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$\mathcal{H}^p$-Boundedness of Weyl multipliers

Jizheng Huang* and Jietong Wang

*Correspondence:
hjzheng@163.com
College of Sciences, North China University of Technology, Beijing, 100144, China

Abstract

In this paper, we will define Hardy spaces $\mathcal{H}^p(\mathbb{C}^n)$ associated with twisted convolution operators and give the atomic decomposition of $\mathcal{H}^p(\mathbb{C}^n)$, where $\frac{2n}{2n+1} < p \leq 1$. Then we consider the boundedness of the Weyl multiplier on $\mathcal{H}^p(\mathbb{C}^n)$.

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Keywords: Hardy space; twisted convolution; Weyl multiplier; square function

1 Introduction

The 'twisted translation' τ_w on $\mathbb{C}^n$ is defined on measurable functions by

$$\begin{aligned}(\tau_w f)(z) &= \exp\left(\frac{1}{2}\sum_{j=1}^{n}(w_j z_j + \bar{w}_j \bar{z}_j)\right) f(z)\\ &= f(z+w)\exp\left(\frac{i}{2}\operatorname{Im}(z\cdot\bar{w})\right)\end{aligned}$$

and the 'twisted convolution' of two functions f and g on $\mathbb{C}^n$ can be defined as

$$\begin{aligned}(f\times g)(z) &= \int_{\mathbb{C}^n} f(w)\tau_{-w}g(z)\,dw\\ &= \int_{\mathbb{C}^n} f(z-w)g(w)\overline{\omega}(z,w)\,dw,\end{aligned}$$

where $\omega(z,w) = \exp(\frac{i}{2}\operatorname{Im}(z\cdot\bar{w}))$.

In order to define the space $\mathcal{H}^p(\mathbb{C}^n)$ associated with 'twisted convolution', we first define the following version maximal operator in terms of twisted convolutions.

Let $\mathcal{B} = \{\varphi \in C^\infty(\mathbb{C}^n) : \operatorname{supp}\varphi \subset B(0,1), \|\varphi\|_\infty \leq 1, \|\nabla\varphi\|_\infty \leq 2\}$, and for $t > 0$, $\varphi_t(z) = t^{-2n}\varphi(\frac{z}{t})$. Given $\sigma \in (0,+\infty]$ and a tempered distribution f, define the grand maximal function

$$M_\sigma f(z) = \sup_{\varphi\in\mathcal{B}, 0<t<\sigma} \left|\varphi_t \times f(z)\right|, \tag{1}$$

where $B(0,1) = \{z \in \mathbb{C}^n : |z| < 1\}$.

Definition 1 Let f be a tempered distribution on $\mathbb{C}^n$ and $\frac{2n}{2n+1} < p \leq 1$, we say that f belongs to the Hardy space $\mathcal{H}^p(\mathbb{C}^n)$ if and only if the grand maximal function

$$M_\sigma f(z) = \sup_{\varphi\in\mathcal{B}, 0<t<\sigma} \left|\varphi_t \times f(z)\right|$$

lies in $L^p(\mathbb{C}^n)$, that is, $\mathcal{H}^p(\mathbb{C}^n) = \{f \in \mathcal{S}'(\mathbb{C}^n) : M_\sigma f \in L^p\}$, where $\mathcal{S}(\mathbb{C}^n)$ denotes the Schwartz space. We set $\|f\|_{\mathcal{H}^p} = \|M_\sigma f\|_{L^p}$.

Remark 1 From Theorem A in [1], we know that for some σ, $0 < \sigma < \infty$, $M_\sigma f \in L^p$ if and only if $M_\infty f \in L^p$, when $\frac{2n}{2n+1} < p \leq 1$, so the $\mathcal{H}^p$ also can be defined as $\{f \in \mathcal{S}'(\mathbb{C}^n) : M_\infty f \in L^p\}$, where $\frac{2n}{2n+1} < p \leq 1$. The case of $p = 1$ has been considered in [1].

