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Stability of weak solutions for the large-scale atmospheric equations

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Abstract
In this paper, we consider the Navier-Stokes equations and temperature equation arising from the evolution process of the atmosphere. Under certain assumptions imposed on the initial data, we show the L^1-stability of weak solutions for the atmospheric equations. Some ideas and delicate estimates are introduced to prove these results.

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1 Introduction and main results

In this paper, we consider the atmospheric motion model under the constant external force and without the effects of topography, and the aerosphere is regarded as a spherical shell encompassing the earth. We introduce a moving frame running with the earth, (λ,θ,p), where $\lambda\in[0,2\pi]$ is the longitude, $\theta\in[0,\pi]$ is the colatitude, and $p\in[p_0,p_s]$ is the atmospheric pressure, which can be used instead of the geocentric distance r because it is strictly monotonically decreasing function for r, where p_s is the atmospheric pressure on the surface of the earth, and $p_0>0$ is the atmospheric pressure at a certain isobaric surface. In the coordinate system consisting of the moving frame and time, the atmospheric state functions are defined by the atmospheric horizontal velocity $V=(v_\lambda,v_\theta)$, the rate of pressure $w=\frac{dp}{dt}$, the temperature T, and the geopotential Φ. All of them satisfy the following system:

$$\begin{cases} \frac{\partial V}{\partial t}+(V\cdot\nabla)V+w\frac{\partial V}{\partial p}+(2\omega\cos\theta+\frac{\cot\theta}{a}v_\lambda)\beta V+\nabla\Phi=\mu_1\Delta V+\nu_1\frac{\partial}{\partial p}(\alpha^2(p)\frac{\partial V}{\partial p}),\\ \frac{\partial T}{\partial t}+(V\cdot\nabla)T+w\frac{\partial T}{\partial p}-\frac{c_0^2 w}{Rp}=\frac{c_0^2\mu_2}{R^2}\Delta T+\frac{c_0^2\nu_2}{R^2}\frac{\partial}{\partial p}(\alpha^2(p)\frac{\partial T}{\partial p})+\frac{\Psi}{c_p},\\ \nabla\cdot V+\frac{\partial w}{\partial p}=0,\\ \frac{\partial\Phi}{\partial p}+\frac{RT}{p}=0, \end{cases}\tag{1.1}$$

with the initial data

$$V|_{t=0}=V_0,\qquad T|_{t=0}=T_0,\qquad \omega|_{t=0}=\omega_0,\tag{1.2}$$

where ω is the angular velocity of the earth; c_0, c_p, and R are the thermodynamics parameters; μ_i and ν_i, $i=1,2$, are the diffusion coefficients; $\alpha(p)\in C[p_0,p_s]$ satisfying $\alpha(p)\geq C_\alpha>0$, namely the diffusion is related to the atmospheric pressure; Ψ is the diabatic heating of the atmosphere, which is a function of (λ,θ,p) and stands for the effect of the constant external force on the atmospheric system. We have $\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in $2\omega\cos\theta\beta V = 2\omega\cos\theta\vec{k}\wedge V$, which denotes the Coriolis force on the atmosphere. The differential operators grad = ∇ and div = $\nabla\cdot$ on the spherical surface have the following form:

$$\begin{cases} \nabla = (\frac{1}{a\sin\theta}\frac{\partial}{\partial\lambda}, \frac{1}{a}\frac{\partial}{\partial\theta}), \\ \nabla\cdot V = \frac{1}{a\sin\theta}\frac{\partial v_\lambda}{\partial\lambda} + \frac{1}{a\sin\theta}\frac{\partial(\sin\theta v_\theta)}{\partial\theta}, \\ \Delta = \frac{1}{a^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{a^2\sin^2\theta}\frac{\partial^2}{\partial\lambda^2}, \end{cases} \tag{1.3}$$

where a is the radius of the earth. The vertical scale of the atmosphere is very much smaller compared with the radius of the earth, so the geocentric distance r is replaced by the radius of the earth in the differential operators. The above equations are studied on $\Omega\times[0,M] := [0,2\pi]\times[0,\pi]\times[p_0,p_s]\times[0,M]$, where $M > 0$.

The boundary conditions without the relief are: All the functions are 2π periodical w.r.t. λ, π periodical w.r.t. θ, and

$$\begin{cases} \frac{v_\lambda}{\partial p}|_{p=p_0} = \frac{v_\theta}{\partial p}|_{p=p_0} = \frac{T}{\partial p}|_{p=p_0} = w|_{p=p_0} = 0, \\ v_\lambda|_{p=p_s} = v_\theta|_{p=p_s} = \frac{T}{\partial p}|_{p=p_s} = w|_{p=p_s} = \Phi|_{p=p_s} = 0. \end{cases} \tag{1.4}$$

There are many important results achieved on the atmospheric problem. Zeng [1], Li and Chou [2] have made important progress on the formulation and the analysis of the models. For different research purposes, different atmospheric models have been investigated by Pedlosky [3], Washington and Parkinson [4], Lions *et al.* [5–8] and references therein. Recently, Chepzhov and Vishik [9] introduced the atmospheric equations considered in this paper. Huang and Guo [10] proved the existence of the weak solutions to the atmospheric equations by the basic differential equation theory and the existence of the corresponding trajectory attractors, from which the existence of the atmospheric global attractors follows. Furthermore, Huang and Guo [11] studied the model of the climate for weather forecasts in which the pressing force of topography on atmosphere and the divergent effect of airflow are included, and they proved the existence and the asymptotic behaviors of the weak solution.

The rest of the paper is as follows. In Section 2, the main results about the L^1-stability of weak solutions to the Navier-Stokes equations and temperature equation are stated. In Section 3, we will give several important *a priori* estimates. Then we will justify the stability of the weak solutions in Section 4. Finally, in Section 5, the conclusion will be given.

