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Regularity of the attractor for strongly damped wave equations with nonlinearity

Hongyan Li*

*Correspondence:
lihongyan580@sohu.com
College of Management, Shanghai
University of Engineering Science,
Shanghai, 201620, China

Abstract

The long time behavior of the solutions for the strongly damped wave equation is considered with nonlinear damping, a nonlinear forcing term, and with a periodic boundary condition. We prove that the global attractor which captures all trajectories in $H^1(\Omega) \times L^2(\Omega)$ is a compact set in $H^2(\Omega) \times H^1(\Omega)$.

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1 Introduction

For $u(x, t) : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$, we consider the strongly damped wave equation on a bounded domain Ω in \mathbb{R}^3 with smooth boundary:

$$u_{tt} - \alpha \Delta u_t - \Delta u + h(u_t) + f(u) = g(x), \quad (1.1)$$

supplemented with the periodic boundary conditions and initial conditions

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \Omega, \quad (1.2)$$

where the strongly damped coefficient α is a positive constant, the damped function $h : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, and the forcing term $f : \mathbb{R} \rightarrow \mathbb{R}$ is a nonlinear term satisfying some growth conditions, $g : \Omega \rightarrow \mathbb{R}$ is the external force.

When $\alpha = 0$, (1.1) reduces to a usual wave equation with nonlinear damping, which arises as an evolutionary mathematical model in various systems (cf. [1, 2]), for example: (i) modeling a continuous Josephson junction with specific h , g and f ; (ii) modeling a hybrid system of nonlinear waves and nerve conduct; and (iii) modeling a phenomenon in quantum mechanics and which has been studied widely by using of the concept of global attractors; see, for example, [2–10] for the linear damping case, and [11–18] for the nonlinear damping case. There are some results on the regularity in these papers; for example, in [7] the authors gave the dimensionality and related properties of the global attractor, and in [14], the authors presented a direct method to establish the optimal regularity of the attractor for the semilinear damped wave equation (when $\alpha = 0$ and $h(u_t) = u_t$) with nonlinearity for the critical growth. But for $\alpha \neq 0$, the case can be complex.

There is a large literature on the asymptotic behavior of solutions to (1.1), (1.2) (cf. [1, 19–23]). For the strongly damped wave equations, [23] had proved the uniform bounded-

ness of the global attractor for very strong damping in $H^1 \times L^2$ and obtained an estimate of the upper bound of the Hausdorff dimension of an attractor for the strongly damped nonlinear wave equation (1.1). Furthermore, when $h(u_t) = 0$, Pata and Zelik in [21] had proved the existence of compact global attractors of optimal regularity, *i.e.*, the compact global attractor on $H^1(\Omega) \times L^2(\Omega)$ is a bounded subset of $H^2 \times H^1$. The aim of this paper is to prove that the dynamical system associated with (1.1) possesses a compact global attractor in $H^2(\Omega) \times H^1(\Omega)$, *i.e.*, we will prove that the global attractor \mathcal{A}_1 in $H^1(\Omega) \times L^2(\Omega)$ is a compact set \mathcal{A}_2 in $H^2(\Omega) \times H^1(\Omega)$. This implies that $\mathcal{A}_1 = \mathcal{A}_2$.

The paper is organized as follows. In Section 2, we present the existence, uniqueness, and continuous dependence of solutions for problem (1.1), and we establish the existence of an absorbing set in $H^1(\Omega) \times L^2(\Omega)$. In Section 3, we prove the existence of the global attractor in $H^1(\Omega) \times L^2(\Omega)$. In Section 4, we establish the regularity of the attractor, *i.e.*, that the global attractor is a compact subset in $H^2(\Omega) \times H^1(\Omega)$.

2 Preliminaries

We assume that $g \in L^2(\Omega) = H$, and $\dot{H}^{-1} = (\dot{H}^1)^*$, with

$$\dot{H}^1 = H^1 \cap \left\{ u \in L^2(\Omega); \int_{\Omega} u(x) dx = 0 \right\},$$

and the functions h, f satisfy the following conditions:

(i) Let $f(u) \in C^1(\mathbb{R}; \mathbb{R})$ satisfy:

(1) The asymptotic sign condition

$$\limsup_{|s| \rightarrow +\infty} \frac{f(s)}{s} \leq 0. \tag{2.1}$$

(2) Let $F(s) = \int_0^s f(\rho) d\rho$, there exist constants $c_1, c_2, \delta, C_\delta > 0$ such that

$$(f(u), u) - c_1 \int_{\Omega} F(u) dx - \delta |u|^2 \geq -C_\delta |\Omega|, \tag{2.2}$$

where $|\cdot|_0$ denotes the absolute value of the number in \mathbb{R} . We have

$$|f'(s)| \leq c_2(1 + |s|^p) \quad \text{with} \quad \begin{cases} 0 \leq p < \infty, & \text{when } n = 1, 2, \\ 0 \leq p < 2, & \text{when } n = 3. \end{cases} \tag{2.3}$$

(ii) There exist two constants $\beta'_1, \beta'_2 \geq 0$ such that

$$h(0) = 0, \quad 0 < \beta'_1 \leq h'(s) \leq \beta'_2 < +\infty, \quad \forall s \in \mathbb{R}. \tag{2.4}$$

Let $Au = -\Delta u$, then the system (1.1)-(1.2) is equivalent to the following initial value problem in $\dot{H}^1(\Omega) \times L^2(\Omega)$:

$$\begin{cases} \dot{Y} = PY + Q(Y), & x \in \Omega, t > 0, \\ Y(0) = Y_0 = (u_0, u_1)^T \in \dot{H} \times L^2, \end{cases} \tag{2.5}$$

where $Y = (u, u_t)^T$, $Q(Y) = (0, -h(u_t) - f(u) + g)^T$,

$$P = \begin{pmatrix} 0 & I \\ -A & -kA \end{pmatrix}.$$

By the assumption (2.1)-(2.4), it is easy to check that the function $Q(Y) : \dot{H}^1 \times L^2 \rightarrow \dot{H}^1 \times L^2$ is continuously differentiable and globally Lipschitz continuous with respect to Y . By the classical theory concerning the existence and uniqueness of the solutions of evolution differential equations (cf. [15]), we have the following lemma (see [23] for details).

