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Operator P-class functions

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Abstract
We introduce and investigate the notion of an operator P-class function. We show that every nonnegative operator convex function is of operator P-class, but the converse is not true in general. We present some Jensen type operator inequalities involving P-class functions and some Hermite-Hadamard inequalities for operator P-class functions.

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1 Introduction and preliminaries

Let $\mathfrak{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with its identity denoted by I. When $\dim \mathcal{H} = n$, we identify $\mathfrak{B}(\mathcal{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$. We denote by $\sigma(J)$ the set of all self-adjoint operators on $\mathcal{H}$ whose spectra are contained in an interval J. An operator $A \in \mathfrak{B}(\mathcal{H})$ is called positive (positive semidefinite for a matrix) if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$ and in such a case we write $A \geq 0$. For self-adjoint operators $A, B \in \mathfrak{B}(\mathcal{H})$, we write $B \geq A$ if $B - A \geq 0$. The Gelfand map $f \mapsto f(A)$ is an isometrical $*$-isomorphism between the C^*-algebra $C(\sigma(A))$ of a complex-valued continuous functions on the spectrum $\sigma(A)$ of a self-adjoint operator A and the C^*-algebra generated by I and A. If $f, g \in C(\sigma(A))$, then $f(t) \geq g(t)$ $(t \in \sigma(A))$ implies that $f(A) \geq g(A)$. A real-valued continuous function f on an interval J is called operator increasing (operator decreasing, resp.) if $A \leq B$ implies $f(A) \leq f(B)$ $(f(B) \leq f(A)$, resp.) for all $A, B \in \sigma(J)$. We recall that a real-valued continuous function f defined on an interval J is operator convex if $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B)$ for all $A, B \in \sigma(J)$ and all $\lambda \in [0,1]$.

A function $f : J \longrightarrow \mathbb{R}$ is said to be of P-class on J or is a P-class function on J if

$$f\bigl(\lambda x + (1-\lambda)y\bigr) \leq f(x) + f(y), \tag{1}$$

where $x, y \in J$ and $\lambda \in [0,1]$; see [1]. Many properties of P-class functions can be found in [1–4]. Note that the set of all P-class functions contains all convex functions and also all nonnegative monotone functions. Every non-zero P-class function is nonnegative valued. In fact, choose $\lambda = 1$ and fix $x_0 \in J$. It follows from (1) that

$$f(x_0) \leq f(x_0) + f(y),$$

where $y \in J$. Thus $0 \leq f(y)$ for all $y \in J$.

For a P-class function f on an interval $[a,b]$,

$$f\left(\frac{a+b}{2}\right) \le 2\int_0^1 f\bigl(ta+(1-t)b\bigr)\,dt \le 2\bigl(f(a)+f(b)\bigr),$$

which is known as the Hermite-Hadamard inequality for the P-class continuous functions; see [3].

In this paper, we introduce and investigate the notion of an operator P-class function and give several examples. We show that if f is a P-class function on $(0,\infty)$ such that $\lim_{t\to\infty} f(t) = 0$, then it is operator decreasing. We also prove that if f is an operator P-class function on an interval J, then

$$f\bigl(C^*AC\bigr) \le 2C^*f(A)C,$$

where $A \in \sigma(J)$ and $C \in \mathfrak{B}(\mathcal{H})$ is an isometry. In addition, we present a Hermite-Hadamard inequality for operator P-class functions.

2 Operator *P*-class functions

In this section, we investigate operator P-class functions and study some relations between the operator P-class functions and the operator monotone functions.

We start our work with the following definition.

Definition 1 Let f be a real-valued continuous function defined on an interval J. We say that f is of operator P-class on J if

$$f\bigl(\lambda A+(1-\lambda)B\bigr) \le f(A)+f(B)$$

for all $A,B \in \sigma(J)$ and all $\lambda \in [0,1]$.

Clearly every nonnegative operator convex function is of operator P-class.