2 Main results

The L^1-stability theory of weak solutions to (1.1) will be considered, and there is a simple version of system (1.1). From $(1.1)_{3,4}$ and the boundary conditions (1.4), we have

$$\Phi(p) = R\int_p^{p_s}\frac{1}{s}T(s)\,ds \tag{2.1}$$

and

$$w(p) = \nabla\cdot\int_p^{p_s}V(s)\,ds, \tag{2.2}$$

which implies

$$\nabla \cdot \int_{p_0}^{p_s} V(s)\,ds = 0. \tag{2.3}$$

Substitute (2.1) and (2.2) into $(1.1)_{1,2}$ and define the unknown function $U := (V, T)$, then we have the simplification of system (1.1):

$$\begin{cases} \frac{\partial V}{\partial t} + (V \cdot \nabla)V + \nabla \cdot \int_p^{p_s} V(s)\,ds \frac{\partial V}{\partial p} + (2\omega\cos\theta + \frac{\cot\theta}{a}\nu_\lambda)\beta V + R\nabla \int_p^{p_s} \frac{1}{s} T(s)\,ds \\ \quad = \mu_1 \Delta V + \nu_1 \frac{\partial}{\partial p}(\alpha^2(p)\frac{\partial V}{\partial p}), \\ \frac{\partial T}{\partial t} + (V \cdot \nabla)T + \nabla \cdot \int_p^{p_s} V(s)\,ds \frac{\partial T}{\partial p} - \frac{c_0^2}{Rp}\nabla \cdot \int_p^{p_s} V(s)\,ds \\ \quad = \frac{c_0^2\mu_2}{R^2}\Delta T + \frac{c_0^2\nu_2}{R^2}\frac{\partial}{\partial p}(\alpha^2(p)\frac{\partial T}{\partial p}) + \frac{\Psi}{c_p}, \\ U|_{t=0} = (\nu_\lambda, \nu_\theta, T)|_{t=0} = U_0 = (\nu_{\lambda 0}, \nu_{\theta 0}, T_0), \\ U(\lambda, \theta, p) = U(\lambda + 2\pi, \theta, p) = U(\lambda, \theta + \pi, p), \\ \frac{\partial U}{\partial p}|_{p=p_0} = 0, \qquad V|_{p=p_s} = 0, \qquad \frac{\partial T}{\partial p}|_{p=p_s} = 0. \end{cases} \tag{2.4}$$

Denote

$$L(U) := \left(-\mu_1 \Delta V - \nu_1 \frac{\partial}{\partial p}\left(\alpha^2(p)\frac{V}{\partial p}\right), -\frac{c_0^2\mu_2}{R^2}\Delta T - \frac{c_0^2\nu_2}{R^2}\frac{\partial}{\partial p}\left(\alpha^2(p)\frac{\partial T}{\partial p}\right)\right), \tag{2.5}$$

$$\begin{aligned} B(U,U) := \Bigg(&(V \cdot \nabla)V + \nabla \cdot \int_p^{p_s} V(s)\,ds \frac{\partial V}{\partial p} \\ &+ \left(2\omega\cos\theta + \frac{\cot\theta}{a}\nu_\lambda\right)\beta V + R\nabla \int_p^{p_s} \frac{1}{s} T(s)\,ds, \\ &(V \cdot \nabla)T + \nabla \cdot \int_p^{p_s} V(s)\,ds \frac{\partial T}{\partial p} \\ &- \frac{c_0^2}{Rp}\nabla \cdot \int_p^{p_s} V(s)\,ds \Bigg)^{\mathrm{T}} \end{aligned} \tag{2.6}$$

and

$$F := \left(0, \frac{\Psi}{c_p}\right), \tag{2.7}$$

then we show the definition of weak solutions of system (2.4).

Definition 2.1 (Definition of weak solution) For any $M > 0$, U is said to be a weak solution of system (2.4) on $\Omega \times [0, M]$, if U has the following regularities:

$$U \in L^\infty\big(0, M; L^2(\Omega)\big) \cap L^2\big(0, M; H^1(\Omega)\big), \tag{2.8}$$

and satisfies the equations in the sense of distributions

$$U_t + L(U) + B(U,U) = F, \quad \text{in } D'\big((0, M) \times \Omega\big). \tag{2.9}$$

Namely, we have for all $\varphi = (\varphi_{\nu_\lambda}, \varphi_{\nu_\theta}, \varphi_T) = (\varphi_V, \varphi_T) \in C^\infty(0, M; C_0^\infty(\Omega))$ and $\varphi(M, \cdot) = 0$,

$$\big(U_0, \varphi(0, \cdot)\big) + \int_0^M (U, \varphi_t)\,dt - \int_0^M \big(a(U, \varphi) + b(U, U, \varphi) - (F, \varphi)\big)\,dt = 0, \tag{2.10}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$,

$$\begin{aligned} a(U,\varphi) &= \mu_1 \int_\Omega \nabla V \cdot \nabla \varphi_V \, d\sigma \, dp + \frac{c_0^2 \mu_2}{R^2} \int_\Omega \nabla T \cdot \nabla \varphi_T \, d\sigma \, dp \\ &\quad + \nu_1 \int_\Omega \alpha^2(p) \frac{\partial V}{\partial p} \cdot \frac{\partial \varphi_V}{\partial p} \, d\sigma \, dp \\ &\quad + \frac{c_0^2 \nu_2}{R^2} \int_\Omega \alpha^2(p) \frac{\partial T}{\partial p} \frac{\partial \varphi_T}{\partial p} \, d\sigma \, dp \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} b(U,U,\varphi) &= \int_\Omega \big((V\cdot\nabla) V \cdot \varphi_V + (V\cdot\nabla) T \varphi_T \big) \, d\sigma \, dp \\ &\quad + \int_\Omega \left(\left(\nabla \cdot \int_p^{p_s} V(s) \, ds \right) \frac{\partial V}{\partial p} \cdot \varphi_V + \left(\nabla \cdot \int_p^{p_s} V(s) \, ds \right) \frac{\partial T}{\partial p} \varphi_T \right) d\sigma \, dp \\ &\quad + \int_\Omega \left(R \nabla \int_p^{p_s} \frac{1}{s} T(s) \, ds \cdot \varphi_V - \frac{c_0^2}{Rp} \left(\nabla \cdot \int_p^{p_s} V(s) \, ds \right) \varphi_T \right) d\sigma \, dp \\ &\quad + \int_\Omega \left(2\omega \cos\theta + \frac{\cot\theta}{a} \nu_\lambda \right) (\nu_\theta \varphi_{\nu_\lambda} - \nu_\lambda \varphi_{\nu_\theta}) \, d\sigma \, dp, \end{aligned} \tag{2.12}$$

where

$$\int_\Omega f \, d\sigma \, dp := \int_{p_0}^{p_s} \int_0^\pi \int_0^{2\pi} f a^2 \sin\theta \, d\lambda \, d\theta \, dp. \tag{2.13}$$

Then we can state the main results of the present paper as follows.