Lemma 2.1 *Consider the initial value problem (2.5) in $\dot{H}^1 \times L^2$. If (2.1)-(2.4) hold, then, for any $Y_0 \in \dot{H}^1 \times L^2$, there exists a unique continuous function $Y(\cdot) = Y(\cdot, Y_0) \in C(\mathbb{R}_+; \dot{H}^1 \times L^2)$ such that $Y(0, Y_0) = Y_0$ and $Y(t)$ satisfies the integral equation*

$$Y(t, Y_0) = e^{Pt} Y_0 + \int_0^t e^{P(t-\tau)} Q(Y(\tau)) d\tau.$$

$Y(t, Y_0)$ is called the mild solution of (2.5), and $Y(t, Y_0)$ is jointly continuous in t and Y_0 , and

$$(u, u_t) \in C(\mathbb{R}_+; \dot{H}^1(\Omega)) \times [C(\mathbb{R}_+; L^2(\Omega)) \cap L^2((0, T^*); \dot{H}^1(\Omega))], \quad \forall T^* > 0.$$

For any $t \geq 0$, we can introduce a map

$$S(t) : Y_0 = (u_0, u_1) \rightarrow (u, u_t) = Y(t, Y_0), \quad S(t) : \dot{H}^1 \times L^2 \rightarrow \dot{H}^1 \times L^2,$$

where $Y(t, Y_0)$ is the solution of (2.5), and then $\{S(t) | t \geq 0\}$ is a continuous semigroup on $\dot{H}^1 \times L^2$.

Consider the map $G : \dot{H}^1 \times L^2 \rightarrow \dot{H}^1 \times L^2$ defined as

$$G(u(t), u_t(t)) = e^{Pt}(u_0, u_1) + \int_0^t e^{P(t-\tau)} (0, g - h(u_t(\tau)) - f(u(\tau))) d\tau. \quad (2.6)$$

For some $\lambda > 0$, let

$$A_1 = A + \lambda I. \quad (2.7)$$

Let

$$\begin{aligned} (u, v) &= \int_{\Omega} uv dx, & |u| &= (u, u)^{\frac{1}{2}}, \quad \forall u, v \in L^2, \\ ((u, v)) &= \int_{\Omega} A_1 u \cdot v dx, & \|u\| &= ((u, u))^{\frac{1}{2}}, \quad \forall u, v \in H^1(\Omega). \end{aligned}$$

Obviously, A_1 is symmetric and positive definite, so we have the following Poincaré type inequality:

$$\|y\| \geq \sqrt{\lambda} |y|, \quad y \in L^2. \quad (2.8)$$

Let $h_1(u_t) = h(u_t) - \alpha\lambda u_t$, then (1.1) can be written in the equivalent form

$$u_{tt} + \alpha A_1 u_t + A_1 u + h_1(u_t) - \lambda u + f(u) = g, \tag{2.9}$$

and by (2.4), $h_1(u_t)$ satisfies

$$h_1(0) = 0, \quad -\alpha\lambda < \beta_1 \leq h_1'(\theta) \leq \beta_2 < +\infty, \quad \forall \theta \in \mathbb{R}, \tag{2.10}$$

where $\beta_1 = \beta_1' - \alpha\lambda$, $\beta_2 = \beta_2' - \alpha\lambda$ are two constants.

To construct an attractor for (1.1), we make the following assumptions. Let $\varphi = (u, v)^T$, $v = u_t + ku$, where k is chosen as

$$k = \frac{\alpha\lambda + \beta_1}{4 + 2(\alpha\lambda + \beta_1)\alpha + \beta_2^2/\lambda_1}. \tag{2.11}$$

Equation (2.9) can be written as

$$\varphi_t + H(\varphi) = F(\varphi), \quad \varphi(0) = (u_0, v_0 = u_1 + ku_0)^T, \quad t \geq 0, \tag{2.12}$$

where

$$F(\varphi) = \begin{pmatrix} 0 \\ \lambda u - f(u) + g \end{pmatrix},$$

$$H(\varphi) = \begin{pmatrix} ku - v \\ A_1 u - k(\alpha A_1 - k)u + (\alpha A_1 - k)v \end{pmatrix} + \begin{pmatrix} 0 \\ h_1(v - ku) \end{pmatrix}.$$

We define a new weighted inner product and norm in $E = \dot{H}^1 \times L^2$ as

$$(\varphi, \psi)_E = \mu((u_1, u_2)) + (v_1, v_2), \quad \|\varphi\|_E = (\varphi, \varphi)_E^{1/2}, \tag{2.13}$$

for any $\varphi = (u_1, v_1)^T, \psi = (u_2, v_2)^T \in E$, where μ is chosen as

$$\mu = \frac{4 + (\alpha\lambda + \beta_1)\alpha + \beta_2^2/\lambda}{4 + 2(\alpha\lambda + \beta_1)\alpha + \beta_2^2/\lambda} \in \left(\frac{1}{2}, 1\right). \tag{2.14}$$

Obviously, the norm $\|\cdot\|_E$ in (2.13) is equivalent to the usual norm $|\cdot|_{\dot{H}^1 \times L^2}$ in E .