Example 1 Let $f(t) = t^{-r}$ $(0 \le r \le 1)$ be defined on $(1,\infty)$. It follows from the operator concavity of t^r $(0 \le r \le 1)$ [5] and the arithmetic-harmonic mean inequality that

$$\begin{aligned}\bigl(\lambda A+(1-\lambda)B\bigr)^{-r} &\le \bigl(\lambda A^r+(1-\lambda)B^r\bigr)^{-1} \quad (\text{by the concavity of } t^r)\\ &\le \lambda A^{-r}+(1-\lambda)B^{-r} \quad (\text{by the arithmetic-harmonic mean inequality})\\ &\le A^{-r}+B^{-r},\end{aligned}$$

where $A,B \in \sigma((1,\infty))$ and $\lambda \in [0,1]$. Thus f is an operator P-class function on $(1,\infty)$.

In addition, every operator P-class f on an interval J is of operator Q-class in the sense that

$$f\bigl(\lambda A+(1-\lambda)B\bigr) \le \frac{f(A)}{\lambda}+\frac{f(B)}{1-\lambda}$$

for all $A,B \in \sigma(J)$ and $\lambda \in (0,1)$; see [6]. In the next example, we show the converse is not true, in general.

Example 2 The function $f(t) = 4 - t^2$ defined on $[-\sqrt{3}, \sqrt{3}]$ is of operator Q-class; see [7, Example 2.1]. We put $\lambda = \frac{1}{2}$, $A = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$ and $B = \begin{pmatrix} -\frac{3}{2} & 0 \\ 0 & -\frac{3}{2} \end{pmatrix}$. Then $f(\lambda A + (1-\lambda)B) = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \nleq f(A) + f(B) = \begin{pmatrix} \frac{14}{4} & 0 \\ 0 & \frac{14}{4} \end{pmatrix}$. Hence f is not of operator P-class.

Example 3 Let $\alpha > 0$ and f be a continuous function on the interval $[\alpha, 2\alpha]$ into itself. It follows from

$$f\bigl(\lambda A + (1-\lambda)B\bigr) \le 2\alpha \le f(A) + f(B) \quad \bigl(A, B \in \sigma\bigl([\alpha, 2\alpha]\bigr), \lambda \in [0,1]\bigr)$$

that f is of operator P-class on $[\alpha, 2\alpha]$.

Example 4 Let g be a nonnegative continuous function on an interval $[a,b]$ and $\alpha = \sup_{x,y\in[a,b],t\in[x,y]} |g(t) - g(x) - g(y)|$. We put $f(t) = g(t) + \alpha$. Then

$$\begin{aligned} f\bigl(\lambda A + (1-\lambda)B\bigr) &= g\bigl(\lambda A + (1-\lambda)B\bigr) + \alpha \\ &\le \bigl(g(A) + \alpha\bigr) + \bigl(g(B) + \alpha\bigr) = f(A) + f(B), \end{aligned}$$

where $A, B \in \sigma([a,b])$ and $\lambda \in [0,1]$. Hence f is an operator P-class function.

Next, we explore some relations between operator P-class functions and operator monotone functions. In fact, we have the following.

Theorem 1 *If f is an operator P-class function on the interval $(0,\infty)$ such that $\lim_{t\to\infty} f(t) = 0$, then f is operator decreasing.*

Proof Let $0 < A \le B$. Fix $\varepsilon > 0$. We put $C = B - A + \varepsilon$. Let $\theta > 0$. It follows from $\lim_{t\to\infty} f(t) = 0$ that there exists $M > 0$ such that $f(t) \le \theta$ for all $t \ge M$. We may assume that the spectrum of the strictly positive operator C is contained in $[\alpha, \beta]$ for some $0 < \alpha < \beta$. It follows from $\lim_{\lambda\to 1^-} \frac{\lambda}{1-\lambda} = \infty$ that there exists $\delta > 0$ such that $\frac{\lambda}{1-\lambda} \ge \frac{M}{\alpha}$ for all $\lambda \in (1-\delta, 1)$. Hence $\sigma(\frac{\lambda}{1-\lambda}C) \subseteq [M, \infty)$ for all $\lambda \in (1-\delta, 1)$. Now, by the functional calculus for the positive operator $\frac{\lambda}{1-\lambda}C$, we have $f(\frac{\lambda}{1-\lambda}C) \le \theta$ for all $\lambda \in (1-\delta, 1)$. Thus $\langle f(\frac{\lambda}{1-\lambda}C)x, x\rangle \le \theta\|x\|^2$ for all $\lambda \in (1-\delta, 1)$ and $x \in \mathcal{H}$. Since $\lambda(B+\varepsilon) = \lambda A + (1-\lambda)(\frac{\lambda}{1-\lambda})C$ and f is P-class we have

