RESEARCH   **Open Access**

A fuzzy semi-infinite optimization problem

Abd El-Monem A Megahed[*]

[*]Correspondence:
a_megahed15@yahoo.com
Permanent address: Department of
Basic Science, Faculty of Computers
and Informatics, Suez Canal
University, Ismalia, Egypt
Present address: Department of
Mathematics, College Of Science,
Majmaah University, Al-Zulfi, KSA

Abstract

In this paper, we present a fuzzy semi-infinite optimization problem. Moreover, we will deduce the Fritz-John and Kuhn-Tucker necessary conditions of this problem. Finally, a numerical example is given to illustrate the results.

MSC: 90C34; 90C05; 90C70; 90C30; 90C46

Keywords: fuzzy; semi-infinite; optimization; necessary conditions

1 Introduction

In many practical problems, we might have information containing some uncertainty, which is treated in this paper as fuzzy information; the considered semi-infinite optimization problem with fuzzy information is a fuzzy semi-infinite optimization problem.

Fuzzy set theory was introduced into conventional linear programming by Zimmermann [1], fuzzy mathematical programming was presented in [2], and fuzzy programming and linear programming with several objective functions were presented by [3].

Optimality conditions of a nonlinear programming problem with fuzzy parameters are established, and a fuzzy function is defined, with its differentiability, convexity, and some important properties being studied in [4]; the fuzzy solution of optimization problems and the incentive solution of the optimization problems are presented which explain that the solution of optimization problems is a generalization of the solutions in the case of crisp optimization problems [5]. As regards fuzzy mathematical programming: theory, application, and an extension are presented in [6].

This paper is organized as follows: In Section 2, the formulation of the problem of semi-infinite optimization is considered. In Section 3, the main section of the paper, we will study a fuzzy semi-infinite optimization problem, and the Fritz-John and Kuhn-Tucker necessary conditions. Finally, the conclusion is drawn in Section 5.

2 A semi-infinite programming problem

A semi-infinite programming problem is an optimization problem in which finitely many variables appear with infinitely many constraints [7, 8], and we consider a generalized semi-infinite optimization problem (GSIP) [9, 10] of the form

$$
\begin{aligned}
&\min_{x} f(x) \text{ s.t. } x \in M, \\
M = &\left\{ x \in R^n / h_i(x) = 0, i \in I, G(x,y) \geq 0 \atop \text{for all } y \in Y(x) \right\}, \\
&\text{where } Y(x) = \left\{ y \in R^r / u_k(x,y) = 0, k \in K, v_l(x,y) \geq 0, l \in L \right\}.
\end{aligned}
\tag{2.1}
$$

Springer

I, K, and L are finite index sets with $|l| < n$ and $|K| < r$ (where $|\cdot|$ denotes the cardinality), all appearing functions are real valued and continuously differentiable, and the set $Y(\overline{x})$ is compact for each $\overline{x} \in R^n$, and the set-valued mapping $Y : x \in R^n \to Y(x) \subset R^n$ is upper semi-continuous at each $\overline{x} \in R^n$.

For the special case that the set $Y(x) = Y$ does not depend on the variable x, this problem is a common semi-infinite problem (SIP). The generalized semi-infinite and bi-level optimization problem are presented by Stein and Still [11]. Bi-level problems are of the following form.

(BL):

$$\min_x f(x) \text{ s.t. } x \in M \text{ and } y \text{ is a solution of} \tag{2.2}$$

$$\min_y G(x, y)$$
$$\text{s.t. } y \in Y(x).$$

The generalized semi-infinite programming on generic local minimizers was introduced by Gunzel *et al.* [12]. The feasible set in generalized semi-infinite optimization is presented by Jongen *et al.* in [13]. Furthermore, the linear and linearized generalized semi-infinite optimization problems were introduced by Rukmann [14]. In [15], a first-order optimality condition in generalized semi-infinite programming is introduced. Still discussed the optimality conditions for generalized semi-infinite programming problems in [16].