Let $\frac{2n}{2n+1} < p \leq 1$, an atom for $\mathcal{H}^p(\mathbb{C}^n)$ centered at $z \in \mathbb{C}^n$ is a function $a(z)$ which satisfies:

(1) $\operatorname{supp} a \subset B(z,r)$,

(2) $\|a\|_\infty \leq (2r)^{-\frac{2n}{p}}$,

(3) $\int a(w)\overline{\omega}(z,w)\,dw = 0$.

The atomic norm of f can be defined as

$$\|f\|_{\text{atom}} = \inf\left\{\left(\sum_{j=1}^\infty |\lambda_j|^p\right)^{\frac{1}{p}} : f = \sum \lambda_j a_j\right\},$$

where the infimum is taken over all decompositions of $f = \sum \lambda_j a_j$ in the sense of $\mathcal{S}'(\mathbb{C}^n)$ and a_j are atoms.

In this paper, we will first give the atomic decomposition of $\mathcal{H}^p(\mathbb{C}^n)$ as follows.

Theorem 1 *Let $f \in \mathcal{S}'(\mathbb{C}^n)$ and $\frac{2n}{2n+1} < p \leq 1$, then $f \in \mathcal{H}^p(\mathbb{C}^n)$ if and only if $f = \sum \lambda_j a_j$, where a_j are atoms. Moreover, $\|f\|_{\mathcal{H}^p} \sim \|f\|_{\text{atom}}$.*

Remark 2 The case $p = 1$ has been proved in [1], so we will consider the case $\frac{2n}{2n+1} < p < 1$ in this paper.

The boundedness of the Weyl multiplier has been considered by many authors (*cf.* [2] and [3]). In this paper, we will consider the boundedness of the Weyl multiplier on $\mathcal{H}^p(\mathbb{C}^n)$. We first give some notations for Weyl multipliers. On $\mathbb{C}^n$ consider the $2n$ linear differential operators

$$Z_j = \frac{\partial}{\partial_{z_j}} + \frac{1}{4}\bar{z}_j, \qquad \bar{Z}_j = \frac{\partial}{\partial_{\bar{z}_j}} - \frac{1}{4}z_j, \quad j = 1,2,\ldots,n. \tag{2}$$

Together with the identity they generate a Lie algebra h^n which is isomorphic to the $2n+1$ dimensional Heisenberg algebra. The only nontrivial commutation relations are

$$[Z_j, \bar{Z}_j] = -\frac{1}{2}I, \quad j = 1,2,\ldots,n. \tag{3}$$

The operator L defined by

$$L = -\frac{1}{2}\sum_{j=1}^n (Z_j\bar{Z}_j + \bar{Z}_j Z_j)$$

is nonnegative, self-adjoint, and elliptic. Therefore it generates a diffusion semigroup $\{T_t^L\}_{t>0} = \{e^{-tL}\}_{t>0}$. There exists an irreducible projective representation W of $\mathbb{C}^n$ into a separable Hilbert space H_W such that

$$W(z+v) = \omega(z,v)W(z)W(v).$$

Given a function f in $L^1(\mathbb{C}^n)$ its Weyl transform $\tau(f)$ is a bounded operator on H_W defined by

$$\tau(f) = \int_{\mathbb{C}^n} f(z)W(z)\,dz. \tag{4}$$

Let $H = -\Delta + |x|^2$ be the Hermite operator, then we have $\tau(Lf) = \tau(f)H$ or more generally

$$\tau\big(\phi(L)f\big) = \tau(f)\phi(H). \tag{5}$$

We say that a bounded operator M on $L^2(\mathbb{R}^n)$ is a Weyl multiplier on $L^p(\mathbb{R}^n)$ if the operator T_M initially defined on $L^1 \cup L^p$ by

$$\tau(T_M f) = \tau(f)M$$

extends to a bounded operator on $L^p(\mathbb{C}^n)$. In [4], the author considered multipliers of the form $\phi(H)$ and proved the L^p-boundedness of $\phi(H)$.

In this paper, we will prove the following.