Theorem 2.1 (Stability of weak solutions) *For any $M > 0$, let $U^n = (\nu_\lambda^n, \nu_\theta^n, T^n) = (V^n, T^n)$ be a sequence of weak solution of system* (2.4) *subject to the initial data*

$$U^n|_{t=0} = (\nu_\lambda^n, \nu_\theta^n, T^n)|_{t=0} = U_0^n = (\nu_{\lambda 0}^n, \nu_{\theta 0}^n, T_0^n), \tag{2.14}$$

and U_0^n be such that

$$U_0^n \to U_0 = (\nu_{\lambda 0}, \nu_{\theta 0}, T_0) \in L^1(\Omega), \tag{2.15}$$

where $U_0 \in L^2(\Omega)$ and satisfies the following upper bound uniformly with respect to $n \in \mathbb{N}$:

$$\int_\Omega |U_0^n|^2 \, d\sigma \, dp = \int_\Omega \left(|\nu_{\lambda 0}^n|^2 + |\nu_{\theta 0}^n|^2 + |T_0^n|^2 \right) d\sigma \, dp < C, \tag{2.16}$$

where $C > 0$ denotes a constant, and we assume that $\Psi \in H^{-2}(\Omega)$.

Then, up to a subsequence, still denoted by the same symbol, we have

$$U^n \to U \in L^2\big(0,T; L^2(\Omega)\big), \tag{2.17}$$

where $U = (\nu_\lambda, \nu_\theta, T)$ is a weak solution of system (2.4) *with the initial data $U_0 = (\nu_{\lambda 0}, \nu_{\theta 0}, T_0)$.*

Remark 2.1 Note that from (2.17), we can find that if

$$U_0^n \to U_0 \in L^1(\Omega), \tag{2.18}$$

then

$$U^n \to U \in L^1\bigl(0,T;L^1(\Omega)\bigr), \tag{2.19}$$

which is the L^1-stability of weak solutions for the system.

Remark 2.2 Furthermore if

$$U_0^n \to U_0 \quad \text{a.e.} \tag{2.20}$$

and

$$U_0^n, U_0 \in L^2(\Omega), \tag{2.21}$$

from the Egorov theorem, we have for $\epsilon > 0$ the following. Let $\delta = \epsilon^2$, then there exists a domain $\Omega_\delta \subset \Omega$, such that $|\Omega/\Omega_\delta| < \delta$, and for all $(\lambda, \theta, p) \in \Omega_\delta$, $\exists N > 0$, for $\forall n > N$, we have

$$\bigl|U_0^n \to U_0\bigr| < \epsilon. \tag{2.22}$$

Then for $\forall n > N$ we have

$$\int_\Omega \bigl|U_0^n - U_0\bigr|\,d\sigma\,dp = \int_{\Omega_\delta} \bigl|U_0^n - U_0\bigr|\,d\sigma\,dp + \int_{\Omega/\Omega_\delta} \bigl|U_0^n - U_0\bigr|\,d\sigma\,dp, \tag{2.23}$$

where we have

$$\int_{\Omega_\delta} \bigl|U_0^n - U_0\bigr|\,d\sigma\,dp \le |\Omega_\delta|\epsilon \le C\epsilon \tag{2.24}$$

and

$$\begin{aligned}\int_{\Omega/\Omega_\delta} \bigl|U_0^n - U_0\bigr|\,d\sigma\,dp &\le \biggl(\int_{\Omega/\Omega_\delta} 1\,d\sigma\,dp\biggr)^{\frac{1}{2}} \biggl(\int_{\Omega/\Omega_\delta} \bigl|U_0^n - U_0\bigr|^2\,d\sigma\,dp\biggr)^{\frac{1}{2}} \\ &\le C\delta^{\frac{1}{2}} = C\epsilon,\end{aligned} \tag{2.25}$$

and from (2.24) and (2.25), we can find

$$U_0^n \to U_0 \in L^1(\Omega), \tag{2.26}$$

thus, we have from Remark 2.1

$$U^n \to U \in L^1\bigl(0,T;L^1(\Omega)\bigr), \tag{2.27}$$

which implies that

$$U^n \to U \quad \text{a.e.,} \tag{2.28}$$

which means the weak solutions are stable almost everywhere.

3 The *a priori* estimates

Next, we will give the *a priori* estimates for the weak solution U^n to system (2.4). Firstly, from a direct calculation, we can establish the following lemma; we omit the proof.

Lemma 3.1

$$\int_\Omega \nabla\phi \cdot \nabla\psi \, d\sigma \, dp = -\int_\Omega \Delta\phi \cdot \psi \, d\sigma \, dp, \tag{3.1}$$

where ϕ and ψ can be vector-valued functions, or the scalar functions,

$$\int_\Omega \phi \cdot \nabla\psi \, d\sigma \, dp = -\int_\Omega \nabla \cdot \phi\psi \, d\sigma \, dp, \tag{3.2}$$

where ϕ is a vector-valued function, and ψ is a scalar function.

Then we have the usual energy inequality as follows.

Lemma 3.2 *Let $T > 0$. Under the assumptions of Theorem* 2.1, *we have for the weak solution U^n to system* (2.4)

$$\|U^n\|^2_{L^2(\Omega)} + \int_0^t \|U^n\|^2_{H^1(\Omega)} \, d\tau \le C\big(1 + \|\Psi\|^2_{H^{-1}(\Omega)}\big), \quad t \in [0, M], \tag{3.3}$$

where $C > 0$ denotes a constant dependent on the initial data and time M and independent of n.

Proof Take the inner product of (2.4) with U^n, integrating on Ω, we have

$$\big(U^n_t, U^n\big) + \big(L\big(U^n\big), U^n\big) + \big(B\big(U^n, U^n\big), U^n\big) = \big(F, U^n\big), \tag{3.4}$$

and using the boundary conditions, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \big(V^{n2} + T^{n2}\big)\, d\sigma \, dp + \mu_1 \int_\Omega \big|\nabla V^n\big|^2 \, d\sigma \, dp + \nu_1 \int_\Omega \alpha^2(p)\left(\frac{\partial V^n}{\partial p}\right)^2 d\sigma \, dp \\
&\quad + \frac{c_0^2\mu_2}{R^2}\int_\Omega \big|\nabla T^n\big|^2 \, d\sigma \, dp + \frac{c_0^2\nu_2}{R^2}\int_\Omega \alpha^2(p)\left(\frac{\partial T^n}{\partial p}\right)^2 d\sigma \, dp = \int_\Omega \frac{\Psi}{c_p} T^n \, d\sigma \, dp,
\end{aligned} \tag{3.5}$$