Lemma 2.2 (see [23], Lemma 1) *For any $\varphi = (u, v)^T \in E$, if*

$$-\alpha\lambda < \beta_1 \leq \beta_2 < +\infty, \quad \beta_2 \geq |\beta_1| + \min\{1/\alpha, (\alpha\lambda + \beta_1)/2\}, \tag{2.15}$$

hold, then

$$(H(\varphi), \varphi)_E \geq \sigma \|\varphi\|_E^2 + \frac{\alpha}{2} \|v\|^2 + \frac{\beta_1}{2} |v|^2 \geq \sigma \|\varphi\|_E^2 + \frac{\alpha\lambda + \beta_1}{2} |v|^2, \tag{2.16}$$

where

$$\sigma = \frac{\alpha\lambda + \beta_1}{\gamma_1 + \sqrt{\gamma_1\gamma_2}}, \quad \gamma_1 = 4 + (\alpha\lambda + \beta_1)\alpha + \frac{\beta_2^2}{\lambda}, \quad \gamma_2 = (\alpha\lambda + \beta_1)\alpha + \frac{\beta_2^2}{\lambda}. \tag{2.17}$$

Proposition 2.1 *The semigroup $\{S(t)|t \geq 0\}$ possesses an absorbing set $\mathcal{B} \subset H^1 \times L^2$.*

Proof Let $\varphi = (u, v)^T \in E$ be the solution of (2.12). Taking the inner product $(\cdot, \cdot)_E$ of (2.12) with φ , we have

$$(\varphi_t, \varphi) + (H(\varphi), \varphi) = (F(\varphi), \varphi)$$

and

$$\frac{1}{2} \frac{d}{dt} \|\varphi\|_E^2 + (H(\varphi), \varphi)_E + (f(u), v) - \lambda(u, v) = (g, v). \tag{2.18}$$

By $v = u_t + ku$, we have

$$(f(u), v) = (f(u), u_t + ku) = \frac{d}{dt} \int_{\Omega} F(u) dx + k(f(u), u), \tag{2.19}$$

by (2.18) and (2.19), we have

$$\frac{1}{2} \frac{d}{dt} \left[\|\varphi\|_E^2 + 2 \int_{\Omega} F(u) dx \right] + (H(\varphi), \varphi)_E + k(f(u), u) - \lambda(u, v) = (g, v). \tag{2.20}$$

Let $y(t) = \|\varphi\|_E^2 + 2 \int_{\Omega} F(u) dx + 2C_{\delta}|\Omega| \geq \frac{1}{4} \|\varphi\|_E^2 \geq 0$. By (2.15) and (2.2), there exists $\delta > 0$ such that

$$\begin{aligned} & (H(\varphi), \varphi)_E + k(f(u), u) - \lambda(u, v) \\ & \geq \sigma \|\varphi\|_E^2 + \frac{\alpha\lambda + \beta_1}{2} |v|^2 + kc_1 \int_{\Omega} F(u) dx + k\delta |u|^2 - kC_{\delta}|\Omega| - \lambda|u||v| \\ & \geq \sigma \|\varphi\|_E^2 + \frac{\alpha\lambda + \beta_1}{2} |v|^2 + kc_1 \int_{\Omega} F(u) dx \\ & \quad + \frac{k\delta}{\lambda\mu} \mu |u|^2 - kC_{\delta}|\Omega| - \frac{\sqrt{\lambda}}{2\sqrt{\mu}} \|\varphi\|_E^2 \\ & \geq \left(\sigma + \frac{k\delta}{\lambda\mu} - \frac{\sqrt{\lambda}}{2\sqrt{\mu}} \right) \|\varphi\|_E^2 + kc_1 \int_{\Omega} F(u) dx - kC_{\delta}|\Omega| + \frac{\alpha\lambda + \beta_1}{2} |v|^2 \\ & \geq \frac{1}{2} \rho y - k(C_{\delta}|\Omega| + c_1 C_{\delta}|\Omega|) + \frac{\alpha\lambda + \beta_1}{2} |v|^2, \end{aligned} \tag{2.21}$$

where $\rho = \min\{kc_1, (\sigma + \frac{k\delta}{\lambda\mu} - \frac{\sqrt{\lambda}}{2\sqrt{\mu}})\}$. By (2.20) and (2.21),

$$\begin{aligned} \frac{d}{dt} y + \rho y & \leq 2kC_{\delta}|\Omega|(1 + c_1) + 2(g, v) - (\alpha\lambda + \beta_1)|v|^2 \\ & \leq 2kC_{\delta}|\Omega|(1 + c_1) + \frac{1}{\alpha\lambda + \beta_1} |g|^2. \end{aligned} \tag{2.22}$$

Using the Gronwall lemma, we have

$$\|\varphi(t)\|_E^2 \leq 4y(0)e^{-2\rho t} + 4 \left(\frac{|g|^2}{(\alpha\lambda + \beta_1)\rho} + \frac{2kC_{\delta}|\Omega|(1 + c_1)}{\rho} \right).$$

Following [24] and [8], it follows from (2.1) that, for each $\kappa > 0$, there is a constant $C_\kappa > 0$ such that, for each $u \in L^2(\Omega)$,

$$(f(u), u) = \int_{\Omega} f(u)u \, dx \leq \kappa |u|^2 + C_\kappa. \tag{2.23}$$