Theorem 2 *Let*

$$\Delta_+\phi(N) = \phi(N+1) - \phi(N) \quad \textit{and} \quad \Delta_-\phi(N) = \phi(N) - \phi(N+1).$$

Suppose that the function ϕ satisfies

$$\big|\Delta_-^k\Delta_+^m\phi(N)\big| \le CN^{-(k+m)} \tag{6}$$

with k, m positive integers such that $k+m = 0,1,\dots,\nu$, where $\nu = n+1$ when n is odd and $\nu = n+2$ when n is even. Then $\phi(H)$ is a Weyl multiplier on $\mathcal{H}^p(\mathbb{C}^n)$, where $\frac{2n}{2n+1} < p \le 1$.

Remark 3 The case $p = 1$ has been proved in [3].

Throughout the article, we will use A and C to denote the positive constants, which are independent of main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \le \frac{B_1}{B_2} \le C$.

The paper is organized as follows. In Section 2, we will give the proof of Theorem 1. Theorem 2 will be proved in Section 3.

2 Atomic decomposition for $\mathcal{H}^p(\mathbb{C}^n)$

The local Hardy space $h^p(\mathbb{C}^n)$ has been defined in [5]; let $f \in \mathcal{S}'(\mathbb{C}^n)$ and write

$$f_\sigma^*(z) = \sup_{\varphi\in\mathcal{B},0<t<\sigma} \big|\varphi_t * f(z)\big|.$$

Proposition 1 *When $\frac{2n}{2n+1} < p < 1$, the following conditions are equivalent*:

(I_1) $f \in h^p(\mathbb{C}^n)$;

(I_2) $f^*_\sigma(z) \in L^p(\mathbb{C}^n)$;

(I_3) $f = \sum \lambda_j a_j$, *where* $\sum_{j=1}^n |\lambda_j|^p < \infty$, $\operatorname{supp} a_j \subset B(z_j, r_j)$, $\|a_j\|_\infty \leq (2r_j)^{-\frac{2n}{p}}$, *and* $\int a_j(z)\,dz = 0$, *whenever* $r_j < \sigma$.

Consider a partition of $\mathbb{C}^n$ into a mesh of balls $B_j = B(z_j, \frac{\sigma}{2})$, $j = 1, 2, \ldots$, and construct a $\mathbb{C}^\infty$ partition of unity φ_j such that $\operatorname{supp}\varphi_j \subset B(z_j, \sigma)$. The proof of the following lemma is quite similar to Theorem 2.2 in [1], so we omit it.

Lemma 1 *Let* $\frac{2n}{2n+1} < p < 1$, *assume* $M_\sigma f(z) \in L^p$, *then* $g_j(z) = f(z)\varphi_j(z)\overline{\omega}(z_j, z) \in h^p$, $j = 1, 2, \ldots$, *moreover, there exists* $C > 0$ *such that*

$$\sum_{j=1}^{\infty} \|g_j\|_{h^p} \leq C \big\| M_\sigma f(z) \big\|_p.$$

By Proposition 1 and Lemma 1, we know that every element in $\mathcal{H}^p(\mathbb{C}^n)$ can be written as $f = \sum \lambda_j a_j$, where

(I) $(\sum_{j=1}^\infty |\lambda_j|^p)^{\frac{1}{p}} \leq C\|f\|_{\mathcal{H}^p}$;

(II) a_j is supported in $B(z_j, r_j)$, and $\|a_j\|_\infty \leq (2r_j)^{-\frac{2n}{p}}$;

(III) whenever $r_j < \sigma$, there exists ξ_j such that $|\xi_j - z_j| \leq 2\sigma$ and $\int a_j(z)\overline{\omega}(\xi_j, z)\,dz = 0$.

This is not yet the atomic decomposition for $\mathcal{H}^p$. In order to obtain it we must first replace condition (III) with a centered cancellation property.

