Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH ARTICLE

Open Access

A new order-preserving average function on a quotient space of strictly monotone functions and its applications

Yasuo Nakasuji[1*] and Sin-Ei Takahasi[2]

*Correspondence: ef20019@hotmail.co.jp
[1]The Open University of Japan, Chiba, 261-8586, Japan
Full list of author information is available at the end of the article

Abstract

We introduce an order in a quotient space of strictly monotone continuous functions on a real interval and show that a new average function on this quotient space is order-preserving. We also apply this new order-preserving function to derive a finite form of Jensen type inequality with negative weights.

MSC: Primary 39B62; secondary 26B25; 26A51

Keywords: Jensen's inequality; strictly monotone function; order-preserving average function

1 Introduction and main results

This is meant as a continuation of our paper [1] related to Jensen's inequality [2]. The reader should refer to the recent paper of József [3] on Jensen's inequality. Further the paper is related to the notion of quasi-arithmetic means, so the reader should refer to the recent paper of Janusz [4].

Let I be a finite closed interval $[m,M]$ on $\mathbf{R}$ and $C(I)$ the space of all continuous real-valued functions defined on I. Moreover, let $C^+_{\mathrm{sm}}(I)$ (resp. $C^-_{\mathrm{sm}}(I)$) be the set of all functions in $C(I)$ which are strictly monotone increasing (resp. decreasing) on I. Put

$$C_{\mathrm{sm}}(I) = C^+_{\mathrm{sm}}(I) \cup C^-_{\mathrm{sm}}(I).$$

Then $C_{\mathrm{sm}}(I)$ is equal to the space of all strictly monotone continuous functions on I. For any $\varphi, \psi \in C_{\mathrm{sm}}(I)$, we write $\varphi \cong \psi$ if there exist two numbers $a, b \in \mathbf{R}$ such that $\varphi(x) = a\psi(x) + b$ for all $x \in I$. Then it is clear that $\cong$ is an equivalence relation in $C_{\mathrm{sm}}(I)$. Let $\tilde{C}_{\mathrm{sm}}(I)$ be the quotient space of $C_{\mathrm{sm}}(I)$ by $\cong$ and we denote by $\tilde{\varphi}$ the coset of $\varphi \in C_{\mathrm{sm}}(I)$. We introduce an order $\preceq$ in $\tilde{C}_{\mathrm{sm}}(I)$ in the next section (see Theorem 2).

Let (Ω, μ) be a probability space and f a function in $L^1(\Omega, \mu)$ such that $f(\omega) \in I$ for almost all $\omega \in \Omega$. Then we see that $\varphi \circ f \in L^1(\Omega, \mu)$ for all $\varphi \in C_{\mathrm{sm}}(I)$ because φ is a bounded continuous function and μ is a finite measure. Put

$$M_\varphi(f) = \varphi^{-1}\left(\int \varphi \circ f \, d\mu\right)$$

for each $\varphi \in C_{\mathrm{sm}}(I)$. Then [1, Theorem 1] which gives a new interpretation of Jensen's inequality is restated as $\tilde{\varphi} \preceq \tilde{\psi} \Rightarrow M_\varphi(f) \le M_\psi(f)$. In this paper, we give a new order-preserving average function $N_{[I,f]}$ on the quotient space $\tilde{C}_{\mathrm{sm}}(I)$, according to this idea. We also apply this function $N_{[I,f]}$ to derive a finite form of Jensen type inequality with negative weights.

Let φ be an arbitrary function of $C_{\mathrm{sm}}(I)$. Since $\varphi(I)$ is an interval of $\mathbf{R}$ and μ is a probability measure on Ω, it follows that

$$\varphi(m) + \varphi(M) - \int \varphi \circ f \, d\mu \in \varphi(I),$$

and hence we have

$$\varphi^{-1}\left(\varphi(m) + \varphi(M) - \int \varphi \circ f \, d\mu\right) \in I.$$

Note that a simple computation implies that if $\varphi, \psi \in C_{\mathrm{sm}}(I)$ satisfy $\tilde{\varphi} = \tilde{\psi}$, then

$$\varphi^{-1}\left(\varphi(m) + \varphi(M) - \int \varphi \circ f \, d\mu\right) = \psi^{-1}\left(\psi(m) + \psi(M) - \int \psi \circ f \, d\mu\right)$$

holds. Then denote by $N_{[I,f]}(\tilde{\varphi})$ the above value.

In this case, our main result can be stated as follows.