$$f\bigl(\lambda(B+\varepsilon)\bigr) \le f(A) + f\left(\left(\frac{\lambda}{1-\lambda}\right)C\right)$$

for all $\lambda \in (1-\delta, 1)$. Hence

$$\bigl\langle f\bigl(\lambda(B+\varepsilon)\bigr)x, x\bigr\rangle \le \bigl\langle f(A)x, x\bigr\rangle + \left\langle f\left(\frac{\lambda}{1-\lambda}C\right)x, x\right\rangle \le \bigl\langle f(A)x, x\bigr\rangle + \theta\|x\|^2,$$

where $\lambda \in (1-\delta, 1)$ and $x \in \mathcal{H}$. As $\lambda \to 1^-$ and then $\theta \to 0^+$ we obtain $\langle f(B+\varepsilon)x, x\rangle \le \langle f(A)x, x\rangle$ for all $x \in \mathcal{H}$. As $\varepsilon \to 0^+$, we conclude that $f(B) \le f(A)$. □

3 Jensen operator inequality for operator *P*-class functions

In this section, we present a Jensen operator inequality for operator P-class functions. We start with the following result in which we utilized the well-known technique of [8].

Theorem 2 *Let f be an operator P-class function on an interval J, $A \in \sigma(J)$, and $C \in \mathfrak{B}(\mathcal{H})$ be an isometry. Then*

$$f(C^*AC) \leq 2C^*f(A)C. \tag{2}$$

Proof Let $X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathfrak{B}(\mathcal{H} \oplus \mathcal{H})$ for some $B \in \sigma(J)$ and let $U = \begin{pmatrix} C & D \\ 0 & -C^* \end{pmatrix}$ and $V = \begin{pmatrix} C & -D \\ 0 & C^* \end{pmatrix}$, where $D = \sqrt{1_{\mathcal{H}} - CC^*}$. Now we can easily conclude from the two facts $C^*D = \sqrt{1_{\mathcal{H}} - CC^*}C = 0$ and $DC = C\sqrt{1_{\mathcal{H}} - C^*C} = 0$ that U and V are unitary operators in $\mathfrak{B}(\mathcal{H} \oplus \mathcal{H})$. Further,

$$U^*XU = \begin{pmatrix} C^*AC & C^*AD \\ DAC & DAD + CBC^* \end{pmatrix}$$

and

$$V^*XV = \begin{pmatrix} C^*AC & -C^*AD \\ -DAC & DAD + CBC^* \end{pmatrix}.$$

Using the operator P-class property of f we have

$$\begin{aligned} \begin{pmatrix} f(C^*AC) & 0 \\ 0 & f(DAD + CBC^*) \end{pmatrix} &= f\begin{pmatrix} C^*AC & 0 \\ 0 & DAD + CBC^* \end{pmatrix} \\ &= f\left(\frac{U^*XU + V^*XV}{2}\right) \\ &\leq f(U^*XU) + f(V^*XV) \\ &= 2\begin{pmatrix} C^*f(A)C & 0 \\ 0 & Df(A)D + Cf(B)C^* \end{pmatrix}. \end{aligned}$$

Therefore

$$f(C^*AC) \leq 2C^*f(A)C. \qquad \square$$

Applying Theorem 2 we have some inequalities including the subadditivity.