Lemma 2 *Let* $\frac{2n}{2n+1} < p < 1$ *and* $a(z)$ *be a function supported on* $B = B(z_0, r)$, $r < \sigma$ *such that*

(I_1) $\|a\|_\infty \leq (2r)^{-\frac{2n}{p}}$;

(I_2) $\int a_j(z)\overline{\omega}(\xi, z)\,dz = 0$ *for some* ξ, $|\xi - z_0| \leq 2\sigma$. *If* σ *is sufficiently small,* $a(z)$ *can be decomposed as* $a(z) = \sum \lambda_j \alpha_j(z)$, *where*
- (a) $\sum_{j=1}^\infty |\lambda_j|^p \leq C$;
- (b) $\operatorname{supp}\alpha_j \subset B(z_j, r_j)$, $\|\alpha_j\|_\infty \leq (2r_j)^{-\frac{2n}{p}}$;
- (c) $\int \alpha_j(z)\overline{\omega}(z_j, z)\,dz = 0$, *whenever* $r_j < \sigma$.

Proof Write $a(z) = g^{(1)}(z) + b^{(1)}(z)$, where

$$b^{(1)}(z) = \left(\frac{1}{|B|}\int_B a(w)\overline{\omega}(z_0, w)\,dw\right)\chi_B(z)\omega(z_0, z).$$

Then the function $\frac{1}{2}g^{(1)}$ satisfies (b) and (c); on the other hand

$$\begin{aligned}
\big|b^{(1)}(z)\big| &= \frac{1}{|B|}\left|\int_B a(w)\big(\overline{\omega}(z_0, w - z_0) - \overline{\omega}(\xi, w - z_0)\big)\,dw\right| \\
&\leq \frac{1}{|B|}\int_B \big|a(w)\big|\big|\big(\overline{\omega}(z_0, w - z_0) - \overline{\omega}(\xi, w - z_0)\big)\big|\,dw \\
&\leq \frac{\|a\|_\infty}{|B|}\int_B \big|\big(\overline{\omega}(z_0, w - z_0) - \overline{\omega}(\xi, w - z_0)\big)\big|\,dw \\
&\leq C \cdot r^{1-\frac{2n}{p}} \cdot \sigma.
\end{aligned}$$

Let $q = \frac{2np}{2n-p}$, since $\frac{2n}{2n+1} < p < 1$, we have $q > 1$, hence

$$\|b^{(1)}(z)\|_q \le C\left(\int_B \left|r^{1-\frac{2n}{p}} \cdot \sigma\right|^q dz\right)^{\frac{1}{q}} \le C\sigma.$$

Since supp $b^{(1)}(z) \subset B(z_0,\sigma)$ and $\overline{\omega}(z_0,z)b^{(1)}(z) \in h^p$, we have

$$\begin{aligned}\|\overline{\omega}(z_0,z)b^{(1)}(z)\|_{h^p} &\le C(\sigma)\left\|\left(\overline{\omega}(z_0,z)b^{(1)}(z)\right)_1^*\right\|_p \\ &= C(\sigma)\left(\int_B \left|\left(\overline{\omega}(z_0,z)b^{(1)}(z)\right)_1^*\right|^p dz\right)^{\frac{1}{p}}.\end{aligned}$$

Let $l = \frac{q}{p}$ and $I = \{\int_B |(\overline{\omega}(z_0,z)b^{(1)}(z))_1^*|^p\, dz\}^{\frac{1}{p}}$, then

$$\begin{aligned}I^p &\le \left\{\int_B \left|\left(\overline{\omega}(z_0,z)b^{(1)}(z)\right)_1^*\right|^q dz\right\}^{\frac{1}{l}} \cdot (2r)^{\frac{2n}{l'}} \\ &= \left\|\left(\overline{\omega}(z_0,z)b^{(1)}(z)\right)_1^*\right\|_q^{\frac{q}{l}} \cdot (2r)^{\frac{2n}{l'}},\end{aligned}$$

thus

$$I \le \left\|\left(\overline{\omega}(z_0,z)b^{(1)}(z)\right)_1^*\right\|_q \cdot (2r)^{\frac{2n(q-p)}{pq}} \le C\cdot\sigma\cdot(1+\sigma)^{\frac{2n+p}{p}}.$$

