(x) = \frac{1}{3}, \qquad \beta(x) = -\frac{x^3-3x}{12}. \tag{5.8}$$

Then, by (1.12),

$$A(x) = \frac{4 + 33x^2 - 30x^4 + 5x^6}{24}. \tag{5.9}$$

Thus

$$c(x) = \frac{2}{x^2 + 1 - 2A(x)} = \frac{24}{8 - 21x^2 + 30x^4 - 5x^6}. \tag{5.10}$$

Therefore

$$B_2(x) = \frac{1}{\sqrt{2c(x)}} = \sqrt{\frac{8 - 21x^2 + 30x^4 - 5x^6}{48}} \tag{5.11}$$

and the extremal functions to Problem (B_x) are

$$C\left(\frac{4 + 33x^2 - 30x^4 + 5x^6 + 8P_2(t) - 2(x^3 - 3x)(P_3(t) - 6P_1(t)) - 24|t - x|}{8 - 21x^2 + 30x^4 - 5x^6} + 1\right). \tag{5.12}$$

□

Proof of Corollary 1.4 By (1.7) and (1.9), we have

$$\begin{aligned} Q_{2n+1}(t) &= A(x) + \frac{x^3 - 3x}{2}P_1(t) + \sum_{k=2}^{2n}(2k+1)p_k(x)P_k(t) \\ &\quad + \alpha(x)P_{2n+1}(t) + \beta(x)P_{2n+2}(t). \end{aligned} \tag{5.13}$$

Then by (1.11)-(1.12), and (1.10),

$$\alpha(x) = -\frac{1}{(n+1)(2n+1)}\left[\frac{x^3 - 3x}{2} + \sum_{1 \le k \le n-1}(k+1)(2k+1)(4k+3)p_{2k+1}(x)\right], \tag{5.14}$$

$$\beta(x) = \frac{1}{(n+1)(2n+3)}\left[1 - \sum_{k=1}^{n}k(2k+1)(4k+1)p_{2k}(x)\right], \tag{5.15}$$

$$\begin{aligned} A(x) &= -\frac{x^3 - 3x}{2}P_1(x) - \sum_{k=2}^{2n}(2k+1)p_k(x)P_k(x) \\ &\quad + \frac{1}{(n+1)(2n+1)}\left[\frac{x^3 - 3x}{2} + \sum_{1 \le k \le n-1}(k+1)(2k+1)(4k+3)p_{2k+1}(x)\right]P_{2n+1}(x) \\ &\quad - \frac{1}{(n+1)(2n+3)}\left[1 - \sum_{k=1}^{n}k(2k+1)(4k+1)p_{2k}(x)\right]P_{2n+2}(x), \end{aligned} \tag{5.16}$$

$$\begin{aligned} \frac{1}{2c(x)} &= \frac{x^2 + 1}{4} - \frac{A(x)}{2} = \frac{(x^2 - 1)^2}{4} + \frac{1}{2}\sum_{k=2}^{2n}(2k+1)p_k(x)P_k(x) \\ &\quad - \frac{1}{2(n+1)(2n+1)}\left[\frac{x^3 - 3x}{2} + \sum_{1 \le k \le n-1}(k+1)(2k+1)(4k+3)p_{2k+1}(x)\right]P_{2n+1}(x) \\ &\quad + \frac{1}{2(n+1)(2n+3)}\left[1 - \sum_{k=1}^{n}k(2k+1)(4k+1)p_{2k}(x)\right]P_{2n+2}(x). \end{aligned} \tag{5.17}$$

Finally, (1.19) and (1.20) follow from direct calculations. We get the proof. □

Proof of Corollary 1.5 By (1.7) and (1.9), we have

$$Q_{2n+2}(t) = A(x) + \frac{x^3-3x}{2}P_1(t) + \sum_{k=2}^{2n+1}(2k+1)p_k(x)P_k(t) + \alpha(x)P_{2n+2}(t) + \beta(x)P_{2n+3}(t). \tag{5.18}$$

Then by (1.11)-(1.12), and (1.10),

$$\alpha(x) = \frac{1}{(n+1)(2n+3)}\left[1 - \sum_{k=1}^{n} k(2k+1)(4k+1)p_{2k}(x)\right], \tag{5.19}$$

$$\beta(x) = -\frac{1}{(n+2)(2n+3)}\left[\frac{x^3-3x}{2} + \sum_{k=1}^{n}(k+1)(2k+1)(4k+3)p_{2k+1}(x)\right], \tag{5.20}$$