Theorem 1 *$N_{[I,f]}$ is an order-preserving real-valued function on the quotient space $\tilde{C}_{\mathrm{sm}}(I)$ with order $\preceq$, that is, $\tilde{\varphi} \preceq \tilde{\psi} \Rightarrow N_{[I,f]}(\tilde{\varphi}) \le N_{[I,f]}(\tilde{\psi})$.*

The above theorem easily implies the following result, which is a finite form of Jensen type inequality with negative weights.

Corollary 1 *Let $\varphi, \psi \in C_{\mathrm{sm}}(I)$ with $\tilde{\varphi} \preceq \tilde{\psi}$ and $t_1, \ldots, t_n \in \mathbf{R}$ with $\sum_{i=1}^n t_i = 1$, $0 < t_1, t_n < 1$, and $t_2, \ldots, t_{n-1} < 0$. Then*

$$\varphi^{-1}\left(\sum_{i=1}^n t_i \varphi(x_i)\right) \le \psi^{-1}\left(\sum_{i=1}^n t_i \psi(x_i)\right)$$

holds for all $x_1, \ldots, x_n \in I$ with $x_1 \le x_2, \ldots, x_{n-1} \le x_n$.

Finally, we give concrete examples of Corollary 1.

2 An order in the quotient space $\tilde{C}_{\mathrm{sm}}(I)$

Let us start with the following two lemmas.

Lemma 1 *Let $\varphi \in C_{\mathrm{sm}}(I)$. Then:*

(i) *φ is increasing and convex on I if and only if φ^{-1} is increasing and concave on $\varphi(I)$.*
(ii) *φ is increasing and concave on I if and only if φ^{-1} is increasing and convex on $\varphi(I)$.*
(iii) *φ is decreasing and convex on I if and only if φ^{-1} is decreasing and convex on $\varphi(I)$.*
(iv) *φ is decreasing and concave on I if and only if φ^{-1} is decreasing and concave on $\varphi(I)$.*

Proof Straightforward. □

Lemma 2

(i) *If φ is a convex function on I and ψ is an increasing convex function on $\varphi(I)$, then $\psi \circ \varphi$ is convex on I.*
(ii) *If φ is a convex function on I and ψ is a decreasing concave function on $\varphi(I)$, then $\psi \circ \varphi$ is concave on I.*
(iii) *If φ is a concave function on I and ψ is an increasing concave function on $\varphi(I)$, then $\psi \circ \varphi$ is concave on I.*
(iv) *If φ is a concave function on I and ψ is a decreasing convex function on $\varphi(I)$, then $\psi \circ \varphi$ is convex on I.*

Proof Straightforward. □

For any $\varphi, \psi \in C_{\mathrm{sm}}(I)$, we write $\varphi \preceq \psi$ if any of the following four conditions holds:
(i) $\varphi, \psi \in C^{+}_{\mathrm{sm}}(I)$ and $\varphi \circ \psi^{-1}$ is concave on $\psi(I)$.
(ii) $\varphi \in C^{-}_{\mathrm{sm}}(I)$, $\psi \in C^{+}_{\mathrm{sm}}(I)$ and $\varphi \circ \psi^{-1}$ is convex on $\psi(I)$.
(iii) $\varphi, \psi \in C^{-}_{\mathrm{sm}}(I)$ and $\varphi \circ \psi^{-1}$ is convex on $\psi(I)$.
(iv) $\varphi \in C^{+}_{\mathrm{sm}}(I)$, $\psi \in C^{-}_{\mathrm{sm}}(I)$ and $\varphi \circ \psi^{-1}$ is concave on $\psi(I)$.

Remark Lemma 1 guarantees that the above $\varphi \preceq \psi$ is a restatement of the concepts appearing in [1, Lemma 3].

Lemma 3 *Let $\varphi, \varphi', \psi, \psi' \in C_{\mathrm{sm}}(I)$. If $\varphi \cong \varphi'$, $\psi \cong \psi'$, and $\varphi \preceq \psi$, then $\varphi' \preceq \psi'$.*