Then we have $\|\overline{\omega}(z_0,z)b^{(1)}(z)\|_{h^p} \le C\cdot\sigma\cdot(1+\sigma)^{\frac{2n+p}{p}}$. Let σ be small enough such that $C\cdot\sigma\cdot(1+\sigma)^{\frac{2n+p}{p}} < \frac{1}{2}$. By Proposition 1, we have

$$b^{(1)}(z) = \sum_{j=1}^{\infty} \eta_j^{(1)} a_j^{(1)}(z), \quad \text{where } \sum_{j=1}^{\infty} \left|\eta_j^{(1)}\right|^p \le \frac{1}{2}.$$

The functions $a_j^{(1)}(z)$ are as in (b) and (c). We can now decompose the function $a_j^{(1)}(z)$ whose support is contained in a ball $B(z_j,r_j)$, with $r_j < \sigma$, as we did for $a(z)$, thus

$$b^{(1)}(z) = g^{(2)}(z) + b^{(2)}(z),$$

where $g^{(2)}(z) = \sum \lambda_j^{(2)} \alpha_j^{(2)}(z)$, $\alpha_j^{(2)}(z)$ satisfy (b) and (c), and $\sum_{j=1}^{\infty} |\lambda_j^{(2)}|^p \le \frac{1}{2}$. Moreover,

$$b^{(2)}(z) = \sum \eta_j^{(2)} a_j^{(2)}(z),$$

where the $a_j^{(2)}(z)$ are as in (b) and (c) and

$$\sum_{j=1}^{\infty} \left|\eta_j^{(2)}\right|^p \le \frac{1}{4}.$$

So we can construct sequences $b^{(k)}$ and $g^{(k)}$ such that

$$b^{(k)} = g^{(k+1)} + b^{(k+1)},$$

$g^{(k+1)} = \sum \lambda_j^{(k+1)} \alpha_j^{(k+1)}(z)$, where the $\alpha_j^{(k+1)}(z)$ satisfy (b) and (c), and also

$$\sum_{j=1}^{\infty} \left|\lambda_j^{(k+1)}\right|^p \leq \frac{1}{2^{k-1}},$$

$b^{(k+1)} = \sum \eta_j^{(k+1)} a_j^{(k+1)}(z)$, where the $a_j^{(k+1)}(z)$ satisfy (b) and (c), and also

$$\sum_{j=1}^{\infty} \left|\eta_j^{(k+1)}\right|^p \leq \frac{1}{2^{k+1}}.$$

This shows that $a(z) = \sum_k g^{(k)}(z)$ and gives the proof of Lemma 2. □

Lemma 3 *There exists $C > 0$ such that for any $\mathcal{H}^p$-atom, we have $\|M_\infty a(z)\|_p < C$, where $\frac{2n}{2n+1} < p < 1$.*

Proof Without loss of generality, we can assume that $a(z)$ is an atom supported on $B(0,r)$. Let $\varphi \in \mathcal{B}$, $t > 0$, then

$$\varphi_t \times a(z) = \int \left[\varphi_t(z-w) - \varphi_t(z)\right] a(w)\overline{\omega}(z,w)\, dw + \varphi_t(z)\hat{a}(-iz).$$

If $|z| > 2r$ and $\varphi_t \times a(z) \neq 0$, we have $t > |z| - r > \frac{1}{2}z$, so that

$$\left|\varphi_t \times a(z)\right| \leq C_1 r \frac{r^{\frac{2n(p-1)}{p}}}{|z|^{2n+1}} + C_2 \frac{|\hat{a}(-iz)|}{|z|^{2n}}.$$

Let $I_1 = r\frac{r^{\frac{2n(p-1)}{p}}}{|z|^{2n+1}}$, and $I_2 = \frac{|\hat{a}(-iz)|}{|z|^{2n}}$, then $|\varphi_t \times a(z)| \leq C(I_1 + I_2)$. Therefore

$$\int_{|z|>2r} \left|M_\infty a(z)\right|^p dz \leq \int_{|z|>2r} |I_1 + I_2|^p\, dz \leq C\left(\int_{|z|>2r} |I_1|^p\, dz + \int_{|z|>2r} |I_2|^p\, dz\right).$$

First we have

$$\int_{|z|>2r} |I_1|^p\, dz \leq C_1 \int_{|z|>2r} \frac{r^{2n(p-1)+p}}{|z|^{(2n+1)p}}\, dz \leq C_1'.$$