Note that (2.2) and (2.23) imply

$$\begin{aligned} c_1 \int_{\Omega} F(u) \, dx &\leq (f(u), u) - \delta |u|^2 + C_\delta |\Omega| \\ &\leq \kappa |u|^2 + C_\kappa - \delta |u|^2 + C_\delta |\Omega| \leq c(1 + |u|^2). \end{aligned} \tag{2.24}$$

For any bounded set B of E , where $\sup_{\varphi \in B} \|\varphi\|_E < r$, if $\varphi(0) \in B$, there exists $c_2 = c_2(r) > 0$ such that $y(0) = \|\varphi(0)\|_E^2 + 2 \int_{\Omega} F(u_0) \, dx + 2C_\delta |\Omega| \leq c_2$. Therefore, for the solution $\varphi(t) = (u(t), v(t))^T$ of (2.12) with $\varphi(0) \in B$,

$$\|\varphi(t)\|_E^2 \leq 4c_2(r)e^{-2\rho t} + 4 \left(\frac{|g|^2}{(\alpha\lambda + \beta_1)\rho} + \frac{2kC_\delta |\Omega|(1 + c_1)}{\rho} \right).$$

Taking

$$M_0 = 8 \left(\frac{|g|^2}{(\alpha\lambda + \beta_1)\rho} + \frac{2kC_\delta |\Omega|(1 + c_1)}{\rho} \right)$$

completes the proof. □

3 Existence of the global attractor in $\dot{H}^1 \times L^2$

Theorem 3.1 (see [2, I.1.1]) *Let $\{S(t)|t \geq 0\}$ be a continuous semigroup on $\dot{H}^1 \times L^2$ that possesses an absorbing ball in $\dot{H}^1 \times L^2$. Let us assume that, for any $t \geq 0$,*

$$S(t) = S_1(t) + S_2(t),$$

where:

- For every bounded set B , there exists t_0 that depends on B , such that $\bigcup_{t \geq t_0} S_1(t)B$ is relatively compact in $\dot{H}^1 \times L^2$.
- For every bounded set B ,

$$\limsup_{t \rightarrow \infty} \left(\sup_{\theta \in B} \|S_2(t)\theta\|_{\dot{H}^1 \times L^2} \right) = 0.$$

Then $S(t)_{t \geq 0}$ possesses a global attractor \mathcal{A} that is compact in $\dot{H}^1 \times L^2$.

Remark 3.1 (see [10]) We characterize such an attractor \mathcal{A} as

$$\begin{aligned} \mathcal{A} = \{ &(u_0, u_1) \in \dot{H}^1 \times L^2, (u_0^n, u_1^n)_n \subset \mathcal{B}, \exists (t_n) \rightarrow \infty, \\ &\text{such that } \|S(t_n)(u_0^n, u_1^n) - (u_0, u_1)\|_{\dot{H}^1 \times L^2} \rightarrow 0 \}. \end{aligned}$$

Theorem 3.2 *The semigroup $\{S(t)|t \geq 0\}$ possesses a global attractor \mathcal{A} in $\dot{H}^1 \times L^2$.*

Proof We consider $g_\epsilon \in C_0^\infty$ and $\int_0^1 g_\epsilon(x) dx = 0$ such that

$$|g - g_\epsilon|_{L^2} < \epsilon, \tag{3.1}$$

and we introduce the splitting $(u, v) = (u_1, v_1) + (u_2, v_2) + (u_3, v_3)$ where (u_1, v_1) satisfies

$$\begin{cases} u_{1,t} + ku_1 - v_1 = 0, \\ v_{1,t} + A_1u_1 - k(\alpha A_1 - k)u_1 + (\alpha A_1 - k)v_1 + h_1(v_1 - ku_1) + f(u) - \lambda u_1 = g_\epsilon, \\ u_1(0) = 0, \quad v_1(0) = 0, \end{cases} \tag{3.2}$$

(u_2, v_2) satisfies

$$\begin{cases} u_{2,t} + ku_2 - v_2 = 0, \\ v_{2,t} + A_1u_2 - k(\alpha A_1 - k)u_2 + (\alpha A_1 - k)v_2 \\ \quad + h_1(v_1 - ku_1 + v_2 - ku_2) - h_1(v_1 - ku_1) + f(u_1 + u_2) - f(u_1) - \lambda u_2 = g - g_\epsilon, \\ u_2(0) = 0, \quad v_2(0) = 0, \end{cases} \tag{3.3}$$

and (u_3, v_3) is the solution of

$$\begin{cases} u_{3,t} + ku_3 - v_3 = 0, \\ v_{3,t} + A_1u_3 - k(\alpha A_1 - k)u_3 + (\alpha A_1 - k)v_3 \\ \quad + h_1(v - ku) - h_1(v_1 - ku_1 + v_2 - ku_2) + f(u) - f(u_1 + u_2) - \lambda u_3 = 0, \\ u_3(0) = u_0, \quad v_3(0) = v_0. \end{cases} \tag{3.4}$$

We now define the families of maps $\{S_k^1(t) | t \geq 0\}$ and $\{S_k^2(t) | t \geq 0\}$ in $\dot{H}^1 \times L^2$, where

$$S_k^1(t)(u_0, v_0) = (u_1(t), v_1(t)) + (u_2(t), v_2(t)), \quad S_k^2(t)(u_0, v_0) = (u_3(t), v_3(t)).$$

