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Convergence theorems for modified generalized f-projections and generalized nonexpansive mappings

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Abstract

The purpose of this paper is to study a sequence of modified generalized f-projections in a reflexive, smooth, and strictly convex Banach space and show that Mosco convergence of their ranges implies their pointwise convergence to the generalized f-projection onto the limit set. Furthermore, we prove a strong convergence theorem for a countable family of α-nonexpansive mappings in a uniformly convex and smooth Banach space using the properties of a modified generalized f-projection operator. Our main results generalize the results of Ziming Wang, Yongfu Su, and Jinlong Kang and enrich the research contents of α-nonexpansive mappings.

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1 Introduction

Let E be a real Banach space and C be a nonempty closed convex subset of E. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

Lots of iterative schemes for nonexpansive mappings have been introduced (see [1–3]); furthermore, many strong convergence theorems for nonexpansive mappings have been proved. On the other hand, there are many nonlinear mappings which are more general than the nonexpansive mapping. Compared to the existing problem of a fixed point of those mappings, the iterative methods for finding a fixed point are also very useful in studying the fixed point theory and the theory of equations in other fields.

In 2007, Gobel and Pineda [4] introduced and studied a new mapping, called α-nonexpansive mapping. The mapping is more general than the nonexpansive mapping.

Definition 1.1 For a given multi-index, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfies $\alpha_i \geq 0$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \alpha_i = 1$. A mapping $T : C \to C$ is said to be α-nonexpansive if

$$\sum_{i=1}^{n} \alpha_i \left\| T^i x - T^i y \right\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

In order to show that the class of α-nonexpansive mappings is more general than the one of nonexpansive mappings, we give an example [4].

Example 1.2 Let $E = R^1$, and

$$T(x) = \begin{cases} 0 & \text{if } x = 0; \\ \frac{1}{x} & \text{if } x \in (0, +\infty). \end{cases}$$

Then T is not nonexpansive but α-nonexpansive.

Proof Obviously, T is not nonexpansive. Taking $x = \frac{1}{2}$, $y = 0$, by the definition of Tx, we have

$$\|Tx - Ty\| = |2 - 0| > \left|\frac{1}{2} - 0\right| = \|x - y\|.$$

On the other hand, for every $x, y \in [0, +\infty)$, we have

$$\|T^2x - T^2y\| = \|x - y\|.$$

Therefore, we can affirm that

$$0\|Tx - Ty\| + \|T^2x - T^2y\| = \|x - y\|,$$

where $\alpha = (\alpha_1, \alpha_2) = (0, 1)$. Then T is an α-nonexpansive mapping but not a nonexpansive one. □

If T is a nonexpansive self-mapping, we can imply that T must be an α-nonexpansive one, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) = (\frac{1}{n}, \ldots, \frac{1}{n})$.

For technical reasons, we always assume that the first coefficient α_1 is nonzero, that is, $\alpha_1 > 0$. In this case the mapping T satisfies the Lipschitz condition

$$\|Tx - Ty\| \leq \frac{1}{\alpha_1}\|x - y\|, \quad \forall x, y \in C.$$

For the α-nonexpansive mapping T, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n)$, it is obvious that the mapping

$$T_\alpha x = \sum_{i=1}^{n} \alpha_i T^i x, \quad \forall x \in C \tag{1.3}$$

is nonexpansive. However, the nonexpansiveness of T_α is much weaker than (1.2), for instance, it does not entail the continuity of T (see [4]).

In 2010, Klin-eam and Suantai [5] introduced the relation of fixed point sets between an α-nonexpansive operator and a T_α operator. They gave the following theorem.

Theorem 1.3 (see Theorem 3.1 of Klin-eam and Suantai [5]) *Let C be a closed convex subset of a Banach space E and for all $n \in N$, let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that $\alpha_i \geq 0$, $i = 1, 2, \ldots, n$, $\alpha_1 > 0$, and $\sum_{i=1}^{n} \alpha_i = 1$. Let T be an α-nonexpansive mapping from C into itself. If $\alpha_1 > \frac{1}{\sqrt[n-1]{2}}$, then $F(T) = F(T_\alpha)$, where $F(T)$ is the fixed point set of T.*

At the same time, they have succeeded in proving the demiclosedness principle for the α-nonexpansive mappings.