3 A fuzzy semi-infinite programming problem
3.1 Problem formulation
A fuzzy semi-infinite programming problem is defined as

$$\left. \begin{array}{c} \widetilde{\min}_x f(x) \text{ s.t. } x \in M, \\ \text{where } M = \left\{ \begin{array}{c} x \in R^n / h_i(x) = 0, i \in I, G(x, y) \geq 0 \\ \text{for all } y \in Y(x) \end{array} \right\}, \\ Y(x) = \left\{ y \in R^r / u_k(x, y) = 0, k \in K, v_l(x, y) \geq 0, l \in L \right\}. \end{array} \right\} \tag{3.1}$$

All functions of this problem have the same properties as the problem (2.1). For $\overline{x} \in M$ denote the index sets of the active inequality constraints by

$$Y_0(\overline{x}) = \left\{ y \in Y(\overline{x}) / G(\overline{x}, y) = 0 \right\}, \tag{3.2}$$

$$L_0(\overline{x}, \overline{y}) = \left\{ l \in L / v_l(\overline{x}, \overline{y}) = 0 \text{ for } \overline{y} \in Y(\overline{x}) \right\}.$$
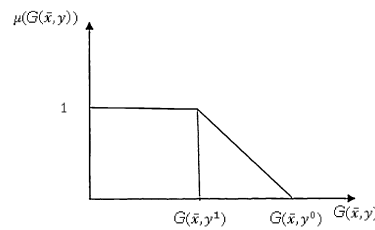
3.2 Lower level problem
Consider the following lower level problem:

$$\left. \begin{array}{c} \widetilde{\min}_y G(\overline{x}, y) \\ \text{s.t. } y \in Y(\overline{x}). \end{array} \right\} \tag{3.3}$$

The fuzzy requirements of the lower level problem (3.3) can be quantified by electing a membership function $\mu(G(\overline{x}, y))$ (Figure 1) which is differentiable in the open interval

Figure 1 The membership function $\mu(G(x,y))$.

$G(\overline{x},y^1) < G(\overline{x},y) < G(\overline{x},y^0)$, where $\mu(G(\overline{x},y))$ is defined by

$$\mu\big(G(\overline{x},y)\big) = \begin{cases} 1, & G(\overline{x},y) \leq G(\overline{x},y^1), \\ \frac{G(\overline{x},y)-G(\overline{x},y^0)}{G(\overline{x},y^1)-G(\overline{x},y^0)}, & G(\overline{x},y^1) \leq G(\overline{x},y) \leq G(\overline{x},y^0), \\ 0, & G(\overline{x},y) \geq G(\overline{x},y^0), \end{cases} \tag{3.4}$$

where $G(\overline{x},y^0)$ and $G(\overline{x},y^1)$ denote the values of the objective function of the lower level problem (3.3) with the degree of the membership function 0 and 1, respectively, *i.e.*, $G(\overline{x},y^0)$ is an undesirable value and $G(\overline{x},y^1)$ is a desirable value of the objective function $G(\overline{x},y)$.

Definition 1 The α-level set of the fuzzy goal $G(\overline{x},y)$ is defined as the ordinary set $L_\alpha(G(\overline{x},y))$; for the value of $G(\overline{x},y)$ the degree of its membership function exceeds the level α, *i.e.*

$$L_\alpha\big(G(\overline{x},y)\big) = \big\{G(\overline{x},y)/\mu\big(G(\overline{x},y)\big) \geq \alpha, \alpha \in (0,1)\big\},$$

where α is the least acceptable degree of the required value. For certain degrees α the problem (3.3) can be transformed into the following equivalent form:

$$\left.\begin{aligned} &\min_y G(\overline{x},y) \\ &\text{s.t. } y \in Y(\overline{x}), \\ &\mu(G(\overline{x},y)) \geq \alpha. \end{aligned}\right\} \tag{3.5}$$

If $\overline{x} \in M$ and $\overline{y} \in Y_0(\overline{x})$, then $\overline{y}$ is a minimizer of the problem (3.5).