$$\begin{aligned} A(x) = {} & -\frac{x^3-3x}{2}P_1(x) - \sum_{k=2}^{2n+1}(2k+1)p_k(x)P_k(x) \\ & - \frac{1}{(n+1)(2n+3)}\left[1 - \sum_{k=1}^{n} k(2k+1)(4k+1)p_{2k}(x)\right]P_{2n+2}(x) \\ & + \frac{1}{(n+2)(2n+3)}\left[\frac{x^3-3x}{2} + \sum_{k=1}^{n}(k+1)(2k+1)(4k+3)p_{2k+1}(x)\right] \\ & \times P_{2n+3}(x), \end{aligned} \tag{5.21}$$

$$\begin{aligned} \frac{1}{2c(x)} = {} & \frac{x^2+1}{4} - \frac{A(x)}{2} \\ = {} & \frac{(x^2-1)^2}{4} + \frac{1}{2}\sum_{k=2}^{2n+1}(2k+1)p_k(x)P_k(x) \\ & + \frac{1}{2(n+1)(2n+3)}\left[1 - \sum_{k=1}^{n} k(2k+1)(4k+1)p_{2k}(x)\right]P_{2n+2}(x) \\ & - \frac{1}{2(n+2)(2n+3)}\left[\frac{x^3-3x}{2} + \sum_{k=1}^{n}(k+1)(2k+1)(4k+3)p_{2k+1}(x)\right] \\ & \times P_{2n+3}(x). \end{aligned} \tag{5.22}$$

Finally, (1.21) and (1.22) follow from direct calculations. We get the proof. □

Proof of Corollary 1.6 First, we get (1.24) from (1.14), (1.16), (1.19), and (1.22).
By (4.7), $p_{2k+1}(0) = 0$ ($k = 1, 2, \ldots$). Thus, if $n \geq 2$, we get from (1.18) and (1.21)

$$\begin{aligned} B_{2n+1}^2(0) = {} & B_{2n+2}^2(0) \\ = {} & \frac{1}{4} + \frac{1}{2}\sum_{k=1}^{n}(4k+1)p_{2k}(0)P_{2k}(0) \\ & + \frac{1}{2(n+1)(2n+3)}\left[1 - \sum_{k=1}^{n} k(2k+1)(4k+1)p_{2k}(0)\right]P_{2n+2}(0). \end{aligned} \tag{5.23}$$

Moreover, using (4.6)-(4.7), we get (1.25):

$$\begin{aligned}
B_{2n+1}^2(0) = B_{2n+2}^2(0) &= \frac{1}{4} + \frac{1}{2} \times \frac{5}{8} \times \left(-\frac{1}{2}\right) - \frac{1}{2}\sum_{k=2}^{n}(4k+1)\frac{(2k-3)!!}{(2k+2)!!}\frac{(2k-1)!!}{(2k)!!} \\
&\quad - \frac{(-1)^n}{2(n+1)(2n+3)}\left[1 - \frac{15}{8} + \sum_{k=2}^{n}(-1)^k k(2k+1)(4k+1)\frac{(2k-3)!!}{(2k+2)!!}\right]\frac{(2n+1)!!}{(2n+2)!!} \\
&= \frac{3}{32} - \sum_{k=2}^{n}\frac{(4k+1)}{2^{4k+2}(k+1)(2k-1)}\binom{2k}{k}^2 \\
&\quad + (-1)^n \frac{\binom{2n+2}{n+1}}{2^{2n+2}\binom{2n+3}{2}}\left[\frac{7}{16} - \sum_{k=2}^{n}\frac{(-1)^k k(4k+1)}{2^{2k+3}(2k-1)}\binom{2k+2}{k+1}\right], \quad n \geq 2.
\end{aligned}$$

Now (1.26) follows directly from (1.25). □

Competing interests
The author declares that they have no competing interests.

Acknowledgements
This work was supported in part by 973 Program (No. 2011CB808002) and NSFC (No. 11371104).

Received: 24 December 2013 Accepted: 23 April 2014 Published: 29 December 2014

References

1. Kalyabin, GA: Sharp constants in one-dimensional inequalities of Poincaré type. Math. Notes **90**(4), 634-636 (2011) (in Russian). Translation in Math. Notes, **90**(3-4), 615-618 (2011)
2. Gerasimov, IV, Nazarov, AI: Best constant in a three-parameter Poincaré inequality. J. Math. Sci. **179**(1), 80-99 (2011)
3. Oshime, Y, Watanabe, K: The best constant of L^p Sobolev inequality corresponding to Dirichlet boundary value problem II. Tokyo J. Math. **34**(1), 115-133 (2011)
4. Talenti, G: Best constant in Sobolev inequality. Ann. Mat. Pura Appl. (4) **110**, 353-372 (1976)
5. Berkovitz, LD: Optimal Control Theory. Springer, New York (1983)
6. Yosida, K: Functional Analysis, 6th edn. Springer, Berlin (1980)

doi:10.1186/1029-242X-2014-523
Cite this article as: Lou: Sharp constants for inequalities of Poincaré type: an application of optimal control theory. *Journal of Inequalities and Applications* 2014 **2014**:523.