Theorem 1.4 (see Theorem 3.4 of Klin-eam and Suantai [5]) *Let C be a closed convex subset of a Banach space E and for all $n \in N$, let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that $\alpha_i \geq 0$, $i = 1, 2, \ldots, n$, $\alpha_1 > 0$, and $\sum_{i=1}^n \alpha_i = 1$. Let T be an α-nonexpansive mapping from C into itself. If $\alpha_1 > \frac{1}{\sqrt[n-1]{2}}$, if $\{x_n\} \subset C$ converges weakly to x and $\{x_n - Tx_n\}$ converges strongly to 0 as $n \to \infty$, then $x \in F(T)$.*

Recently, Wang *et al.* [6] proposed the following hybrid algorithm for an α-nonexpansive mapping in a Banach space:

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \in N. \end{cases} \tag{1.4}$$

As we know that if C is a nonempty closed convex subset of a Hilbert space H and recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$, and the unique point $P_C x \in C$ satisfies the property $\|x - P_C x\| = \min_{y \in C} \|x - y\|$, it is well known that P_C is nonexpansive. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. We consider the functional defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \quad \forall x, y \in E,$$

where J is the normalized duality mapping and the Banach space is smooth. In this connection, Alber [7] introduced a generalized projection Π_C from E to C as follows:

$$\Pi_C(x) = \arg\min_{y \in C} \phi(y, x), \quad \forall x \in E.$$

It is obvious from the definition of functional ϕ that

$$\big(\|y\| - \|x\|\big)^2 \leq \phi(y, x) \leq \big(\|y\| + \|x\|\big)^2, \quad \forall x, y \in E.$$

If E is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$ and Π_C becomes the metric projection of E onto C. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x).$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(y, x)$ and strict monotonicity of the normalized duality mapping J [8]. It is well known that the metric projection operator plays an important role in nonlinear functional analysis, optimization theory, fixed point theory, nonlinear programming, game theory, variational inequality, and complementarity problems, *etc.* [8, 9]. In 1994, Alber [7] introduced and studied the generalized projections from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces. Moreover, Alber [8] presented some applications of the generalized projections to approximately solve variational inequalities and von Neumann intersection problem in Banach spaces. In 2005, Li [9] extended the generalized projection operator from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces and studied some properties of the generalized projection operator with applications to solve the variational inequality in Banach spaces. Later, Wu and Huang [10] introduced a new generalized f-projection operator in Banach spaces. They extended the definition of generalized projection operators introduced by Abler [7] and proved some properties of the generalized f-projection operator. In 2009, Fan *et al.* [11] presented some basic results for the generalized f-projection operator and discussed the existence of solutions and approximation of the solutions for generalized variational inequalities in noncompact subsets of Banach spaces.

The purpose of this paper is to study a sequence of modified generalized f-projections in a reflexive, smooth, and strictly convex Banach space and show that Mosco convergence of their ranges implies their pointwise convergence to the generalized f-projection onto the limit set. Furthermore, we prove strong convergence theorem for a countable family of α-nonexpansive mappings in a uniformly convex and smooth Banach space using the properties of a modified generalized f-projection operator. Our main results generalize the results of Wang *et al.* [6] and enrich the research contents of α-nonexpansive mappings.

2 Preliminaries

A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if for each $\epsilon > 0$ there is $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x-y\| \geq \epsilon$, $\|x+y\| \leq 2(1-\delta)$ holds. The space E is said to be smooth if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S(E) = \{x \in E : \|x\| = 1\}$. And E is said to be uniformly smooth if the limit (2.1) exists uniformly for all $x, y \in S(E)$.