which implies

$$\begin{aligned}
\frac{d}{dt}\big\|U^n\big\|^2_{L^2(\Omega)} + C\big\|U^n\big\|_{H^1(\Omega)} &\le C\|\Psi\|^2_{H^{-1}(\Omega)} + \epsilon C\big\|T^n\big\|^2_{H^1(\Omega)} \\
&\le C\|\Psi\|^2_{H^{-1}(\Omega)} + \epsilon C\big\|U^n\big\|^2_{H^1(\Omega)},
\end{aligned} \tag{3.6}$$

where $\epsilon > 0$ is a small constant such that we have

$$\frac{d}{dt}\|U^n\|^2_{L^2(\Omega)} + C\|U^n\|^2_{H^1(\Omega)} \le C\|\Psi\|^2_{H^{-1}(\Omega)}, \tag{3.7}$$

after the integration with respect to $t \in [0, M]$, we have

$$\begin{aligned}\|U^n\|^2_{L^2(\Omega)} + \int_0^t \|U^n\|^2_{H^1(\Omega)}\,d\tau &\le \|U_0^n\|^2_{L^2(\Omega)} + C\|\Psi\|^2_{H^{-1}(\Omega)} \\ &\le C\bigl(1 + \|\Psi\|^2_{H^{-1}(\Omega)}\bigr),\end{aligned} \tag{3.8}$$

where $C > 0$ denotes a constant dependent of the initial data and time M and independent of n. □

4 Proof of main results

With the help of the *a priori* estimates in (3.3), we have the following estimates for the sequence of weak solutions U^n:

$$U^n \in L^\infty\bigl(0, M; L^2(\Omega)\bigr) \cap L^2\bigl(0, M; H^1(\Omega)\bigr), \tag{4.1}$$

then we will prove the main results, in order to address the convergence of sequence of the weak solution; a lemma of the compactness result will be given first.

Lemma 4.1 (Lion's compactness result) *Suppose E_0, E, E_1 are Banach spaces, $E_0 \hookrightarrow\hookrightarrow E \hookrightarrow E_1$, which means E_0 is compactly embedded in E, E is embedded in E_1, and $p_1 > 1$. Denote*

$$W_{2,p_1}(0, M; E_0, E_1) = \bigl\{\psi \mid \psi \in L^2(0, M; E_0), \psi_t \in L^{p_1}(0, M; E_1)\bigr\} \tag{4.2}$$

as the Banach space with the norm

$$\|\psi\|_{W_{2,p_1}} = \|\psi\|_{L^2(0,M;E_0)} + \|\psi\|_{L^{p_1}(0,M;E_1)}, \tag{4.3}$$

then

$$W_{2,p_1}(0, M; E_0, E_1) \hookrightarrow\hookrightarrow L^2(0, M; E). \tag{4.4}$$

Then we will give the proof of the stability of the weak solutions.

Lemma 4.2 *Let U^n be the weak solution sequence of system* (2.4). *Then, up to a subsequence, we have*

$$U^n \to U, \quad \text{in } L^2\bigl(0, M; L^2(\Omega)\bigr) \tag{4.5}$$

and

$$\int_0^M \bigl(U^n, \varphi_t\bigr)\,dt \to \int_0^M (U, \varphi_t)\,dt, \tag{4.6}$$

for all $\varphi = (\varphi_{v_\lambda}, \varphi_{v_\theta}, \varphi_T) = (\varphi_V, \varphi_T) \in C^\infty(0, M; C_0^\infty(\Omega))$ and $\varphi(M, \cdot) = 0$.

Proof From (4.1), we have the following estimates for the test function $\phi = (\phi_{v_\lambda}, \phi_{v_\theta}, \phi_T) = (\phi_V, \phi_T) \in H^2(\Omega)$:

$$\begin{aligned}
\bigl(L(U^n),\phi\bigr) &= \mu_1 \int_\Omega \nabla V^n \cdot \nabla\phi_V \, d\sigma\, dp + \frac{c_0^2\mu_2}{R^2}\int_\Omega \nabla T^n \cdot \nabla \phi_T \, d\sigma\, dp \\
&\quad + \nu_1 \int_\Omega \alpha^2(p)\frac{\partial V^n}{\partial p}\cdot \frac{\partial \phi_V}{\partial p}\, d\sigma\, dp + \frac{c_0^2\nu_2}{R^2}\int_\Omega \alpha^2(p)\frac{\partial T^n}{\partial p}\frac{\partial \phi_T}{\partial p}\, d\sigma\, dp \\
&\le C\|U^n\|_{H^1(\Omega)}\|\phi\|_{H^1(\Omega)} \le C\|U^n\|_{H^1(\Omega)},
\end{aligned} \tag{4.7}$$

which implies

$$\bigl\|L(U^n)\bigr\|_{H^{-2}(\Omega)} \le C\|U^n\|_{H^1(\Omega)}, \tag{4.8}$$

and we have

$$\int_0^M \bigl\|L(U^n)\bigr\|^2_{H^{-2}(\Omega)}\, dt \le C\int_0^M \|U^n\|^2_{H^1(\Omega)}\, dt \le C, \tag{4.9}$$

where $C > 0$ denotes a constant independent of n, namely,

$$L(U^n) \in L^2\bigl(0,M;H^{-2}(\Omega)\bigr). \tag{4.10}$$

Next, we can find that

$$\begin{aligned}
\bigl(B(U^n,U^n),\phi\bigr) &= \int_\Omega \bigl((V^n\cdot\nabla)V^n\cdot\phi_V + (V^n\cdot\nabla)T^n\phi_T\bigr)\, d\sigma\, dp \\
&\quad + \int_\Omega \biggl(\biggl(\nabla\cdot\int_p^{p_s} V^n(s)\, ds\biggr)\frac{\partial V^n}{\partial p}\cdot\phi_V \\
&\quad + \biggl(\nabla\cdot\int_p^{p_s} V^n(s)\, ds\biggr)\frac{\partial T^n}{\partial p}\phi_T\biggr)\, d\sigma\, dp \\
&\quad + \int_\Omega \biggl(R\nabla\int_p^{p_s}\frac{1}{s}T^n(s)\, ds\cdot\phi_V - \frac{c_0^2}{Rp}\biggl(\nabla\cdot\int_p^{p_s} V^n(s)\, ds\biggr)\phi_T\biggr)\, d\sigma\, dp \\
&\quad + \int_\Omega \biggl(2\omega\cos\theta + \frac{\cot\theta}{a}v_\lambda^n\biggr)\bigl(v_\theta^n\phi_{v_\lambda} - v_\lambda^n\phi_{v_\theta}\bigr)\, d\sigma\, dp,
\end{aligned} \tag{4.11}$$