By Hardy's inequality (*cf.* [6, Theorem 7.22, p.341]), we get

$$\int_{|z|>2r} |I_2|^p\, dz \leq C_2 \int_{|z|>2r} \frac{|\hat{a}(-iz)|^p}{|z|^{2np}}\, dz \leq C_2'.$$

We also have

$$\begin{aligned}\int_{|z|\leq 2r} \left|M_\infty a(z)\right|^p dz &\leq \left(\int_{|z|\leq 2r} \left|M_\infty a(z)\right| dz\right)^{\frac{1}{p}} \cdot (2r)^{2n(1-p)} \\ &\leq C\|a\|_\infty^p (4r)^{2np}(4r)^{2n(1-p)} \leq C.\end{aligned}$$

This completes the proof of Lemma 3. □

Proof of Theorem 1 By Lemma 2 and Lemma 3, we can obtain the proof of Theorem 1. □

3 The boundedness of the Weyl multiplier on $\mathcal{H}^p(\mathbb{C}^n)$

In order to prove Theorem 2, we need to give some characterizations for $\mathcal{H}^p(\mathbb{C}^n)$. Let $K_t^L(z)$ be the heat kernel of $\{T_t^L\}_{t>0}$, then we can get (*cf.* [4])

$$K_t(z) = (4\pi)^{-n}(\sinh t)^{-n}e^{-\frac{1}{4}|z|^2(\coth t)}. \tag{7}$$

It is easy to prove that the heat kernel $K_t(z)$ has the following estimates (*cf.* [3]).

Lemma 4 *There exists a positive constant $C > 0$ such that*

(i) $|K_t(z)| \le Ct^{-n}e^{-C\frac{|z|^2}{t}}$;
(ii) $|\nabla K_t(z)| \le Ct^{-n-\frac{1}{2}}e^{-C\frac{|z|^2}{t}}$.

Let $Q_t^k(z)$ be the twisted convolution kernel of $Q_t^k = t^{2k}\partial_s^k T_s^L|_{s=t^2}$, then

$$Q_t^k(z) = t^{2k}\partial_s^k K_s(z)|_{s=t^2}.$$

Lemma 5 *There exist constants $C, C_k > 0$ such that*

(i) $|Q_t^k(z)| \le C_k t^{-2n}e^{-Ct^{-2}|z|^2}$;
(ii) $|\nabla Q_t^k(z) - Q_t^k(w)| \le C_k t^{-2n-1}e^{-Ct^{-2}|z|^2}|z-w|$.

In the following, we define the Lusin area integral operator by

$$\left(S_L^k f\right)(z) = \left(\int_0^{+\infty}\int_{|z-w|<t}\left|Q_t^k f(w)\right|^2\frac{dw\,dt}{t^{2n+1}}\right)^{1/2}$$

and the Littlewood-Paley g-function

$$\mathcal{G}_L^k(f)(z) = \left(\int_0^{\infty}\left|Q_t^k f(z)\right|^2\frac{dt}{t}\right)^{1/2}.$$

We also consider the g_λ^*-function associated with L defined by

$$g_{\lambda,k}^* f(x) = \left(\int_0^{\infty}\int_{\mathbb{C}^n}\left(\frac{t}{t+|z-w|}\right)^{2\lambda n}\left|Q_t^k f(w)\right|^2\frac{dw\,dt}{t^{2n+1}}\right)^{1/2}.$$

We have the following lemma, whose proof is standard (*cf.* [3]).

Lemma 6

(i) *The operators S_L^k and $\mathcal{G}_L^k$ are isometries on $L^2(\mathbb{C}^n)$.*
(ii) *When $\lambda > 1$, there exists a constant $C > 0$, such that*

$$C^{-1}\|f\|_{L^2} \le \left\|g_{\lambda,k}^* f\right\|_{L^2} \le C\|f\|_{L^2}.$$

Now we can prove the following lemma.