First step: We prove that (u_1, v_1) is bounded in $H^2 \times H^1$. The system (3.2) can be written as

$$\varphi_t^1 + H_1(\varphi^1) = F_1(\varphi^1), \quad \varphi^1(0) = (0, 0)^T, \quad t \geq 0, \tag{3.5}$$

where

$$\begin{aligned} F_1(\varphi^1) &= \begin{pmatrix} 0 \\ -f(u_1) + \lambda u_1 + g_\epsilon \end{pmatrix}, \\ H_1(\varphi^1) &= \begin{pmatrix} ku_1 - v_1 \\ A_1u_1 - k(\alpha A_1 - k)u_1 + (\alpha A_1 - k)v_1 + h_1(v_1 - ku_1) \end{pmatrix}. \end{aligned} \tag{3.6}$$

Similar to Proposition 2.1, we obtain

$$\|\varphi^1\|_E^2 \leq \frac{1}{\rho} C(|g_\epsilon|_{H^{-1}}, \lambda, \alpha, \delta, \kappa, k, \beta_1, \beta_2) = M_1. \tag{3.7}$$

Now we multiply (3.2) by (A_1u_1, A_1v_1) and integrate on Ω to obtain

$$\frac{1}{2} \frac{d}{dt} \|A_1^{\frac{1}{2}} \varphi^1\|_E^2 + \sigma \|A_1^{\frac{1}{2}} \varphi^1\|_E^2 + \frac{\alpha\lambda + \beta_1}{2} |A_1^{\frac{1}{2}} v_1|^2 \leq (-f(u_1) + \lambda u_1, A_1 v_1) + (g_\epsilon, A_1 v_1), \tag{3.8}$$

with

$$\begin{aligned} (-f(u_1), A_1 v_1) &= (-A_1^{\frac{1}{2}} f(u_1), A_1^{\frac{1}{2}}(u_{1,t} + ku_1)) \\ &= -\frac{d}{dt} (A_1^{\frac{1}{2}} f(u), A_1^{\frac{1}{2}} u_1) + (A_1^{\frac{1}{2}} f'(u_1) u_{1,t}, A_1^{\frac{1}{2}} u_1) - k(A_1^{\frac{1}{2}} f(u), A_1^{\frac{1}{2}} u_1). \end{aligned}$$

Then (3.8) can be rewritten as

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} [\|A_1^{\frac{1}{2}} \varphi^1\|_E^2 + 2(A_1^{\frac{1}{2}} f(u_1), A_1^{\frac{1}{2}} u_1)] + \sigma \|A_1^{\frac{1}{2}} \varphi^1\|_E^2 + k(A_1^{\frac{1}{2}} f(u_1), A_1^{\frac{1}{2}} u_1) \\ &\leq -\frac{\alpha\lambda + \beta_1}{2} |A_1^{\frac{1}{2}} v_1|^2 + (A_1^{\frac{1}{2}} f'(u_1) u_{1,t}, A_1^{\frac{1}{2}} u_1) + \lambda(A_1^{\frac{1}{2}} u_1, A_1^{\frac{1}{2}} v_1) + (A_1^{\frac{1}{2}} g_\epsilon, A_1^{\frac{1}{2}} v_1), \end{aligned}$$

i.e.,

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} [\|A_1^{\frac{1}{2}} \varphi^1\|_E^2 + 2(A_1^{\frac{1}{2}} f(u_1), A_1^{\frac{1}{2}} u_1)] + \left(\sigma - \frac{\sqrt{\lambda}}{2\sqrt{\mu}}\right) \|A_1^{\frac{1}{2}} \varphi^1\|_E^2 + k(A_1^{\frac{1}{2}} f(u_1), A_1^{\frac{1}{2}} u_1) \\ &\leq (A_1^{\frac{1}{2}} f'(u_1) u_{1,t}, A_1^{\frac{1}{2}} u_1) + \frac{A_1^{\frac{1}{2}} g_\epsilon}{2(\alpha\lambda + \beta_1)}; \end{aligned} \tag{3.9}$$

for the first term on the right-hand side of (3.9), we have

$$\begin{aligned} (A_1^{\frac{1}{2}} f'(u_1) u_{1,t}, A_1^{\frac{1}{2}} u_1) &\leq c_3 |A_1^{\frac{1}{2}} u_{1,t}| |u_1|_6^p |A_1^{\frac{1}{2}} u_1|_{\frac{6}{3-p}} \\ &\leq c_3 |A_1^{\frac{1}{2}} u_{1,t}| |u_1|_{H^1}^p |A_1^{\frac{1}{2}} u_1|_{\frac{2-p}{2}} |A_1^{\frac{1}{2}} u_1|_{\frac{p}{2}} \\ &\leq c_3 \lambda^{\frac{p-2}{4}} |A_1^{\frac{1}{2}} u_{1,t}| |u_1|_{H^1}^p |A_1 u_1| \\ &\leq c_4 |A_1 u_1|^2 + c_4 \lambda^{\frac{p-4}{2}} |A_1^{\frac{1}{2}} u_{1,t}|^2 |u_1|_{H^1}^{2p} \\ &\leq c_4 |A_1 u_1|^2 + c_5 \lambda^{\frac{p-4}{2}} (1 + |u_1|_{H^1}^6 + |v_1|_{H^1}^6) \\ &\leq c_6 \|A_1^{\frac{1}{2}} \varphi^1\|_E^2 + c_5 \lambda^{\frac{p-4}{2}} (1 + |u_1|_{H^1}^6 + |v_1|^6). \end{aligned} \tag{3.10}$$