and we have

$$\begin{aligned}
&\int_\Omega \bigl((V^n\cdot\nabla)V^n\cdot\phi_V + (V^n\cdot\nabla)T^n\phi_T\bigr)\, d\sigma\, dp \\
&\quad \le C\biggl(\int_\Omega |\nabla U^n|^2\, d\sigma\, dp\biggr)^{\frac{1}{2}}\biggl(\int_\Omega |U^n|^2|\phi|^2\, d\sigma\, dp\biggr)^{\frac{1}{2}} \\
&\quad \le C\|U^n\|_{H^1(\Omega)}\biggl(\int_\Omega |U^n|^3\, d\sigma\, dp\biggr)^{\frac{1}{3}}\biggl(\int_\Omega |\phi|^6\, d\sigma\, dp\biggr)^{\frac{1}{6}} \\
&\quad \le C\|U^n\|^{\frac{3}{2}}_{H^1(\Omega)}\|U^n\|^{\frac{1}{2}}_{L^2(\Omega)}\|\phi\|_{H^1(\Omega)} \\
&\quad \le C\|U^n\|^{\frac{3}{2}}_{H^1(\Omega)}\|U^n\|^{\frac{1}{2}}_{L^2(\Omega)},
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
&\int_\Omega \left(\left(\nabla\cdot\int_p^{p_s} V^n(s)\,ds\right)\frac{\partial V^n}{\partial p}\cdot\phi_V+\left(\nabla\cdot\int_p^{p_s} V^n(s)\,ds\right)\frac{\partial T^n}{\partial p}\phi_T\right)d\sigma\,dp \\
&\quad=\int_\Omega \nabla\cdot V^n V^n\cdot\phi_V\,d\sigma\,dp+\int_\Omega \nabla\cdot V^n T^n\phi_T\,d\sigma\,dp \\
&\qquad-\int_\Omega\left(\nabla\cdot\int_p^{p_s} V^n(s)\,ds\right)V^n\cdot\frac{\partial\phi_V}{\partial p}\,d\sigma\,dp \\
&\qquad-\int_\Omega\left(\nabla\cdot\int_p^{p_s} V^n(s)\,ds\right)T^n\frac{\partial\phi_T}{\partial p}\,d\sigma\,dp \\
&\quad\le C\left(\int_\Omega\left|\nabla U^n\right|^2 d\sigma\,dp\right)^{\frac12}\left(\int_\Omega\left|U^n\right|^2|\phi|^2\,d\sigma\,dp\right)^{\frac12} \\
&\qquad+C\left(\int_\Omega\left|\nabla U^n\right|^2 d\sigma\,dp\right)^{\frac12}\left(\int_\Omega\left|U^n\right|^2\left|\frac{\partial\phi}{\partial p}\right|^2 d\sigma\,dp\right)^{\frac12} \\
&\quad\le C\left\|U^n\right\|_{H^1(\Omega)}\left(\int_\Omega\left|U^n\right|^3 d\sigma\,dp\right)^{\frac13}\left(\int_\Omega|\phi|^6\,d\sigma\,dp\right)^{\frac16} \\
&\qquad+C\left\|U^n\right\|_{H^1(\Omega)}\left(\int_\Omega\left|U^n\right|^3 d\sigma\,dp\right)^{\frac13}\left(\int_\Omega\left|\frac{\partial\phi}{\partial p}\right|^6 d\sigma\,dp\right)^{\frac16} \\
&\quad\le C\left\|U^n\right\|_{H^1(\Omega)}^{\frac32}\left\|U^n\right\|_{L^2(\Omega)}^{\frac12}\|\phi\|_{H^1(\Omega)}+C\left\|U^n\right\|_{H^1(\Omega)}^{\frac32}\left\|U^n\right\|_{L^2(\Omega)}^{\frac12}\|\phi\|_{H^2(\Omega)} \\
&\quad\le C\left\|U^n\right\|_{H^1(\Omega)}^{\frac32}\left\|U^n\right\|_{L^2(\Omega)}^{\frac12},
\end{aligned}\tag{4.13}$$

$$\begin{aligned}
&\int_\Omega\left(R\nabla\int_p^{p_s}\frac1s T^n(s)\,ds\cdot\phi_V-\frac{c_0^2}{Rp}\left(\nabla\cdot\int_p^{p_s}V^n(s)\,ds\right)\phi_T\right)d\sigma\,dp \\
&\quad\le C\left(\int_\Omega\left|\nabla U^n\right|^2 d\sigma\,dp\right)^{\frac12}\left(\int_\Omega|\phi|^2\,d\sigma\,dp\right)^{\frac12} \\
&\quad\le C\left\|U^n\right\|_{H^1(\Omega)}\|\phi\|_{H^1(\Omega)}\le C\left\|U^n\right\|_{H^1(\Omega)}
\end{aligned}\tag{4.14}$$

and

$$\begin{aligned}
&\int_\Omega\left(2\omega\cos\theta+\frac{\cot\theta}{a}v_\lambda^n\right)\left(v_\theta^n\phi_{v_\lambda}-v_\lambda^n\phi_{v_\theta}\right)d\sigma\,dp \\
&\quad\le C\left(\int_\Omega\left(1+\left|U^n\right|^2\right)d\sigma\,dp\right)^{\frac12}\left(\int_\Omega\left|U^n\right|^2|\phi|^2\,d\sigma\,dp\right)^{\frac12} \\
&\quad\le C\left(1+\left\|U^n\right\|_{L^2(\Omega)}\right)\left(\int_\Omega\left|U^n\right|^3 d\sigma\,dp\right)^{\frac13}\left(\int_\Omega|\phi|^6\,d\sigma\,dp\right)^{\frac16} \\
&\quad\le C\left\|U^n\right\|_{H^1(\Omega)}^{\frac12}\left(1+\left\|U^n\right\|_{L^2(\Omega)}^{\frac32}\right)\|\phi\|_{H^1(\Omega)} \\
&\quad\le C\left\|U^n\right\|_{H^1(\Omega)}^{\frac12}\left(1+\left\|U^n\right\|_{L^2(\Omega)}^{\frac32}\right),
\end{aligned}\tag{4.15}$$