Corollary 1 *Let f be operator P-class on an interval J, $A_j \in \sigma(J)$ $(1 \leq j \leq n)$, and $C_j \in \mathfrak{B}(\mathcal{H})$ $(1 \leq j \leq n)$, where $\sum_{j=1}^n C_j^*C_j = 1$. Then*

$$f\left(\sum_{j=1}^n C_j^*A_jC_j\right) \leq 2\sum_{j=1}^n C_j^*f(A_j)C_j.$$

Proof Let

$$\tilde{A} = \tilde{A} = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \cdots & \\ & & & A_n \end{pmatrix} \in \mathfrak{B}(\mathcal{H} \oplus \cdots \oplus \mathcal{H}), \qquad \tilde{C} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} \in \mathfrak{B}(\mathcal{H} \oplus \cdots \oplus \mathcal{H}).$$

It follows from $\tilde{C}^*\tilde{C} = 1$ and (2) that

$$f\left(\sum_{j=1}^n C_j^* A_j C_j\right) = f\left(\tilde{C}^*\tilde{A}\tilde{C}\right) \le 2\tilde{C}^* f(\tilde{A})\tilde{C} = 2\sum_{j=1}^n C_j^* f(A_j) C_j. \qquad \square$$

Corollary 2 *Let f be operator P-class on $[0,\infty)$ such that $f(0) = 0$, $A \in \sigma([0,\infty))$, and $C \in \mathfrak{B}(\mathcal{H})$ be a contraction. Then*

$$f\left(C^*AC\right) \le 2C^* f(A)C.$$

Proof For every contraction $C \in \mathfrak{B}(\mathcal{H})$, we put $D = \sqrt{1_{\mathcal{H}} - C^*C}$. It follows from $C^*C + D^*D = 1_{\mathcal{H}}$ and (2) that

$$f\left(C^*AC\right) = f\left(C^*AC + D^*0D\right) \le 2f\left(C^*AC\right) + 2f\left(D^*0D\right) = 2C^*f(A)C. \qquad \square$$

Corollary 3 *Let f be operator P-class on $[0,\infty)$ such that $f(0) = 0$ and $A, B \in \sigma((0,\infty))$ such that $A \le B$. Then*

$$A^{-1}f(A) \le 2B^{-1}f(B).$$

Proof Let $A, B \in \sigma((0,\infty))$ such that $0 < A \le B$. We put $C = B^{-1/2}A^{1/2}$. Then $CC^* = B^{-1/2}AB^{-1/2} \le 1_{\mathcal{H}}$, so C is a contraction. It follows from (2) that

$$f(A) = f\left(C^*BC\right) \le 2C^*f(B)C = 2A^{1/2}B^{-1/2}f(B)B^{-1/2}A^{1/2}.$$

Therefore

$$A^{-1}f(A) \le 2B^{-1}f(B). \qquad \square$$

In the following theorem, we obtain the Choi-Davis-Jensen type inequality for operator P-class functions.

Theorem 3 *Let Φ be a unital positive linear map on $\mathfrak{B}(\mathcal{H})$, $A \in \sigma(J)$ and f be operator P-class on an interval J. Then*

$$f\left(\Phi(A)\right) \le 2\Phi\left(f(A)\right). \tag{3}$$

Proof Let $A \in \sigma(J)$. We put Ψ the restriction of Φ to the C^*-algebra $\mathcal{C}^*(A, I)$ generated by I and A. Then Ψ is a unital completely positive map on $\mathcal{C}^*(A, I)$. The celebrated Stinespring dilation theorem [9, Theorem 1] states that there exist an isometry $V : \mathcal{H} \longrightarrow \mathcal{H}$ and a unital $*$-homomorphism $\pi : \mathcal{C}^*(A, I) \longrightarrow \mathfrak{B}(\mathcal{H})$ such that $\Psi(A) = V^*\pi(A)V$. Hence

$$\begin{aligned} f\left(\Phi(A)\right) &= f\left(\Psi(A)\right) = f\left(V^*\pi(A)V\right) \le 2V^*f\left(\pi(A)\right)V \quad \text{(by (2))} \\ &= 2V^*\pi\left(f(A)\right)V = 2\Psi\left(f(A)\right) = 2\Phi\left(f(A)\right). \qquad \square \end{aligned}$$

We will show that the constant 2 is the best possible such one in the following example.