Remark 2.1 The following basic properties of a Banach space E can be found in Cioranescu [12]:

(i) if E is uniformly convex, then E is reflexive and strictly convex;
(ii) a Banach space E is uniformly smooth if and only if E^* is uniformly convex;
(iii) each uniformly convex Banach space E has the Kadec-Klee property, *i.e.*, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

Let E be a real Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in E.$$

Many properties of the normalized duality mapping J can be found in Takahashi [13] or Vainberg [14]. We list some properties below for easy reference:

(i) J is a monotone and bounded operator in arbitrary Banach spaces;
(ii) J is a strictly monotone operator in strictly convex Banach spaces;

(iii) J is a continuous operator in smooth Banach spaces;
(iv) J is a uniformly continuous operator on each bounded set in uniformly smooth Banach spaces;
(v) J is a bijection in smooth, reflexive, and strictly convex Banach spaces;
(vi) J is the identity operator in Hilbert spaces.

Next, we recall the concept of generalized f-projector operator, together with its properties. Let $G : C \times E^* \to R \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, \varphi) = \|\xi\|^2 - 2\langle \xi, \varphi\rangle + \|\varphi\|^2 + 2\rho f(\xi), \tag{2.2}$$

where $\xi \in C$, $\varphi \in E^*$, ρ is a positive number and $f : C \to R \cup \{+\infty\}$ is proper, convex, and lower semi-continuous. From the definitions of G and f, it is easy to see the following properties:
(i) $G(\xi, \varphi)$ is convex and continuous with respect to φ when ξ is fixed;
(ii) $G(\xi, \varphi)$ is convex and lower semi-continuous with respect to ξ when φ is fixed.

Definition 2.2 ([10]) Let E be a real Banach space with its dual E^*. Let C be a nonempty, closed, and convex subset of E. We say that $\Pi_C^f : E^* \to 2^C$ is a generalized f-projection operator if

$$\Pi_C^f \varphi = \Bigl\{ u \in C : G(u, \varphi) = \inf_{\xi \in C} G(\xi, \varphi) \Bigr\}, \quad \forall \varphi \in E^*. \tag{2.3}$$

For the generalized f-projection operator, Wu and Huang [10] proved the following basic properties.

Lemma 2.3 ([10]) *Let E be a real reflexive Banach space with its dual E^*, and let C be a nonempty, closed, and convex subset of E. Then the following statements hold*:
(i) *$\Pi_C^f \varphi$ is a nonempty closed convex subset of C for all $\varphi \in E^*$.*
(ii) *If E is smooth, then for all $\varphi \in E^*$, $x \in \Pi_C^f \varphi$ if and only if*

$$\langle x - y, \varphi - Jx\rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C.$$

(iii) *If E is strictly convex and $f : C \to R \cup \{+\infty\}$ is positive homogeneous* (*i.e.*, $f(tx) = tf(x)$ *for all* $t > 0$ *such that* $tx \in C$, *where* $x \in C$), *then* Π_C^f *is a single-valued mapping.*

Fan *et al.* [11] showed that the condition f is positive homogeneous, which appeared in Lemma 2.3, can be removed.

Lemma 2.4 ([11]) *Let E be a real reflexive Banach space with its dual E^*, and let C be a nonempty, closed, and convex subset of E. Then if E is strictly convex, then Π_C^f is a single-valued mapping.*

Recall that J is a single-valued mapping when E is a smooth Banach space. There exists a unique element $\varphi \in E^*$ such that $\varphi = Jx$ for each $x \in E$. This substitution in (2.2) gives

$$G(\xi, Jx) = \|\xi\|^2 - 2\langle \xi, Jx\rangle + \|x\|^2 + 2\rho f(\xi). \tag{2.4}$$

Now, we consider the second generalized f-projection operator in a Banach space.

Definition 2.5 Let E be a real Banach space and C be a nonempty, closed, and convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is a generalized f-projection operator if

$$\Pi_C^f(x) = \Big\{ u \in C : G(u, Jx) = \inf_{\xi \in C} G(\xi, Jx) \Big\}, \quad \forall x \in E.$$

We know that the following lemmas hold for the operator Π_C^f.