Proof Assume that $\varphi \cong \varphi'$, $\psi \cong \psi'$, and $\varphi \preceq \psi$. Then we must show $\varphi' \preceq \psi'$. Since $\varphi \cong \varphi'$, $\psi \cong \psi'$, we can write φ' and ψ' as follows:

$$\varphi' = a\varphi + b \quad \text{and} \quad \psi' = c\psi + d$$

for some $a, b, c, d \in \mathbf{R}$. Then we have $a \neq 0$ and $c \neq 0$. Put

$$\zeta(x) = ax + b \quad \text{and} \quad \eta(x) = cx + d$$

for each $x \in \mathbf{R}$. In the case of $\varphi, \psi \in C^{+}_{\mathrm{sm}}(I)$ and $a, c > 0$, we find that $\varphi \circ \psi^{-1}$ is concave on $\psi(I)$ because $\varphi \preceq \psi$. Then $\zeta \circ \varphi \circ \psi^{-1}$ is increasing and concave on $\psi(I)$ from Lemma 2-(iii) and hence $\varphi' \circ \psi'^{-1} = \zeta \circ \varphi \circ \psi^{-1} \circ \eta^{-1}$ is also concave on $\psi'(I)$ from Lemma 2-(iii). However, since $\varphi', \psi' \in C^{+}_{\mathrm{sm}}(I)$, we obtain $\varphi' \preceq \psi'$ as required. Moreover, we can easily see that $\varphi' \preceq \psi'$ holds in the other 15 cases:

$$\bigl[\varphi \in C^{+}_{\mathrm{sm}}(I), \psi \in C^{-}_{\mathrm{sm}}(I), a > 0, c > 0\bigr], \qquad \ldots,$$
$$\bigl[\varphi \in C^{-}_{\mathrm{sm}}(I), \psi \in C^{-}_{\mathrm{sm}}(I), a < 0, c < 0\bigr].$$ □

For any $\tilde{\varphi}, \tilde{\psi} \in \tilde{C}_{\mathrm{sm}}(I)$, we write $\tilde{\varphi} \preceq \tilde{\psi}$ by the same notation if $\varphi \preceq \psi$ holds. This is well defined by Lemma 3. In this case, we have the following.

Theorem 2 *$\preceq$ is an order relation in $\tilde{C}_{\mathrm{sm}}(I)$.*

Proof We show the theorem by dividing into three steps.

(I) It is evident that $\preceq$ satisfies the reflexivity.

(II) Assume that $\tilde{\varphi} \preceq \tilde{\psi}$ and $\tilde{\psi} \preceq \tilde{\varphi}$. Then $\varphi \preceq \psi$ and $\psi \preceq \varphi$ hold. In the case of $\varphi, \psi \in C^+_{\mathrm{sm}}(I)$, we find that $\varphi \circ \psi^{-1}$ is concave on $\psi(I)$ and $\psi \circ \varphi^{-1}$ is concave on $\varphi(I)$. Since $\psi \circ \varphi^{-1}$ is increasing and concave on $\varphi(I)$, it follows from Lemma 1-(ii) that $\varphi \circ \psi^{-1} = (\psi \circ \varphi^{-1})^{-1}$ is convex on $\psi(I)$. Therefore $\varphi \circ \psi^{-1}$ is affine on $\psi(I)$ and hence $\varphi \cong \psi$, that is, $\tilde{\varphi} = \tilde{\psi}$. By the same method, we can easily see that $\tilde{\varphi} = \tilde{\psi}$ holds in the other three cases:

$$\left[\varphi \in C^+_{\mathrm{sm}}(I), \psi \in C^-_{\mathrm{sm}}(I)\right], \qquad \left[\varphi \in C^-_{\mathrm{sm}}(I), \psi \in C^+_{\mathrm{sm}}(I)\right] \quad \text{and} \quad \left[\varphi, \psi \in C^-_{\mathrm{sm}}(I)\right].$$

Therefore $\preceq$ satisfies the symmetry law.

(III) Assume that $\tilde{\varphi} \preceq \tilde{\psi}$ and $\tilde{\psi} \preceq \tilde{\lambda}$. Then $\varphi \preceq \psi$ and $\psi \preceq \lambda$ hold. In the case of $\varphi, \psi, \lambda \in C^+_{\mathrm{sm}}(I)$, we find that $\varphi \circ \psi^{-1}$ is increasing and concave on $\psi(I)$ and $\psi \circ \lambda^{-1}$ is concave on $\lambda(I)$. Then it follows from Lemma 2-(iii) that $\varphi \circ \lambda^{-1} = (\varphi \circ \psi^{-1}) \circ (\psi \circ \lambda^{-1})$ is concave on $\lambda(I)$, and hence $\varphi \preceq \lambda$, that is, $\tilde{\varphi} \preceq \tilde{\lambda}$ holds. By the same method, we can easily see that $\tilde{\varphi} \preceq \tilde{\lambda}$ holds in the other seven cases:

$$\begin{aligned} &\left[\varphi \in C^+_{\mathrm{sm}}(I), \psi \in C^+_{\mathrm{sm}}(I), \lambda \in C^-_{\mathrm{sm}}(I)\right], \qquad \ldots, \\ &\left[\varphi \in C^-_{\mathrm{sm}}(I), \psi \in C^-_{\mathrm{sm}}(I), \lambda \in C^-_{\mathrm{sm}}(I)\right]. \end{aligned}$$