from (4.12)-(4.15), we have

$$\begin{aligned}
\left\|B\left(U^n,U^n\right)\right\|_{H^{-2}(\Omega)}\le{}& C\left\|U^n\right\|_{H^1(\Omega)}^{\frac32}\left\|U^n\right\|_{L^2(\Omega)}^{\frac12}+C\left\|U^n\right\|_{H^1(\Omega)} \\
&+C\left\|U^n\right\|_{H^1(\Omega)}^{\frac12}\left(1+\left\|U^n\right\|_{L^2(\Omega)}^{\frac32}\right)
\end{aligned}\tag{4.16}$$

and

$$\begin{aligned}
&\int_0^M \|B(U^n,U^n)\|_{H^{-2}(\Omega)}^{\frac{4}{3}}\,dt \\
&\quad\le C\int_0^M \Big(\|U^n\|_{H^1(\Omega)}^2\|U^n\|_{L^2(\Omega)}^{\frac{2}{3}} + \|U^n\|_{H^1(\Omega)}^{\frac{4}{3}} \\
&\qquad + \|U^n\|_{H^1(\Omega)}^{\frac{2}{3}}\big(1+\|U^n\|_{L^2(\Omega)}^{\frac{3}{2}}\big)^{\frac{4}{3}}\Big)\,dt \\
&\quad\le C\int_0^M \Big(\|U^n\|_{H^1(\Omega)}^2 + \|U^n\|_{H^1(\Omega)}^{\frac{4}{3}} + \|U^n\|_{H^1(\Omega)}^{\frac{2}{3}}\Big)\,dt \\
&\quad\le C\int_0^M \|U^n\|_{H^1(\Omega)}^2 t + C \le C,
\end{aligned}\tag{4.17}$$

where $C > 0$ denotes a constant independent of n, namely,

$$B(U^n,U^n) \in L^{\frac{4}{3}}\big(0,M;H^{-2}(\Omega)\big).\tag{4.18}$$

Finally, we have

$$(F,\phi) = \int_\Omega \frac{\Psi}{c_p}\phi_T\,d\sigma\,dp \le C\|\Psi\|_{H^{-2}(\Omega)}\|\phi\|_{H^2(\Omega)} \le C\|\Psi\|_{H^{-2}(\Omega)}\tag{4.19}$$

and

$$F \in L^\infty\big(0,M;H^{-2}(\Omega)\big).\tag{4.20}$$

Then we have from (4.10), (4.18), and (4.20)

$$U_t^n \in L^{\frac{4}{3}}\big(0,M;H^{-2}(\Omega)\big),\tag{4.21}$$

which together with Lemma 4.1 and $U^n \in L^2(0,M;H^1(\Omega))$ gives

$$U^n \to U \in L^2\big(0,M;L^2(\Omega)\big),\tag{4.22}$$

and using (4.22), we can prove (4.6) holds. □

Lemma 4.3 *Let U^n be the weak solution sequence of system* (2.4). *Then, up to a subsequence, we have*

$$\int_0^M a\big(U^n,\varphi\big)\,dt \to \int_0^M a(U,\varphi)\,dt,\tag{4.23}$$

for all $\varphi = (\varphi_{v_\lambda},\varphi_{v_\theta},\varphi_T) = (\varphi_V,\varphi_T) \in C^\infty(0,M;C_0^\infty(\Omega))$ *and* $\varphi(M,\cdot) = 0$.

Proof As $\frac{\partial U^n}{\partial\lambda}$, $\frac{\partial U^n}{\partial\theta}$ and $\frac{\partial U^n}{\partial p} \in L^2(0,M;L^2(\Omega))$, thus, we have

$$\frac{\partial U^n}{\partial\lambda} \rightharpoonup \frac{\partial U}{\partial\lambda},\qquad \frac{\partial U^n}{\partial\theta} \rightharpoonup \frac{\partial U}{\partial\theta},\qquad \frac{\partial U^n}{\partial p} \rightharpoonup \frac{\partial U}{\partial p} \in L^2\big(0,M;L^2(\Omega)\big),\tag{4.24}$$

which means that the sequences converge weakly; then we have

$$\begin{aligned}
&\int_0^M a(U^n,\varphi)\,dt \\
&= \mu_1 \int_0^M \int_\Omega \nabla V^n \cdot \nabla\varphi_V \,d\sigma\,dp\,dt + \frac{c_0^2\mu_2}{R^2}\int_0^M \int_\Omega \nabla T^n \cdot \nabla\varphi_T \,d\sigma\,dp\,dt \\
&\quad + \nu_1 \int_0^M \int_\Omega \alpha^2(p)\frac{\partial V^n}{\partial p}\cdot\frac{\partial\varphi_V}{\partial p}\,d\sigma\,dp\,dt + \frac{c_0^2\nu_2}{R^2}\int_0^M \int_\Omega \alpha^2(p)\frac{\partial T^n}{\partial p}\frac{\partial\varphi_T}{\partial p}\,d\sigma\,dp\,dt \\
&\to \mu_1 \int_0^M \int_\Omega \nabla V \cdot \nabla\varphi_V \,d\sigma\,dp\,dt + \frac{c_0^2\mu_2}{R^2}\int_0^M \int_\Omega \nabla T \cdot \nabla\varphi_T \,d\sigma\,dp\,dt \\
&\quad + \nu_1 \int_0^M \int_\Omega \alpha^2(p)\frac{\partial V}{\partial p}\cdot\frac{\partial\varphi_V}{\partial p}\,d\sigma\,dp\,dt + \frac{c_0^2\nu_2}{R^2}\int_0^M \int_\Omega \alpha^2(p)\frac{\partial T}{\partial p}\frac{\partial\varphi_T}{\partial p}\,d\sigma\,dp\,dt \\
&= \int_0^M a(U,\varphi)\,dt.
\end{aligned} \tag{4.25}$$

□

Lemma 4.4 *Let U^n be the weak solution sequence of system* (2.4). *Then, up to a subsequence, we have*

$$\int_0^M b(U^n,U^n,\varphi)\,dt \to \int_0^M b(U,U,\varphi)\,dt, \tag{4.26}$$

for all $\varphi = (\varphi_{v_\lambda},\varphi_{v_\theta},\varphi_T) = (\varphi_V,\varphi_T) \in C^\infty(0,M;C_0^\infty(\Omega))$ and $\varphi(M,\cdot) = 0$.