Lemma 2.6 ([15]) *Let C be a nonempty, closed, and convex subset of a smooth and reflexive Banach space E. Then the following statements hold*:

(i) *$\Pi_C^f x$ is a nonempty closed and convex subset of C for all $x \in E$.*
(ii) *For all $x \in E$, $\hat{x} \in \Pi_C^f x$ if and only if*

$$\langle \hat{x} - y, Jx - J\hat{x} \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C.$$

(iii) *If E is strictly convex, then Π_C^f is a single-valued mapping.*

Now, we introduce a modified generalized f-projection operator. Let $G : C \times E^* \to R$ be a functional defined as follows:

$$G(\xi, \varphi) = \|\xi\|^2 - 2\langle \xi, \varphi \rangle + \|\varphi\|^2 + 2\rho f(\xi), \tag{2.5}$$

where $\xi \in C$, $\varphi \in E^*$, ρ is a positive number and $f : C \to R$ is convex and weakly continuous. From the definitions of G and f, it is easy to see the following properties:

(i) $G(\xi, \varphi)$ is convex and continuous with respect to φ when ξ is fixed;
(ii) $G(\xi, \varphi)$ is convex and weakly lower semi-continuous with respect to ξ when φ is fixed.

Obviously, the other definitions and lemmas hold respectively.

Next, we give the following example [16] which shows that metric projection, generalized projection and generalized f-projection are different.

Example 2.7 Let $X = R^3$ be provided with the norm

$$\|(x_1, x_2, x_3)\| = \sqrt{(x_1^2 + x_2^2)} + \sqrt{(x_2^2 + x_3^2)}.$$

This is a smooth strictly convex Banach space and $C = \{x \in R^3 | x_2 = 0, x_3 = 0\}$ is a closed and convex subset of X. It is a simple computation; we get $P_C(1,1,1) = (1,0,0)$, $\Pi_C(1,1,1) = (2,0,0)$.

We set $\rho = 1$ is a positive number and define $f : C \to R$ by

$$f(x) = -2 - 2\sqrt{5}.$$

Then f is convex and weakly continuous. Simple computations show that

$$\Pi_C^f(1,1,1) = (4,0,0).$$

Let E be a Banach space, and let $C_1, C_2, C_3, \dots$ be a sequence of weakly closed subsets of E. We denote by $s - Li_n C_n$ the set of limit points of $\{C_n\}$, that is, $x \in s - Li_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and that $x_n \in C_n$ for all $n \in N$. Similarly, we denote by $w - Ls_n C_n$ the set of cluster points of $\{C_n\}$, $y \in w - Ls_n C_n$ if and only if there exists $\{y_{n_i}\}$ such that $\{y_{n_i}\}$ converges weakly to y and that $\{y_{n_i}\} \in C_{n_i}$ for all $i \in N$. Using these definitions, we define the Mosco convergence [2] of C_{n_i}. If C_0 satisfies

$$s - Li_n C_n = C_0 = w - Ls_n C_n, \tag{2.6}$$

we say that C_n is a Mosco convergent sequence to C_0 and write

$$C_0 = M - \lim_{n\to\infty} C_n. \tag{2.7}$$

Notice that the inclusion $s - Li_n C_n \subset w - Ls_n C_n$ is always true. Therefore, in order to show the existence of $M - \lim_{n\to\infty} C_n$, it is sufficient to prove $w - Ls_n C_n \subset s - Li_n C_n$. For more details, see [17].