By (3.9)-(3.10), we have

$$\begin{aligned} &\frac{d}{dt} [\|A_1^{\frac{1}{2}} \varphi^1\|_E^2 + 2(A_1^{\frac{1}{2}} f(u_1), A_1^{\frac{1}{2}} u_1)] \\ &\quad + 2\left(\sigma - \frac{\sqrt{\lambda}}{2\sqrt{\mu}} - c_6\right) \|A_1^{\frac{1}{2}} \varphi^1\|_E^2 + 2k(A_1^{\frac{1}{2}} f(u_1), A_1^{\frac{1}{2}} u_1) \\ &\leq 2c_5 \lambda^{\frac{p-4}{2}} (|u_1|_{H^1}^6 + |v_1|^6) + \frac{A_1^{\frac{1}{2}} g_\epsilon}{\alpha\lambda + \beta_1}. \end{aligned} \tag{3.11}$$

Let $\varrho = \min\{\sigma - \frac{\sqrt{\lambda}}{2\sqrt{\mu}} - c_6, k\} \geq 0$, using the Gronwall lemma, we have

$$\begin{aligned} &\|A_1^{\frac{1}{2}} \varphi^1\|_E^2 + 2(A_1^{\frac{1}{2}} f(u_1), A_1^{\frac{1}{2}} u_1) \\ &\leq (\|A_1^{\frac{1}{2}} \varphi^1(0)\|_E^2 + 2(A_1^{\frac{1}{2}} f(u_1(0)), A_1^{\frac{1}{2}} u_1(0))) e^{-2\varrho t} \end{aligned}$$

$$\begin{aligned}
 &+ 2c_5\lambda^{\frac{p-4}{2}} \int_0^t e^{-2\varrho t} (|u_1(s)|_{H^1}^6 + |v_1(s)|^6) ds + \frac{|A_1^{\frac{1}{2}}g_\epsilon|}{(\alpha\lambda + \beta_1)\varrho} \\
 &\leq 2c_5\lambda^{\frac{p-4}{2}} \int_0^t e^{-2\varrho t} (|u_1(s)|_{H^1}^6 + |v_1(s)|^6) ds + \frac{|A_1^{\frac{1}{2}}g_\epsilon|}{(\alpha\lambda + \beta_1)\varrho}.
 \end{aligned}$$

By (3.7), we have

$$\|A_1^{\frac{1}{2}}\varphi^1\|_E^2 + 2(A_1^{\frac{1}{2}}f(u_1), A_1^{\frac{1}{2}}u_1) \leq \frac{1}{\rho\varrho} C^3(|A_1^{\frac{1}{2}}g_\epsilon|, \lambda, \alpha, \delta, \kappa, k, \beta_1, \beta_2).$$

Note that (3.2) implies that

$$(A_1^{\frac{1}{2}}f(u_1), A_1^{\frac{1}{2}}u_1) \leq c_7|A_1^{\frac{1}{2}}u_1|^2 + c_7|A_1^{\frac{1}{2}}u_1| \leq c_8\|A_1^{\frac{1}{2}}\varphi^1\|_E^2.$$

Then we obtain

$$\|A_1^{\frac{1}{2}}\varphi^1\|_E^2 \leq \frac{1}{\rho\varrho} C^3(|A_1^{\frac{1}{2}}g_\epsilon|, \lambda, \alpha, \delta, \kappa, k, \beta_1, \beta_2). \tag{3.12}$$

Proposition 2.1, (3.8), and (3.12) imply that (u_1, v_1) is bounded in $H^2 \times H^1$.

Second step: Let $\varphi^2 = (u_2, v_2)$; we will prove that there exists a $K > 0$ independence on ϵ such that

$$\|\varphi^2\|_E \leq K\epsilon.$$

Multiplying (3.3) by (u_2, v_2) , we thus obtain

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{dt} \|\varphi^2\|_E^2 + \left(\sigma - \frac{2\sqrt{\lambda}}{\sqrt{\mu}} \right) \|\varphi^2\|_E^2 + \frac{\alpha\lambda + \beta_1}{2} |v_2|^2 + (f(u_1 + u_2) - f(u_1), v_2) \\
 &\leq |g - g_\epsilon| |v_2|.
 \end{aligned} \tag{3.13}$$

For the last term on the left-hand side of (3.13), by (2.3), Proposition 2.1, and (3.7), there exists ξ such that

$$\begin{aligned}
 (f(u_1 + u_2) - f(u_1), v_2) &= (f'(u_1 + \xi u_2)u_2, v_2) \\
 &\geq -c_2(1 + |u_1|^p + |u_2|^p)|u_2||v_2| \\
 &\geq -c_{11}(M_0, M_1)|u_2||v_2|.
 \end{aligned} \tag{3.14}$$

By the Gronwall and Poincaré inequalities,

$$\begin{aligned}
 \|\varphi^2\|_E^2 &\leq \frac{2|g - g_\epsilon|^2}{\left(\frac{2\sqrt{\lambda}}{\sqrt{\mu}} + c_{11}(M_0, M_1) - \sigma\right)(\alpha\lambda + \beta_1)} \\
 &\leq \frac{8\epsilon^2}{\left(\frac{2\sqrt{\lambda}}{\sqrt{\mu}} + c_{11}(M_0, M_1) - \sigma\right)(\alpha\lambda + \beta_1)}.
 \end{aligned} \tag{3.15}$$

By (3.12), (3.15), and the following lemma we see that $\{S_k^1(t) | t \geq 0\}$ is compact in $\dot{H}^1 \times L^2$.

Lemma 3.1 (see [25]) *Let X be a complete metric space and A be a subset in X , such that*

$$\forall \epsilon, \quad A \subset K_\epsilon + B(0, C(\epsilon))$$

with $\lim_{\epsilon \rightarrow 0} C(\epsilon) = 0$ and K_ϵ is compact in X , then A is compact in X .