By the Fritz-John conditions there exist coefficients $\overline{\lambda}, \overline{\beta} = (\overline{\beta}_k, k \in K)$, $\overline{\gamma} = (\overline{\gamma}_l, l \in L_0(\overline{x},\overline{y}))$, and $\overline{v}$ satisfying

$$D_y \mathcal{L}^{(\overline{x},\overline{y})}(\overline{x},\overline{y},\overline{\lambda},\overline{\beta},\overline{\gamma},\overline{v}) = 0 \quad \text{and} \quad \overline{\lambda} + \overline{v} + \sum_{k \in K} |\overline{\beta}_k| + \sum_{l \in L_0(\overline{x},\overline{y})} \overline{\gamma}_l = 1, \tag{3.6}$$

$$\overline{\lambda} \geq 0, \overline{\gamma}_l \geq 0, \overline{v} \geq 0, l \in L_0(\overline{x},\overline{y}),$$

where

$$\mathcal{L}^{(\overline{x},\overline{y})}(x,y,\lambda,\beta,\gamma,v) = \lambda G(\overline{x},y) - v\big(\mu\big(G(\overline{x},y)\big) - \alpha\big) - \sum_{k \in K} \beta_i u_k(x,y) - \sum_{l \in L_0(\overline{x},\overline{y})} \gamma_l v_l.$$

In other words, for $\overline{x} \in M$ and $\overline{y} \in Y_0(\overline{x})$ the set $F(\overline{x},\overline{y}) = \{(\lambda,\beta,\gamma,v) \in R \times R^{|k|} \times R^{|L_0(\overline{x},\overline{y})|} \times R/(\lambda,\beta,\gamma,v)$ satisfies (3.6)$\}$ is nonempty and, furthermore, is also compact.

4 Fritz-John conditions

Firstly, we will give some lemmas, definitions, and a proposition which will be used in the proof of the Fritz-John conditions.

Lemma 2 [9] *Let $x^t \in M$, $y^t \in Y(x^t)$ and $x^t \to \overline{x}$. Then the set $\{\|y^t\|, t \in N\}$ is bounded, and $\overline{y} \in Y(\overline{x})$ whenever $y^t \to \overline{y}$.*

Lemma 3 [9] *Let $l = \phi$, and for $\overline{x} \in M$ let $Y_0(\overline{x}) = \phi$. Then $\overline{x}$ is an interior point of M.*

Definition 4 For $\overline{x} \in M$ define

$$V(\overline{x}) = \bigcup_{\overline{y} \in Y_0(\overline{x})} \left\{ D_x \mathcal{L}^{(\overline{x},\overline{y})}(\overline{x}, \overline{y}, \lambda, \beta, \gamma, \nu)/(\alpha, \beta, \gamma) \in F(\overline{x}, \overline{y}) \right\}. \tag{4.1}$$

Lemma 5 [9] *Let $\overline{x} \in M$. Then the set $V(\overline{x})$ is compact.*

Definition 6 [17] For a set V, we define $\mathrm{Conv}(V)$ to denote the convex hull of V, *i.e.* $x \in \mathrm{Conv}(V)$ if and only if

$$x = \sum_{i=1}^{n} \lambda_i x_i, \quad x_i \in V, \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, \tag{4.2}$$

i.e. $\mathrm{Conv}(V)$ consists of all finite convex combinations of the elements of V.

Lemma 7 [18] *Let $V \subset R^n$ be a nonempty compact set. Then there exists a $\xi \in R^n$ with $s^T \xi > 0$ for all $s \in V$ if and only if $0 \notin \mathrm{Conv}(V)$.*

Lemma 8 [19] *Let I_1, I_2 be finite index sets and $s^i \in R^n$, $i \in I_1$, and $z^j \in R^n$, $j \in I_2$. Then either* (i) *or* (ii) *holds.*

  (i) *There are real numbers a_i, $i \in I_1$, $b_j \geq 0$, $j \in I_2$, satisfying*

$$\begin{aligned}
&\sum_{i \in I} a_i s^i + \sum_{j \in I_2} b_j z^j = 0, \\
&\sum_{i \in I_1} |a_i| + \sum_{j \in I_2} b_j = 1.
\end{aligned} \tag{4.3}$$