Proof We have

$$\begin{aligned}
&\int_0^M b(U^n,U^n,\varphi)\,dt \\
&= \int_0^M \int_\Omega \big((V^n\cdot\nabla)V^n\cdot\varphi_V + (V^n\cdot\nabla)T^n\varphi_T\big)\,d\sigma\,dp\,dt \\
&\quad + \int_0^M \int_\Omega \biggl(\biggl(\nabla\cdot\int_p^{p_s} V^n(s)\,ds\biggr)\frac{\partial V^n}{\partial p}\cdot\varphi_V \\
&\quad + \biggl(\nabla\cdot\int_p^{p_s} V^n(s)\,ds\biggr)\frac{\partial T^n}{\partial p}\varphi_T\biggr)\,d\sigma\,dp\,dt \\
&\quad + \int_0^M \int_\Omega \biggl(R\nabla\int_p^{p_s}\frac{1}{s}T^n(s)\,ds\cdot\varphi_V - \frac{c_0^2}{Rp}\biggl(\nabla\cdot\int_p^{p_s} V^n(s)\,ds\biggr)\varphi_T\biggr)\,d\sigma\,dp\,dt \\
&\quad + \int_0^M \int_\Omega \biggl(2\omega\cos\theta + \frac{\cot\theta}{a}v_\lambda^n\biggr)\big(v_\theta^n\varphi_{v_\lambda} - v_\lambda^n\varphi_{v_\theta}\big)\,d\sigma\,dp\,dt,
\end{aligned} \tag{4.27}$$

and we have from (4.5) and (4.24)

$$\begin{aligned}
&\int_0^M \int_\Omega \big((V^n\cdot\nabla)V^n\cdot\varphi_V + (V^n\cdot\nabla)T^n\varphi_T\big)\,d\sigma\,dp\,dt \\
&= \int_0^M \int_\Omega \big((V^n\cdot\nabla)V^n\cdot\varphi_V + (V^n\cdot\nabla)T^n\varphi_T\big)\,d\sigma\,dp\,dt \\
&\quad - \int_0^M \int_\Omega \big((V\cdot\nabla)V^n\cdot\varphi_V + (V\cdot\nabla)T^n\varphi_T\big)\,d\sigma\,dp\,dt
\end{aligned}$$

$$\begin{aligned}
&+\int_0^M\int_\Omega\big((V\cdot\nabla)V^n\cdot\varphi_V+(V\cdot\nabla)T^n\varphi_T\big)\,d\sigma\,dp\,dt\\
&\to\int_0^M\int_\Omega\big((V\cdot\nabla)V\cdot\varphi_V+(V\cdot\nabla)T\varphi_T\big)\,d\sigma\,dp\,dt,
\end{aligned}\tag{4.28}$$

where we use the fact

$$\begin{aligned}
&\int_0^M\int_\Omega\big(\big(V^n\cdot\nabla\big)V^n\cdot\varphi_V+\big(V^n\cdot\nabla\big)T^n\varphi_T\big)\,d\sigma\,dp\,dt\\
&\quad-\int_0^M\int_\Omega\big((V\cdot\nabla)V^n\cdot\varphi_V+(V\cdot\nabla)T^n\varphi_T\big)\,d\sigma\,dp\,dt\\
&\le C\left(\int_0^M\int_\Omega\left|U^n-U\right|^2dp\,dt\right)^{\frac12}\left(\int_0^M\int_\Omega\left|\nabla U^n\right|^2dp\,dt\right)^{\frac12}\to 0.
\end{aligned}\tag{4.29}$$

Using $|\nabla U^n|\in L^2(0,M;L^2(\Omega))$, we have

$$\nabla\cdot\int_p^{p_s}V^n(s)\,ds,\nabla\int_p^{p_s}\frac1s T^n(s)\,ds\in L^2\big(0,M;L^2(\Omega)\big),\tag{4.30}$$

which implies

$$\nabla\cdot\int_p^{p_s}V^n(s)\,ds\rightharpoonup\nabla\cdot\int_p^{p_s}V(s)\,ds,\qquad\nabla\int_p^{p_s}\frac1s T^n(s)\,ds\rightharpoonup\nabla\int_p^{p_s}\frac1s T(s)\,ds,\tag{4.31}$$