Example 5 Let $f(t) = 2 - t^2$ for $t \in [-1,1]$. Then $1 \leq f(t) \leq 2$ and

$$f\big(\lambda A + (1-\lambda)B\big) = 2 - \big(\lambda A + (1-\lambda)B\big)^2 \leq 2 \leq 2 - A^2 + 2 - B^2 = f(A) + f(B),$$

where $A, B \in \sigma([-1,1])$. Hence f is of operator P-class on $[-1,1]$. Now, consider that the unital positive map $\Phi : \mathbb{M}_2 \to \mathbb{M}_2$ is defined by $\Phi(A) = \frac{\operatorname{tr}(A)}{2}I$. Then for the Hermitian matrix $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ we have $\Phi(A) = 0$, $f(\Phi(A)) = 2$, $f(A) = I$, and $\Phi(f(A)) = I$. Therefore $f(\Phi(A)) = 2\Phi(f(A))$. This shows that the coefficient 2 in (2) and (3) is the best.

Example 6 Consider (the nonnegative increasing function and so) P-class function $f(t) = \sqrt{t}$ where $t \in (0,\infty)$. Let the unital positive map $\Psi : \mathbb{M}_2(\mathbb{C}) \to \mathbb{C}$ be defined by $\Psi(A) = a_{22}$ with $A = (a_{ij})_{1 \leq i,j \leq 2}$ and let $A = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}^2$. Then $\Psi(f(A)) = 1$ and $f(\Psi(A)) = \sqrt{8}$. Hence $f(\Psi(A)) \nleq 2\Psi(f(A))$. It follows from (3) that f is not of operator P-class.

We present a Hermite-Hadamard inequality for operator P-class functions in the next theorem.

Theorem 4 *Let Φ be a unital positive linear map on $\mathfrak{B}(\mathcal{H})$ and f be operator P-class on J. Then*

$$f\left(\frac{\Phi(A) + \Phi(B)}{2}\right) \leq 2\int_0^1 f\big(\lambda\Phi(A) + (1-\lambda)\Phi(B)\big)\,d\lambda \leq 4\big(\Phi\big(f(A)\big) + \Phi\big(f(B)\big)\big),$$

where $A, B \in \sigma(J)$ and $\lambda \in [0,1]$.

Proof Let $A, B \in \sigma(J)$ and $\lambda \in [0,1]$. Then

$$\begin{aligned} f\left(\frac{\Phi(A) + \Phi(B)}{2}\right) &= f\left(\frac{\lambda\Phi(A) + (1-\lambda)\Phi(B) + (1-\lambda)\Phi(A) + \lambda\Phi(B)}{2}\right) \\ &\leq f\big(\lambda\Phi(A) + (1-\lambda)\Phi(B)\big) + f\big((1-\lambda)\Phi(A) + \lambda\Phi(B)\big) \\ &\leq 2\big(f\big(\Phi(A)\big) + f\big(\Phi(B)\big)\big). \end{aligned} \tag{4}$$

Integrating both sides of (4) over $[0,1]$ we obtain

$$\begin{aligned} f\left(\frac{\Phi(A) + \Phi(B)}{2}\right) &\leq \int_0^1 f\big(\lambda\Phi(A) + (1-\lambda)\Phi(B)\big)\,d\lambda \\ &\quad + \int_0^1 f\big((1-\lambda)\Phi(A) + \lambda\Phi(B)\big)\,d\lambda \\ &= 2\int_0^1 f\big(\lambda\Phi(A) + (1-\lambda)\Phi(B)\big)\,d\lambda \\ &\leq 2\big(f\big(\Phi(A)\big) + f\big(\Phi(B)\big)\big) \\ &\leq 4\big(\Phi\big(f(A)\big) + \Phi\big(f(B)\big)\big) \quad \text{(by (3))}. \end{aligned}$$