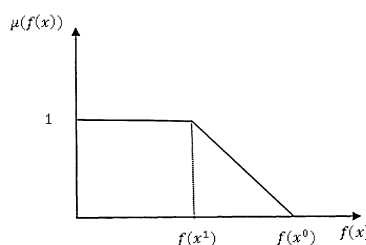 (ii) *The set $\{s^i, i \in I_1\}$ is linearly independent and there exists a $\xi \in R^n$ with*

$$\left(s^i\right)^T \xi = 0, \quad i \in I_1, \qquad \left(z^j\right)^T \xi > 0, \quad j \in I_2. \tag{4.4}$$

Proposition 9 [9] *For $t \in R$ and $y \in R^r$, we define the following functions:*

$$\overline{u}_k(t, y) = u_k(\overline{x} + t\xi, y), \quad k \in K,$$

$$\overline{v}_l(t, y) = v_l(\overline{x} + t\xi, y), \quad l \in L,$$

$$\overline{G}(t, y) = G(\overline{x} + t\xi, y).$$

Then the following conditions are satisfied.

Figure 2 The membership function $\mu(f(x))$.

(i) *The set $\{D\overline{u}_k(0,\overline{y}), k \in K\}$ $(= \{[D_x u_k(\overline{x},\overline{y})\xi, D_y u_k(\overline{x},\overline{y})], k \in K\})$ is linearly independent.*

(ii) *There is a $w \in R^{r+1}$ satisfying*

$$\left.\begin{aligned} D\overline{G}(0,\overline{y})w &> 0, \\ D\overline{u}_k(0,\overline{y})w &= 0, k \in K, \\ D\overline{v}_k(0,\overline{y})w &< 0, l \in L_0(E\overline{x},\overline{y}). \end{aligned}\right\} \tag{4.5}$$

The fuzzy requirements of the problem (3.1) can be quantified by electing a membership function $\mu(f(x))$ (Figure 2) which is differentiable in the open interval $f(x^1) < f(x) < f(x^0)$ where $\mu(f(x))$ is defined by

$$\mu\big(f(x)\big) = \begin{cases} 1, & f(x) \leq f(x^1), \\ \frac{f(x)-f(x^1)}{f(\overline{x}^0)-f(\overline{x}^1)}, & f(x^1) \leq f(x) \leq f(x^0), \\ 0, & f(x) \geq f(x^0), \end{cases} \tag{4.6}$$

where $f(x^0)$ and $f(x^1)$ denote the values of the objective function of the problem (3.1) with the degree of membership function 0 and 1, respectively, *i.e.*, $f(x^0)$ is an undesirable value and $f(x^1)$ is a desirable value of the objective function $f(x)$. For a certain degree α_1 the defuzzification of the problem (3.1) is

$$\min_x f(x) \text{ s.t. } x \in M,$$

$$\text{where } M = \left\{ \begin{aligned} &x \in R^n/h_i(x) = 0, i = 1,\ldots,m, G(x,y) \geq 0 \\ &\qquad\qquad \text{for all } y \in Y(x) \end{aligned} \right\},$$

$$Y(x) = \big\{ y \in R^r/u_k(x,y) = 0, k \in K, v_l(x,y) \geq 0, l \in L \big\}, \tag{4.7}$$

$$\mu\big(f(x)\big) \geq \alpha_1, \quad \alpha_1 \in (0,1),$$

$$\mu\big(G(\overline{x},y)\big) \geq \alpha, \quad \alpha \in (0,1).$$

Theorem 10 *Let $\overline{x}$ be a local minimizer of the problem* (4.7). *Then either $Y_0(\overline{x}) \neq \phi$ and there exist*

$$\left.\begin{aligned} y^j &\in Y_0(\overline{x}), j = 1,\ldots,p, \\ (\lambda^j,\beta^j,\gamma^j,v^j) &\in F(\overline{x},y^j), j = 1,\ldots,p, \\ k \geq 0, \eta \geq 0, \rho_i, i &\in I, \zeta_j \geq 0, j = 1,\ldots,p, \end{aligned}\right\} \tag{4.8}$$