and we have

$$\begin{aligned}
&\int_0^M\int_\Omega\left(\left(\nabla\cdot\int_p^{p_s}V^n(s)\,ds\right)\frac{\partial V^n}{\partial p}\cdot\varphi_V+\left(\nabla\cdot\int_p^{p_s}V^n(s)\,ds\right)\frac{\partial T^n}{\partial p}\varphi_T\right)d\sigma\,dp\,dt\\
&=\int_0^M\int_\Omega\nabla\cdot V^nV^n\cdot\varphi_V\,d\sigma\,dp\,dt+\int_0^M\int_\Omega\nabla\cdot V^nT^n\varphi_T\,d\sigma\,dp\,dt\\
&\quad-\int_0^M\int_\Omega\left(\nabla\cdot\int_p^{p_s}V^n(s)\,ds\right)V^n\cdot\frac{\partial\varphi_V}{\partial p}\,d\sigma\,dp\,dt\\
&\quad-\int_0^M\int_\Omega\left(\nabla\cdot\int_p^{p_s}V^n(s)\,ds\right)T^n\frac{\partial\varphi_T}{\partial p}\,d\sigma\,dp\,dt\\
&=\int_0^M\int_\Omega\nabla\cdot V^nV^n\cdot\varphi_V\,d\sigma\,dp\,dt-\int_0^M\int_\Omega\nabla\cdot V^nV\cdot\varphi_V\,d\sigma\,dp\,dt\\
&\quad+\int_0^M\int_\Omega\nabla\cdot V^nT^n\varphi_T\,d\sigma\,dp\,dt\\
&\quad-\int_0^M\int_\Omega\nabla\cdot V^nT\varphi_T\,d\sigma\,dp\,dt+\int_0^M\int_\Omega\nabla\cdot V^nV\cdot\varphi_V\,d\sigma\,dp\,dt\\
&\quad+\int_0^M\int_\Omega\nabla\cdot V^nT\varphi_T\,d\sigma\,dp\,dt\\
&\quad-\int_0^M\int_\Omega\left(\nabla\cdot\int_p^{p_s}V^n(s)\,ds\right)V^n\cdot\frac{\partial\varphi_V}{\partial p}\,d\sigma\,dp\,dt\\
&\quad+\int_0^M\int_\Omega\left(\nabla\cdot\int_p^{p_s}V^n(s)\,ds\right)V\cdot\frac{\partial\varphi_V}{\partial p}\,d\sigma\,dp\,dt
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^M\int_\Omega\left(\nabla\cdot\int_p^{p_s}V^n(s)\,ds\right)T^n\frac{\partial\varphi_T}{\partial p}\,d\sigma\,dp\,dt\\
&+\int_0^M\int_\Omega\left(\nabla\cdot\int_p^{p_s}V^n(s)\,ds\right)T\frac{\partial\varphi_T}{\partial p}\,d\sigma\,dp\,dt\\
&-\int_0^M\int_\Omega\left(\nabla\cdot\int_p^{p_s}V^n(s)\,ds\right)V\cdot\frac{\partial\varphi_V}{\partial p}\,d\sigma\,dp\,dt\\
&-\int_0^M\int_\Omega\left(\nabla\cdot\int_p^{p_s}V^n(s)\,ds\right)T\frac{\partial\varphi_T}{\partial p}\,d\sigma\,dp\,dt\\
\to{}&\int_0^M\int_\Omega\nabla\cdot VV\cdot\varphi_V\,d\sigma\,dp\,dt+\int_0^M\int_\Omega\nabla\cdot VT\varphi_T\,d\sigma\,dp\,dt\\
&-\int_0^M\int_\Omega\left(\nabla\cdot\int_p^{p_s}V(s)\,ds\right)V\cdot\frac{\partial\varphi_V}{\partial p}\,d\sigma\,dp\,dt\\
&-\int_0^M\int_\Omega\left(\nabla\cdot\int_p^{p_s}V(s)\,ds\right)T\frac{\partial\varphi_T}{\partial p}\,d\sigma\,dp\,dt\\
={}&\int_0^M\int_\Omega\left(\left(\nabla\cdot\int_p^{p_s}V(s)\,ds\right)\frac{\partial V}{\partial p}\cdot\varphi_V\right.\\
&\left.+\left(\nabla\cdot\int_p^{p_s}V(s)\,ds\right)\frac{\partial T}{\partial p}\varphi_T\right)d\sigma\,dp\,dt,
\end{aligned}\tag{4.32}
$$

where we use the fact

$$
\begin{aligned}
&\int_0^M\int_\Omega\nabla\cdot V^nV^n\cdot\varphi_V\,d\sigma\,dp\,dt-\int_0^M\int_\Omega\nabla\cdot V^nV\cdot\varphi_V\,d\sigma\,dp\,dt\\
&\quad+\int_0^M\int_\Omega\nabla\cdot V^nT^n\varphi_T\,d\sigma\,dp\,dt-\int_0^M\int_\Omega\nabla\cdot V^nT\varphi_T\,d\sigma\,dp\,dt\\
&\quad-\int_0^M\int_\Omega\left(\nabla\cdot\int_p^{p_s}V^n(s)\,ds\right)V^n\cdot\frac{\partial\varphi_V}{\partial p}\,d\sigma\,dp\,dt\\
&\quad+\int_0^M\int_\Omega\left(\nabla\cdot\int_p^{p_s}V^n(s)\,ds\right)V\cdot\frac{\partial\varphi_V}{\partial p}\,d\sigma\,dp\,dt\\
&\quad-\int_0^M\int_\Omega\left(\nabla\cdot\int_p^{p_s}V^n(s)\,ds\right)T^n\frac{\partial\varphi_T}{\partial p}\,d\sigma\,dp\,dt\\
&\quad+\int_0^M\int_\Omega\left(\nabla\cdot\int_p^{p_s}V^n(s)\,ds\right)T\frac{\partial\varphi_T}{\partial p}\,d\sigma\,dp\,dt\\
&\le C\left(\int_0^M\int_\Omega\left|U^n-U\right|^2dp\,dt\right)^{\frac12}\left(\int_0^M\int_\Omega\left|\nabla U^n\right|^2dp\,dt\right)^{\frac12}\to 0,
\end{aligned}\tag{4.33}
$$

applying (4.31), we also have

$$
\begin{aligned}
&\int_0^M\int_\Omega\left(R\nabla\int_p^{p_s}\frac{1}{s}T^n(s)\,ds\cdot\varphi_V-\frac{c_0^2}{Rp}\left(\nabla\cdot\int_p^{p_s}V^n(s)\,ds\right)\varphi_T\right)d\sigma\,dp\,dt\\
&\quad\to\int_0^M\int_\Omega\left(R\nabla\int_p^{p_s}\frac{1}{s}T(s)\,ds\cdot\varphi_V-\frac{c_0^2}{Rp}\left(\nabla\cdot\int_p^{p_s}V(s)\,ds\right)\varphi_T\right)d\sigma\,dp\,dt.
\end{aligned}\tag{4.34}
$$

Finally, by means of (4.5), we can prove that

$$\begin{aligned} &\int_0^M \int_\Omega \left(2\omega\cos\theta + \frac{\cot\theta}{a} v_\lambda^n\right)\left(v_\theta^n \varphi_{v_\lambda} - v_\lambda^n \varphi_{v_\theta}\right) d\sigma\, dp\, dt \\ &\quad \to \int_0^M \int_\Omega \left(2\omega\cos\theta + \frac{\cot\theta}{a} v_\lambda\right)(v_\theta \varphi_{v_\lambda} - v_\lambda \varphi_{v_\theta})\, d\sigma\, dp\, dt. \end{aligned} \tag{4.35}$$

Summing (4.28), (4.32), (4.34), and (4.35), we complete the proof of (4.26). □

5 Conclusion

In this paper, the stability of weak solutions for the atmospheric equations is investigated with the constant external force and without the effects of topography; from Theorem 2.1 and Remark 2.1 and Remark 2.2, we show that if $U_0^n \to U_0 \in L^1(\Omega)$, then $U^n \to U \in L^1(0,T;L^1(\Omega))$; if $U_0^n \to U_0$ a.e., then $U^n \to U$ a.e., which means that if the difference of the initial data of two different weak solutions is small almost everywhere, then the difference of this two weak solutions is small almost everywhere as time increases. Furthermore, in the future we will consider the stability of weak solutions to the atmospheric models with the effects of topography, a non-constant external force, radiation heating, and the moist phase transformation, *etc.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
RXL organized and wrote this paper. QCZ examined all the steps of the proofs in this research and gave some advices. All authors read and approved the final manuscript.

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