□

4 Some inequalities for *P*-class functions involving continuous operator fields

Let $\mathcal{A}$ be a C^*-algebra of operators acting on a Hilbert space and let T be a locally compact Hausdorff space. A field $(A_t)_{t \in T}$ of operators in $\mathcal{A}$ is called a continuous field of operators if the mapping $t \mapsto A_t$ is norm continuous on T. If $\mu(t)$ is a Radon measure on T and the function $t \mapsto \|A_t\|$ is integrable, one can form the Bochner integral $\int_T A_t \, d\mu(t)$, which is the unique element in $\mathcal{A}$ such that

$$\varphi\left(\int_T A_t \, d\mu(t)\right) = \int_T \varphi(A_t) \, d\mu(t)$$

for every linear functional φ in the norm dual $\mathcal{A}^*$ of $\mathcal{A}$.

Let $\mathcal{C}(T, \mathcal{A})$ denote the set of bounded continuous functions on T with values in $\mathcal{A}$. It is easy to see that the set $\mathcal{C}(T, \mathcal{A})$ is a C^*-algebra under the pointwise operations and the norm $\|(A_t)_{t \in T}\| = \sup_{t \in T} \|A_t\|$; *cf.* [10].

Assume that there is a field $(\Phi_t)_{t \in T}$ of positive linear mappings $\Phi_t : \mathcal{A} \longrightarrow \mathcal{B}$ from $\mathcal{A}$ to another C^*-algebra $\mathcal{B}$. We say that such a field is continuous if the mapping $t \mapsto \Phi_t(A)$ is continuous for every $A \in \mathcal{A}$. If the C^*-algebras are unital and the field $t \mapsto \Phi_t(I)$ is integrable with integral I, we say that $(\Phi_t)_{t \in T}$ is unital; see [10].

Theorem 5 *Let $f : J \longrightarrow \mathbb{R}$ be an operator P-class function defined on an interval J, and let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. If $(\Phi_t)_{t \in T}$ is a unital field of positive linear mappings $\Phi_t : \mathcal{A} \longrightarrow \mathcal{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ, then*

$$f\left(\int_T \Phi_t(A_t) \, d\mu(t)\right) \le 2 \int_T \Phi_t\big(f(A_t)\big) \, d\mu(t)$$

holds for every bounded continuous field $(A_t)_{t \in T}$ of self-adjoint elements in $\mathcal{A}$ with spectra contained in J.

Proof We consider the unital positive linear map $\Psi : \mathcal{C}(T, \mathcal{A}) \longrightarrow \mathcal{B}$ defined by $\Psi((A_t)_{t \in T}) = \int_T \Phi_t(A_t) \, d\mu(t)$. Let $\tilde{A} = (A_t)_{t \in T} \in \mathcal{C}(T, \mathcal{A})$. It follows from $\sigma(\tilde{A}) \subseteq J$ and (3) that

$$f\big(\Psi\big((A_t)_{t \in T}\big)\big) = f\big(\Psi(\tilde{A})\big) \le 2\Psi\big(f(\tilde{A})\big) = 2\Psi\big(f\big((A_t)_{t \in T}\big)\big) = 2\Psi\big(\big(f(A_t)\big)_{t \in T}\big). \qquad \square$$

In the discrete case, $T = \{1, \ldots, n\}$ in Theorem 5, we get the following result.

Corollary 4 *Let $f : J \longrightarrow \mathbb{R}$ be an operator P-class function defined on an interval J, let $A_j \in \sigma(J)$ $(1 \le j \le n)$ and Φ_j $(1 \le j \le n)$ be unital positive linear maps on $\mathfrak{B}(\mathcal{H})$. Then*

$$f\left(\sum_{j=1}^{n} \Phi_j(A_j)\right) \le 2 \sum_{j=1}^{n} \Phi_j\big(f(A_j)\big).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the manuscript and read and approved the final manuscript.

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