satisfying

$$
\left.
\begin{aligned}
k + \eta + \sum_{i \in I} |\rho_i| + \sum_{j=1}^{p} \zeta_j > 0 \\
\text{as well as} \\
kDf(\overline{x}) - \eta D(\mu(f(\overline{x}) - \alpha_1)) - \sum_{i \in I} \rho_i Dh_i(x) \\
- \sum_{j=1}^{p} \zeta_j D_x \mathcal{L}^{(\overline{x}, y^j)}(\overline{x}, y^j, \lambda^j, \beta^j, \gamma^j, \nu^j) = 0,
\end{aligned}
\right\}
\tag{4.9}
$$

or $Y_0(\overline{x}) = \phi$ *and there exist* $k \geq 0$, $\eta \geq 0$, ρ_i, $i \in I$, *satisfying*

$$
\left.
\begin{aligned}
k + \eta + \sum_{i \in I} |\rho_i| > 0, \\
kDf(\overline{x}) - \eta D(\mu(f(\overline{x}) - \alpha_1)) - \sum_{i \in I} \rho_i Dh_i(x) = 0.
\end{aligned}
\right\}
\tag{4.10}
$$

Proof Let $\overline{x}$ be a local minimizer of the problem (4.7). We distinguish three cases.

Case 1: The set $\{Dh_i(\overline{x}), i \in I\}$ is linearly dependent. Then we are done by choosing $k = 0$, $\eta = 0$, $\zeta = 0$ in (4.9) (if $Y_0(\overline{x}) \neq \phi$), or $k = 0$, $\eta = 0$ in (4.10) (if $Y_0(\overline{x}) = \phi$) as well as a linear combination $\sum_{i \in I} \rho_i Dh_i(x) = 0$ with $\sum_{i \in I} |\rho_i| > 0$.

Case 2: There exists a $\overline{y} \in Y_0(\overline{x})$ and the set $\{Du_k(\overline{x}, \overline{y}) = 0, k \in K\}$ is linearly dependent. Then there exists a linear combination $\sum_{k \in K} \overline{\beta}_k Du_k(\overline{x}, \overline{y}) = 0$, $\sum_{k \in K} |\overline{\beta}_k| = 1$ and therefore we have $\sum_{k \in K} \overline{\beta}_k D_y u_k(\overline{x}, \overline{y}) = 0$, $(0, \beta, 0, 0) \in F(\overline{x}, \overline{y})$ (with $\overline{\beta} = (\overline{\beta}_k, k \in K)$), and

$$
D_x \mathcal{L}^{(\overline{x}, \overline{y})}(\overline{x}, \overline{y}, 0, 0, \beta, 0) = \sum_{k \in K} \overline{\beta}_k Du_k(\overline{x}, \overline{y}) = 0.
\tag{4.11}
$$

Case 3: Neither Case 1 nor Case 2 holds. Then the proof is similar to Theorem 1.1 in [9]. □

5 A constraint qualification

Definition 11 The Mangasarian-Fromovitz constraint qualification (MFCQ) of the problem (4.7) is said to hold at $\overline{x}$ if:

(1) the set $\{Dh_i(\overline{x}), i \in I\}$ is linearly independent and

(2) there exists a $\varpi \in R^n$ such that

$$
\left.
\begin{aligned}
\text{for all } i \in I : Dh_i(\overline{x})\varpi = 0, \\
\text{for all } \overline{y} \in Y_0(\overline{x}) : D_x \mathcal{L}^{(\overline{x}, \overline{y})}(\overline{x}, \overline{y}, \overline{\lambda}, \overline{\beta}, \overline{\gamma}, \overline{\nu})\varpi > 0, \\
\text{for all } (\overline{\lambda}, \overline{\beta}, \overline{\gamma}, \overline{\nu}) \in F(\overline{x}, \overline{y}).
\end{aligned}
\right\}
\tag{5.1}
$$