Therefore $\preceq$ satisfies the transitive law. $\square$

3 Proofs of Theorem 1 and Corollary 1

Let φ be an arbitrary function of $C_{\mathrm{sm}}(I)$. Then an easy observation implies that

$$(-\varphi)^{-1}(y) = \varphi^{-1}(-y) \tag{1}$$

for all $y \in -\varphi(I)$ and that

$$N_{[I,f]}(\widetilde{-\varphi}) = N_{[I,f]}(\tilde{\varphi}). \tag{2}$$

Lemma 4 *Let $\varphi \in C_{\mathrm{sm}}(I)$. If either φ is increasing and concave on I or decreasing and convex on I, then*

$$N_{[I,f]}(\tilde{\varphi}) \le \int \varphi^{-1} \circ \big(\varphi(m) + \varphi(M) - \varphi \circ f\big)\, d\mu \le m + M - \int f\, d\mu$$

holds. If either φ is increasing and convex on I or decreasing and concave on I, then the above inequalities are reversed.

Proof (I) Suppose that φ is increasing and concave on I. Then φ^{-1} is increasing and convex on $\varphi(I)$ by Lemma 1-(ii), and hence the first inequality in Lemma 4 follows from Jensen's inequality. Put

$$\varphi^{\sharp}(x) = \varphi^{-1}\big(\varphi(m) + \varphi(M) - \varphi(x)\big) + x$$

for each $x \in I$. Then it follows from Lemma 2-(i) that $\varphi^\sharp$ is a convex function on I such that $\varphi^\sharp(m) = \varphi^\sharp(M) = m + M$. Therefore we have

$$\varphi^{-1}\bigl(\varphi(m) + \varphi(M) - \varphi\bigl(f(\omega)\bigr)\bigr) \leq m + M - f(\omega) \tag{3}$$

for almost all $\omega \in \Omega$. By integrating (3) with respect to ω, we obtain the second inequality in Lemma 4. We next suppose that φ is decreasing and convex on I. Then $-\varphi$ is increasing and concave on I. Therefore the desired inequality follows from (1), (2), and the above argument.

(II) Suppose that φ is increasing and convex on I. Then φ^{-1} is increasing and concave on $\varphi(I)$ by Lemma 1-(i), and hence the first inequality in Lemma 4 is reversed from Jensen's inequality. Also since $\varphi^\sharp$ is concave on I by Lemma 2-(iii), it follows that the second inequality in Lemma 4 is reversed from a consideration in (I). Similarly for the decreasing and concave case. □

Proof of Theorem 1 Let $\tilde{\varphi}, \tilde{\psi} \in \tilde{C}_{\mathrm{sm}}(I)$ with $\tilde{\varphi} \preceq \tilde{\psi}$, where $\varphi, \psi \in C_{\mathrm{sm}}(I)$.

(I-i) In the case of $\varphi, \psi \in C^+_{\mathrm{sm}}(I)$, we find that $\varphi \circ \psi^{-1}$ is increasing and concave on $\psi(I) = [\psi(m), \psi(M)]$ because $\varphi \preceq \psi$. Therefore we have from Lemma 4

$$\begin{aligned}
&\psi\bigl(N_{[I,f]}(\tilde{\varphi})\bigr) \\
&\quad = \bigl(\psi \circ \varphi^{-1}\bigr)\left(\varphi(m) + \varphi(M) - \int \varphi \circ f \, d\mu\right) \\
&\quad = \bigl(\varphi \circ \psi^{-1}\bigr)^{-1}\left(\bigl(\varphi \circ \psi^{-1}\bigr)\bigl(\psi(m)\bigr) + \bigl(\varphi \circ \psi^{-1}\bigr)\bigl(\psi(M)\bigr) - \int \bigl(\varphi \circ \psi^{-1}\bigr) \circ (\psi \circ f) \, d\mu\right) \\
&\quad = N_{[\psi(I), \psi \circ f]}\bigl(\widetilde{\varphi \circ \psi^{-1}}\bigr) \\
&\quad \leq \psi(m) + \psi(M) - \int \psi \circ f \, d\mu \\
&\quad = \psi\bigl(N_{[I,f]}(\tilde{\psi})\bigr),
\end{aligned}$$

so we obtain $N_{[I,f]}(\tilde{\varphi}) \leq N_{[I,f]}(\tilde{\psi})$ since ψ is strictly increasing on I.