Lemma 7 *Let $\frac{2n}{2n+1} < p < 1$ and $f \in \mathcal{S}'(\mathbb{C}^n)$, then we have:*

(1) *$f \in \mathcal{H}^p(\mathbb{C}^n)$ if and only if its Lusin area integral $S_L^k f \in L^p(\mathbb{C}^n)$. Moreover, we have*

$$\|f\|_{\mathcal{H}^p} \sim \|S_L^k f\|_{L^p}.$$

(2) *$f \in \mathcal{H}^p(\mathbb{C}^n)$ if and only if its Littlewood-Paley g-function $\mathcal{G}_L^k f \in L^p(\mathbb{C}^n)$. Moreover, we have*

$$\|f\|_{\mathcal{H}^p} \sim \|\mathcal{G}_L^k f\|_{L^p}.$$

(3) *$f \in \mathcal{H}^p(\mathbb{C}^n)$ if and only if its $\mathcal{G}_\lambda^*$-function $\mathcal{G}_{\lambda,k}^* f \in L^p(\mathbb{C}^n)$, where $\lambda > 4$. Moreover, we have*

$$\|f\|_{\mathcal{H}^p} \sim \|\mathcal{G}_{\lambda,k}^* f\|_{L^p}.$$

Proof (1) By Lemma 6, we know there exists a constant $C > 0$ such that, for any atom $a(x)$ of $\mathcal{H}^p(\mathbb{C}^n)$, we have

$$\|S_L^k a\|_{L^p} \le C.$$

For the reverse, by Theorem 1, we can prove similarly to Proposition 4.1 in [7].

(2) Firstly, we can prove $\mathcal{G}_L^k$ is uniformly bounded on atoms of $\mathcal{H}^p(\mathbb{C}^n)$. For the reverse, we can prove the following inequality (*cf.* Theorem 7. 28 in [8]):

$$\|S_L^{k+1} f\|_{L^p} \le C\|\mathcal{G}_L^k f\|_{L^p}. \tag{8}$$

Then (2) follows from part (1) and (8).

(3) By $S_L^k f(z) \le (\frac{1}{2})^{2\lambda n} g_{\lambda,k}^* f(z)$, we know $f \in \mathcal{H}^p(\mathbb{C}^n)$ when $g_{\lambda,k}^* f \in L^p(\mathbb{C}^n)$. In the following, we show there exists a constant $C > 0$ such that for any atom $a(z)$ of $\mathcal{H}^p(\mathbb{C}^n)$, we have

$$\|g_{\lambda,k}^* a\|_{L^p} \le C.$$

We assume $a(z)$ is supported in $B(z_0, r)$, then

$$g_{\lambda,k}^* a(z)^2 \le C S_L^k a(z)^2 + \sum_{k=1}^{\infty} 2^{-2k\lambda n} S_L^{2^k} a(z)^2.$$

Then

$$\|g_{\lambda,k}^* a\|_{L^p} \le C_1 \|S_L^k a\|_{L^p} + C_2 \sum_{k=1}^{\infty} 2^{-k\lambda n} \|S_L^{2^k} a\|_{L^p}.$$

By part (1), we have $\|S_L^k a\|_{L^p} \le C$. We can prove (*cf.* [3])

$$\|S_L^{2^k} a\|_{L^p} \le C 2^{4kn}. \tag{9}$$

Therefore, when $\lambda > 4$, we have $\|g_{\lambda,k}^* a\|_{L^p} \le C$. Then Lemma 7 is proved. $\square$

In the following, we give the proof of Theorem 2.

Proof of Theorem 2 Firstly, by Lemma 4.1 in [2], we get

$$\mathcal{G}_L^{k+1}(F)(z) \leq C\mathcal{G}^*_{\frac{k}{n},1}(f)(z),$$

where $F(z) = T_\phi f(z)$, then, by Lemma 7, when $k > 4n$,

$$\|\phi(L)f\|_{\mathcal{H}^p} \leq C\|\mathcal{G}_L^{k+1}(F)(z)\|_{L^p} \leq C\|\mathcal{G}^*_{\frac{k}{n},1}(f)(z)\|_{L^p} \leq C\|f\|_{\mathcal{H}^p}.$$

This completes the proof of Theorem 2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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