Theorem 12 *Let $\overline{x}$ be a local minimizer of the problem* (4.7) *and* (MFCQ) *be satisfied. Then either* $Y_0(\overline{x}) \neq \phi$ *and there exist*

$$
\left.
\begin{aligned}
y^j \in Y_0(\overline{x}), j = 1, \ldots, p, \\
(\lambda^j, \beta^j, \gamma^j, \nu^j) \in F(\overline{x}, y^j), j = 1, \ldots, p, \\
k \geq 0, \eta \geq 0, \rho_i, i \in I, \zeta_j \geq 0, j = 1, \ldots, p,
\end{aligned}
\right\}
\tag{5.2}
$$

satisfying

$$
\left.
\begin{aligned}
k + \eta + \sum_{i \in I} |\rho_i| + \sum_{j=1}^{p} \zeta_j > 0 \\
\text{as well as} \\
kDf(\overline{x}) - \eta D(\mu(f(\overline{x}) - \alpha_1)) - \sum_{i \in I} \rho_i Dh_i(x) \\
- \sum_{j=1}^{p} \zeta_j D_x \mathcal{L}^{(\overline{x}, y^j)}(\overline{x}, y^j, \lambda^j, \beta^j, \gamma^j, \nu^j) = 0,
\end{aligned}
\right\}
\tag{5.3}
$$

or $Y_0(\overline{x}) = \phi$ *and there exist* $k \geq 0$, $\eta \geq 0$, ρ_i, $i \in I$, *satisfying the set*

$$\left. \begin{array}{r} k + \eta + \sum_{i \in I} |\rho_i| > 0, \\ kDf(\overline{x}) - \eta D(\mu(f(\overline{x}) - \alpha_1)) - \sum_{i \in I} \rho_i Dh_i(x) = 0. \end{array} \right\} \tag{5.4}$$

The proof is similar to the proof of Theorem 10 if we choose $k = 1$.

Example

$$\widetilde{\min} f(x) = \left(x_1 - \frac{1}{4} \right)^2 + x_2^2,$$

subject to $G(x, y) = y + x_2$,

$V(x, y) = x_1 - y^2.$

The lower level problem is

$$\widetilde{\min}\, G(x, y) = y + x_2,$$

subject to $V(x, y) = x_1 - y^2.$

The optimal solution of the crisp problem is $(x_1, x_2) = (\frac{1}{4}, 0)$, $y = 0$, and $(\lambda, \gamma) = (0, 1)$; and $f^1(\frac{1}{4}, 0) = 0$, $G^1(x, y) = 0$.

Let $f^0(x) = 1$ and $G^0(x, y) = 1$ be undesirable values of the problem, the membership functions of f and G are defined by

$$\mu(f) = \frac{f - f^0}{f^1 - f^0} = -\left[\left(x_1 - \frac{1}{4} \right)^2 + x_2^2 - 1 \right],$$

$$\mu(G) = \frac{G - G^0}{G^1 - G^0} = 1 - y - x_2.$$

The new problem is

$$\min f(x) = \left(x_1 - \frac{1}{4} \right)^2 + x_2^2,$$

subject to $G(x, y) = y + x_2$,

$V(x, y) = x_1 - y^2,$

$-\left[\left(x_1 - \frac{1}{4} \right)^2 + x_2^2 - 1 \right] \geq \alpha_1, \quad \alpha_1 \in (0, 1),$

$1 - y - x_2 \geq \alpha_2, \quad \alpha_2 \in (0, 1).$

The lower level problem is

$$\min G(x, y) = y + x_2,$$

$V(x, y) = x_1 - y^2,$

$1 - y - x_2 \geq \alpha_2,$

then

$$\mathcal{L}(x, y, \alpha_1, \gamma_3, \gamma_5) = \lambda_1(y + x_2) - \gamma_1(x_1 - y^2) - \gamma_2(-y - x_2 + 1 - \alpha_2),$$

$$D_y\mathcal{L}(x, y, \alpha_1, \gamma_3, \gamma_5) = \lambda_1 + 2y\gamma_1 + \gamma_2 = 0, \quad \text{then } y = -\frac{\lambda_1 + \gamma_2}{2\gamma_1}, \lambda_1 + \gamma_1 + \gamma_2 = 1.$$