(I-ii) In the case of $\varphi \in C^-_{\mathrm{sm}}(I)$ and $\psi \in C^+_{\mathrm{sm}}(I)$, we find that $\varphi \circ \psi^{-1}$ is decreasing and convex on $\psi(I)$ because $\varphi \preceq \psi$. Then $-\varphi, \psi \in C^+_{\mathrm{sm}}(I)$ and $(-\varphi) \circ \psi^{-1}$ is increasing and concave on $\psi(I)$. Therefore we have from (I-i) and (2)

$$N_{[I,f]}(\tilde{\varphi}) = N_{[I,f]}(\widetilde{-\varphi}) \leq N_{[I,f]}(\tilde{\psi}).$$

(I-iii) In the case of $\varphi, \psi \in C^-_{\mathrm{sm}}(I)$, we find that $\varphi \circ \psi^{-1}$ is increasing and convex on $\psi(I)$ because $\varphi \preceq \psi$. Then $\varphi \in C^-_{\mathrm{sm}}(I)$, $-\psi \in C^+_{\mathrm{sm}}(I)$, and $\varphi \circ (-\psi)^{-1}$ is decreasing and convex on $-\psi(I)$ by (1). Therefore we have from (I-ii) and (2)

$$N_{[I,f]}(\tilde{\varphi}) \leq N_{[I,f]}(\widetilde{-\psi}) = N_{[I,f]}(\tilde{\psi}).$$

(I-iv) In the case of $\varphi \in C^+_{\mathrm{sm}}(I)$ and $\psi \in C^-_{\mathrm{sm}}(I)$, we find that $\varphi \circ \psi^{-1}$ is decreasing and concave on $\psi(I)$ because $\varphi \preceq \psi$. Then $-\varphi, \psi \in C^-_{\mathrm{sm}}(I)$ and $-\varphi \circ \psi^{-1}$ is increasing and convex on $\psi(I)$. Therefore we have from (I-iii) and (2)

$$N_{[I,f]}(\tilde{\varphi}) = N_{[I,f]}(\widetilde{-\varphi}) \le N_{[I,f]}(\tilde{\psi}).$$

This completes the proof. □

Remark Let $\varphi, \psi \in C_{\mathrm{sm}}(I)$. We see from Theorem 1 and Lemma 1 that $\psi \preceq \varphi$ and then $N_{[I,f]}(\tilde{\varphi}) \ge N_{[I,f]}(\tilde{\psi})$ if any of the following four conditions holds:

(v) $\varphi, \psi \in C^{+}_{\mathrm{sm}}(I)$ and $\varphi \circ \psi^{-1}$ is convex on $\psi(I)$.
(vi) $\varphi \in C^{-}_{\mathrm{sm}}(I)$, $\psi \in C^{+}_{\mathrm{sm}}(I)$, and $\varphi \circ \psi^{-1}$ is concave on $\psi(I)$.
(vii) $\varphi, \psi \in C^{-}_{\mathrm{sm}}(I)$ and $\varphi \circ \psi^{-1}$ is concave on $\psi(I)$.
(viii) $\varphi \in C^{+}_{\mathrm{sm}}(I)$, $\psi \in C^{-}_{\mathrm{sm}}(I)$, and $\varphi \circ \psi^{-1}$ is convex on $\psi(I)$.

Throughout the remainder of the paper, we assume that $\Omega = I$ and $f(x) = x$ for all $x \in I$.