Third step: Let $\varphi^3 = (u^3, v^3)$, by (3.4) and Lemma 2.2, we have

$$\frac{1}{2} \frac{d}{dt} \|\varphi^3\|_E^2 + \sigma \|\varphi^3\|_E^2 + \frac{\alpha\lambda + \beta_1}{2} |v_3|^2 \leq -(f(u) - f(u_1 + u_2), v_3) + \lambda(u_3, v_3). \quad (3.16)$$

For the right-hand side of (3.16), by (2.1), we have

$$\begin{aligned} -(f(u) - f(u_1 + u_2), v_3) + \lambda(u_3, v_3) &\leq c_{13} |u_3| |v_3| + \lambda |u_3| |v_3| \\ &\leq c_{14} \|u_3\| |v_3| \\ &\leq \frac{\sigma}{2} \|\varphi^3\|_E^2 + \left(\frac{c_{14}^2}{2\sigma\mu} - \frac{\sigma}{2} \right) |v_3|^2. \end{aligned} \quad (3.17)$$

Let $c_{14} < \sqrt{\sigma\mu(\alpha\lambda + \beta_1) + \sigma^2\mu}$. Equations (3.16) and (3.17) lead to

$$\|\varphi^3\|_E^2 \leq (\mu \|u_0\|^2 + |v_0|^2) \exp\{-\sigma t\}. \quad (3.18)$$

In other words we have

$$S(t)(u_0, v_0) \rightarrow 0 \quad \text{in } \dot{H}^1 \times L^2, \quad (3.19)$$

uniformly in bounded sets. Then from Theorem 3.1, (3.18), and the compactness of $\{S_k^1(t) | t \geq 0\}$, for the system (1.1) there exists a global attractor \mathcal{A} in $\dot{H}^1 \times L^2$. \square

Remark 3.2 It is easy to see that the semigroup

$$S_k(t) = S_k^1(t) + S_k^2(t) : (u_0, v_0 = u_1 + ku_0)^T \rightarrow (u(t), u_t(t) + ku(t))^T, \quad E \rightarrow E,$$

defined by (2.12) has the following relation with $S(t)$:

$$S_k(t) = R_k S(t) R_{-k}, \quad (3.20)$$

where R_k is an isomorphism of E :

$$R_k : \{u, u_t\} \rightarrow \{u, u_t + ku\}.$$

Since the semigroup $\{S_k(t) | t \geq 0\}$ defined by (1.1) possesses a global attractor $\mathcal{A}_0 \subset E$, by (3.19), $\{S(t) | t \geq 0\}$ also possesses a global attractor $\mathcal{A} = R_k \mathcal{A}_0$.

4 Regularity of the attractor

In this section, we suppose that $g \in \dot{L}^2$ and prove that the attractor is compact in $H^2 \times H^1$. Let $(u_0, v_0) \in \dot{H}^1 \times L^2$. We set

$$S_k(t)(u_0, v_0) = (p(t), q(t)) + (w(t), \rho(t)), \quad \forall t \geq 0, \quad (4.1)$$

where (p, q) is the solution of

$$\begin{cases} p_t + kp - q = 0, \\ q_t + A_1 p - k(\alpha A_1 - k)p + (\alpha A_1 - k)q + h_1(q - kp) + f(u) - \lambda u = g, \\ p(0) = 0, \quad q(0) = 0, \end{cases} \quad (4.2)$$

(w, ρ) satisfies

$$\begin{cases} w_t + kw - \rho = 0, \\ \rho_t + A_1 w - k(\alpha A_1 - k)w + (\alpha A_1 - k)\rho + h_1(v - ku) - h_1(q - kp) = 0, \\ w(0) = u_0, \quad \rho(0) = v_0. \end{cases} \quad (4.3)$$

We use Remark 3.1 to prove that $\mathcal{A} \subset H^2 \times H^1$. Let $(u_0, v_0) \in \mathcal{A}$. Let $(u_0^n, v_0^n) \in \mathcal{B}$ and a sequence of a real numbers $t_n \rightarrow \infty$ as $n \rightarrow \infty$, such that

$$S_k(t_n)(u_0^n, v_0^n) \rightarrow (u_0, v_0) \quad \text{in } \dot{H}^1 \times L^2 \text{ as } n \rightarrow \infty. \quad (4.4)$$

We also have

$$S_k(t_n)(u_0^n, v_0^n) = (p^n(t_n), q^n(t_n)) + (w^n(t_n), \rho^n(t_n)), \quad \forall n \in \mathbb{N}. \quad (4.5)$$

We deduce from (3.12) and (3.20) that

$$\|(p^n(t_n), q^n(t_n))\|_{H^2 \times H^1}^2 \leq \frac{1}{\rho Q} C^3(|A_1^{\frac{1}{2}} g_\epsilon|, \lambda, \alpha, \delta, \kappa, k, \beta_1, \beta_2) \quad (4.6)$$

and

$$\|(w^n(t_n), \rho^n(t_n))\|_{\dot{H}^1 \times L^2}^2 \leq \|(u_0, v_0)\|_{H^1 \times L^2}^2 \exp(-2\sigma t_n). \quad (4.7)$$

From (4.6), we infer that there exist subsequences $t_{n'}$ and $(u_0^{n'}, v_0^{n'})$, and $(p^0, q^0) \in H^2 \times H^1$ such that

$$(p^{n'}(t_{n'}), q^{n'}(t_{n'})) \rightharpoonup (p^0, q^0) \quad \text{weakly in } H^2 \times H^1. \quad (4.8)$$

From (4.4), (4.5), and (4.7), we have

$$(p^{n'}(t_{n'}), q^{n'}(t_{n'})) \rightarrow (u_0, v_0) \quad \text{in } H^1 \times L^2. \quad (4.9)$$

We conclude that $(u_0, v_0) = (p^0, q^0)$ and then

$$\mathcal{A} \subset H^2 \times H^1. \quad (4.10)$$

In the following, we use the famous argument of [5] to show that the attractor \mathcal{A} is actually a compact set in $H^2 \times H^1$.