Since

$$kDf(\overline{x}) - \gamma_3 D\big(\mu\big(f(\overline{x}) - \alpha_1\big)\big) - \beta_1 D_x\mathcal{L}(x, y, \lambda_1, \gamma_1, \gamma_2) = 0,$$

we have

$$2k\left(x_1 - \frac{1}{4}\right) + 2\gamma_3\left(x_1 - \frac{1}{4}\right) + \beta_1\gamma_1 = 0, \quad x_1 = \frac{1}{4} - \frac{\beta_1\gamma_1}{2k + 2\gamma_3},$$

$$2kx_2 + 2\gamma_3 x_2 - \beta_1(\lambda_1 + \gamma_2) = 0, \quad x_2 = \frac{\beta_1(\lambda_1 + \gamma_2)}{2k + 2\gamma_3};$$

the solution is

$$\left\{ \begin{bmatrix} \beta_1 = 1.0, k = 0.0, \gamma_3 = 0.583, \lambda_1 = 0.2, \\ \gamma_1 = 0.7, \gamma_2 = 0.1, \\ \alpha_1 = 0.21706, \alpha_2 = 0.86272, x_2 = 0.257, \\ x_1 = 0.0457, y = -0.21428 \end{bmatrix}, f = 0.107662, G = 0.043 \right\}.$$

6 Conclusion

In this work, we discussed a fuzzy semi-infinite optimization problem, by considering that the minimum of the objective function is fuzzy $(\widetilde{\min})$. The Fritz-John conditions and the constraint qualification are discussed for this problem. Finally, an illustrative example is given to clarify the results.

References
 1. Zimmermann, H-J: Description and optimization of fuzzy systems. Int. J. Gen. Syst. **2**, 209-215 (1976)
 2. Zimmermann, H-J: Fuzzy mathematical programming. Comput. Oper. Res. **10**, 291-298 (1983)
 3. Zimmermann, H-J: Fuzzy programming and linear programming with several objective functions. Fuzzy Sets Syst. **1**, 45-55 (1978)
 4. Cantão, LAP, Yamakami, A: Nonlinear programming with fuzzy parameters: theory and applications. In: Mohammedian, M (ed.) Proceedings of CIMCA 2003. ISBN:1740880684
 5. Jameed, AF, Sadeghi, A: Solving nonlinear programming problem in fuzzy environment. Int. J. Contemp. Math. Sci. **7**(4), 159-170 (2012)
 6. Luhandjula, MK: Fuzzy mathematical programming: theory, applications and extension. J. Uncertain Syst. **1**(2), 124-136 (2007)
 7. Reemsten, R, Ruckmann, J-J (eds.): Semi-Infinite Programming. Kluwer Academic, Boston (1998)
 8. Hettich, R, Kortank, KO: Semi-infinite programming: theory methods, and applications. SIAM Rev. **35**, 380-429 (1993)
 9. Jongen, HT, Ruckmann, J-J, Stein, O: Generalized semi-infinite optimization: a first order optimality condition and examples. Math. Program. **83**, 145-158 (1998)
10. Weber, G-W: Generalized semi-infinite optimization: on some foundation. Vyčisl. Tehnol. **4**, 41-61 (1999)

11. Stein, O, Still, G: On generalized semi-infinite optimization and bi-level optimization. Eur. J. Oper. Res. **142**, 444-462 (2002)
12. Gunzel, H, Jongen, HT, Stein, O: Generalized semi-infinite programming: on generic local minimizers. J. Glob. Optim. **42**(3), 413-421 (2008)
13. Jongen, HT, Twilt, F, Weber, G-W: Semi infinite optimization: structure and feasibility of the feasible set. J. Optim. Theory Appl. **72**, 529-552 (1992)
14. Rukmann, JJ: On linear and liberalized generalized semi-infinite optimization problem. Ann. Oper. Res. **101**, 191-208 (2001)
15. Ruckmann, J-J, Shapiro, A: First-order optimality conditions in generalized semi-infinite programming. J. Optim. Theory Appl. **101**, 677-691 (1999)
16. Stein, O, Still, G: On optimality conditions for generalized semi- infinite programming problems. J. Optim. Theory Appl. **104**, 443-458 (2000)
17. Bazaraa, MS, Shetty, CM: Nonlinear Programming Theory and Algorithms. Wiley, New York (1979)
18. Cheney, EW: Introduction to Approximation Theory. McGraw-Hill, New York (1966)
19. Jongen, HT, Jonker, P, Twilt, F: Nonlinear Optimization in R^n. II. Transversality, Flows, Parametric Aspects. Peter Lang, Frankfurt am Main (1986)