Proof of Corollary 1 Let $\varphi, \psi \in C_{\mathrm{sm}}(I)$ with $\varphi \preceq \psi$ and $t_1, \dots, t_n \in \mathbf{R}$ with $\sum_{i=1}^{n} t_i = 1$, $0 < t_1$, $t_n < 1$ and $t_2, \dots, t_{n-1} < 0$. Let $x_1, \dots, x_n \in I$ be such that $x_1 \le x_2, \dots, x_{n-1} \le x_n$. Put $s_1 = 1 - t_1, s_2 = -t_2, \dots, s_{n-1} = -t_{n-1}, s_n = 1 - t_n$. Then we have $\sum_{i=1}^{n} s_i = 1$ and $s_1, \dots, s_n > 0$. So

$$\mu \equiv s_1 \delta_{x_1} + \cdots + s_n \delta_{x_n}$$

is a probability measure on I, where δ_x denotes the Dirac measure at $x \in I$. Taking $[x_1, x_n]$ instead of I in Theorem 1, we obtain

$$\varphi^{-1}\left(\varphi(x_1) + \varphi(x_n) - \sum_{i=1}^{n} s_i \varphi(x_i)\right) \le \psi^{-1}\left(\psi(x_1) + \psi(x_n) - \sum_{i=1}^{n} s_i \psi(x_i)\right),$$

which implies the desired inequality

$$\varphi^{-1}\left(\sum_{i=1}^{n} t_i \varphi(x_i)\right) \le \psi^{-1}\left(\sum_{i=1}^{n} t_i \psi(x_i)\right).$$

This completes the proof. □

Remark Let φ, ψ be in $C_{\mathrm{sm}}(I)$ such that any of (v), (vi), (vii), and (viii) holds. Then $\psi \preceq \varphi$ holds from Lemma 1. Therefore if $t_1, \dots, t_n \in \mathbf{R}$ with $\sum_{i=1}^{n} t_i = 1$, $0 < t_1$, $t_n < 1$, and $t_2, \dots, t_{n-1} < 0$, then

$$\varphi^{-1}\left(\sum_{i=1}^{n} t_i \varphi(x_i)\right) \ge \psi^{-1}\left(\sum_{i=1}^{n} t_i \psi(x_i)\right)$$

holds from Corollary 1.

Example 1 Put $\varphi(x) = \log x$ and $\psi(x) = x$ for each positive number $x > 0$. Then Corollary 1 easily implies that

$$\prod_{i=1}^{n} x_i^{t_i} \le \sum_{i=1}^{n} t_i x_i$$

holds for all $t_1, \dots, t_n \in \mathbf{R}$ with $\sum_{i=1}^n t_i = 1$, $0 < t_1, t_n < 1$, and $t_2, \dots, t_{n-1} < 0$, and all positive numbers $x_1, \dots, x_n$ with $x_1 \leq x_2, \dots, x_{n-1} \leq x_n$. This is a geometric-arithmetic mean inequality with negative weights.

Example 2 Put $\varphi(x) = \frac{1}{x}$ and $\psi(x) = \log x$ for each positive number $x > 0$. Then Corollary 1 easily implies that

$$\left(\sum_{i=1}^n \frac{t_i}{x_i}\right)^{-1} \leq \prod_{i=1}^n x_i^{t_i}$$

holds for all $t_1, \dots, t_n \in \mathbf{R}$ with $\sum_{i=1}^n t_i = 1$, $0 < t_1, t_n < 1$, and $t_2, \dots, t_{n-1} < 0$, and all positive numbers $x_1, \dots, x_n$ with $x_1 \leq x_2, \dots, x_{n-1} \leq x_n$. This is a harmonic-geometric mean inequality with negative weights.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this paper. All authors read and approved the final manuscript.

Author details
[1]The Open University of Japan, Chiba, 261-8586, Japan. [2]Toho University, Funabashi, 273-0866, Japan.

Acknowledgements
The authors are grateful to the referees for careful reading of the paper and for helpful suggestions and comments. The second author is partially supported by Grant-in-Aid for Scientific Research, Japan Society for the Promotion of Science.

Received: 23 June 2014 Accepted: 21 October 2014 Published: 06 Nov 2014

References
1. Nakasuji, Y, Kumahara, K, Takahasi, S-E: A new interpretation of Jensen's inequality and geometric properties of φ-means. J. Inequal. Appl. **2011**, 48 (2011)
2. Jensen, J: Sur les fonctions convexes et les inégalités entre les valeurs moyennes. Acta Math. **30**, 175-193 (1906)
3. József, S: On global bounds for generalized Jensen's inequality. Ann. Funct. Anal. **4**(1), 18-24 (2013)
4. Janusz, M: Generalized weighted quasi-arithmetic means and the Kolmogorov-Nagumo theorem. Colloq. Math. **133**(1), 35-49 (2013)

10.1186/1029-242X-2014-450
Cite this article as: Nakasuji and Takahasi: A new order-preserving average function on a quotient space of strictly monotone functions and its applications. *Journal of Inequalities and Applications* 2014, 2014:450