Theorem 4.1 *The semigroup $\{S(t)|t \geq 0\}$ in $\dot{H}^1 \times L^2$ possesses a global attractor \mathcal{A} which is compact subset of $H^2 \times H^1$.*

Proof Multiplying (1.1) by $(A_1 u, A_1 v)$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} [\mu \|A_1^{\frac{1}{2}} u\|^2 + |A_1^{\frac{1}{2}} v|^2] + \sigma (\mu \|A_1^{\frac{1}{2}} u\|^2 + |A_1^{\frac{1}{2}} v|^2) + \frac{\alpha\lambda + \beta_1}{2} |A_1^{\frac{1}{2}} v|^2 \\ & = (g, A_1 v) - (f(u) - \lambda u, A_1 v). \end{aligned} \tag{4.11}$$

We put $G(u, v) = \mu \|A_1^{\frac{1}{2}} u\|^2 + |A_1^{\frac{1}{2}} v|^2$, $K(u, v) = \int_{\Omega} [A_1^{\frac{1}{2}} g \cdot A_1^{\frac{1}{2}} v - (f(u) - \lambda u) A_1 v] dx - \frac{\alpha\lambda + \beta_1}{2} |A_1^{\frac{1}{2}} v|^2$, then we have

$$\frac{1}{2} \frac{d}{dt} G(u, v) + \sigma G(u, v) = K(u, v), \tag{4.12}$$

this shows that

$$G(u, v) = e^{-2\sigma t} G(u_0, v_0) + 2 \int_0^t e^{2\sigma(s-t)} K(u, v) ds. \tag{4.13}$$

We consider a sequence $(a_n, b_n)_{n \in \mathbb{N}} \in \mathcal{A}$, and we may assume that, up to a subsequence,

$$\begin{aligned} (a_n, b_n) & \rightharpoonup (a, b) \quad \text{weakly in } H^2 \times H^1, \\ (a_n, b_n) & \rightarrow (a, b) \quad \text{strongly in } H^1 \times L^2. \end{aligned}$$

We want to prove the strong convergence of (a_n, b_n) in $H^2 \times H^1$; this will give the compactness of \mathcal{A} in $H^2 \times H^1$.

For a given $T \geq 0$ and up to subsequence extraction, we may assume that a.e. in $t \in [0, T]$,

$$\begin{aligned} S(t - T)(a_n, b_n) & \rightharpoonup S(t - T)(a, b) \quad \text{weakly in } H^2 \times H^1, \\ S(t - T)(a_n, b_n) & \rightarrow S(t - T)(a, b) \quad \text{strongly in } H^1 \times L^2. \end{aligned}$$

Using (4.13) for $(u_0, v_0) = S(-T)(a_n, b_n)$, we have

$$\begin{aligned} G(S(t - T)(a_n, b_n)) & = e^{-2\sigma t} G(S(-T)(a_n, b_n)) \\ & \quad + 2 \int_0^t e^{2\sigma(s-t)} K(S(s - T)(a_n, b_n)) ds. \end{aligned} \tag{4.14}$$

By the Lebesgue dominated convergence theorem, similar to (3.8)-(3.12), we have

$$\int_0^t e^{2\sigma(s-t)} K(S(s - T)(a_n, b_n)) ds \rightarrow \int_0^t e^{2\sigma(s-t)} K(S(s - T)(a, b)) ds, \tag{4.15}$$

passing to the limit sup in (4.14) and taking $t = T$ we have

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} G(a_n, b_n) & \leq \overline{\lim}_{n \rightarrow \infty} e^{-2\sigma T} G(S(-T)(a_n, b_n)) \\ & \quad + 2 \int_0^T e^{2\sigma(s-T)} K(S(s - T)(a, b)) ds. \end{aligned} \tag{4.16}$$

The last term of (4.16) follows from (4.14) for $(u_0, v_0) = S(-T)(a, b)$ and $t = T$, and we have

$$I = 2 \int_0^T e^{2\sigma(s-T)} K(S(s-T)(a, b)) ds = G(a, b) - e^{-2\sigma T} G(S(-T)(a, b)). \quad (4.17)$$

We replace I in (4.16), we obtain

$$\overline{\lim}_{n \rightarrow \infty} G(a_n, b_n) \leq G(a, b) - e^{-2\sigma T} (G(S(-T)(a, b)) - \overline{\lim}_{n \rightarrow \infty} G(S(-T)(a_n, b_n))). \quad (4.18)$$

It is a standard matter to prove

$$e^{-2\sigma T} (G(S(-T)(a, b)) - \overline{\lim}_{n \rightarrow \infty} G(S(-T)(a_n, b_n))) \rightarrow 0 \quad \text{when } T \rightarrow \infty. \quad (4.19)$$

It follows from (4.18) that

$$\overline{\lim}_{n \rightarrow \infty} G(a_n, b_n) \leq G(a, b) \leq \underline{\lim}_{n \rightarrow \infty} G(a_n, b_n). \quad (4.20)$$

This completes the proof of lemma. \square

Competing interests

The author declares to have no competing interests.

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