3 Main results

3.1 Generalized Mosco convergence theorems

Theorem 3.1 *Let E be a smooth, reflexive, and strictly convex Banach space and C be a nonempty closed convex subset of E. Let $C_1, C_2, C_3, \dots$ be nonempty closed convex subsets of C, $f : E \to R$ be a convex and weakly continuous mapping with $C \subset \operatorname{int}(D(f))$. If $C_0 = M - \lim_{n\to\infty} C_n$ exists and is nonempty, then C_0 is a closed convex subset of C and, for each $x \in C$, $\{\Pi^f_{C_n} x\}$ converges weakly to $\Pi^f_{C_0} x$.*

Proof It is easy to prove that C_0 is closed and convex if C_n is a closed convex subset of C for each $n \in N$. Fix $x \in C$. For the sake of simplicity, we write x_n instead of $\Pi^f_{C_n} x$ for $n \in N$. Since $C_0 = M - \lim_{n\to\infty} C_n$, we have that for each $y \in C_0$, there exists $\{y_n\} \subset E$ such that $y_n \to y$ as $n \to \infty$ and that $y_n \in C_n$ for each $n \in N$. From Lemma 2.6, we have

$$\langle x_n - y_n, Jx - Jx_n \rangle + \rho f(y_n) - \rho f(x) \ge 0.$$

Hence, we obtain

$$\begin{aligned} 0 &\le \langle x_n - x, Jx - Jx_n \rangle + \langle x - y_n, Jx - Jx_n \rangle + \rho f(y_n) - \rho f(x) \\ &\le -\big(\|x\| - \|x_n\|\big)^2 + \big(\|x\| + \|x_n\|\big)\|x - y_n\| + \rho f(y_n) - \rho f(x), \end{aligned}$$

thus,

$$\big(\|x\| - \|x_n\|\big)^2 \le \big(\|x\| + \|x_n\|\big)\|x - y_n\| + \rho f(y_n) - \rho f(x).$$

Suppose that $\{x_n\}$ is not bounded. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\|x_{n_i}\| \to \infty$. It follows that

$$\frac{\|x\|^2}{\|x_{n_i}\|} - 2\|x\| + \|x_{n_i}\| \le \left(1 + \frac{\|x\|}{\|x_{n_i}\|}\right)\|x - y_{n_i}\| + \frac{\rho f(y_{n_i}) - \rho f(x)}{\|x_{n_i}\|}$$

for a sufficiently large number $i \in N$. As $i \to \infty$, we obtain $+\infty \le \|x - y_{n_i}\| < +\infty$. This is a contradiction. Hence we have that $\{x_n\}$ is bounded.

Since $\{x_n\}$ is bounded, there exists a subsequence, again denoted by $\{x_n\}$, such that it converges weakly to $x_0 \in C$. From the definition of C_0, we get $x_0 \in C_0$.

Now, we prove that $\Pi^f_{C_0} x = x_0$. From weak lower semi-continuity of the norm and weak continuity of f, we have

$$\begin{aligned}\liminf_{n\to\infty} G(x_n, Jx) &= \liminf_{n\to\infty} \|x_n\|^2 - 2\langle x_n, Jx\rangle + \|x\|^2 + \rho f(x_n)\\ &\ge \|x_0\|^2 - 2\langle x_0, Jx\rangle + \|x\|^2 + \rho f(x_0)\\ &= G(x_0, Jx).\end{aligned}$$

On the other hand, we get

$$\begin{aligned}\liminf_{n\to\infty} G(x_n, Jx) &\le \liminf_{n\to\infty} G(y_n, Jx)\\ &= \liminf_{n\to\infty} \|y_n\|^2 - 2\langle y_n, Jx\rangle + \|x\|^2 + \rho f(y_n)\\ &= G(y, Jx).\end{aligned}$$

So,

$$G(x_0, Jx) \le G(y, Jx), \quad \forall y \in C_0,$$

that is,

$$G(x_0, Jx) = \inf_{y\in C_0} G(y, Jx).$$

Hence we get $\Pi^f_{C_0} x = x_0$.

According to our consideration above, each sequence $\{x_n\}$ has, in turn, a subsequence which converges weakly to the unique point $\Pi^f_{C_0} x$. Therefore, the sequence $\{x_n\}$ converges weakly to $\Pi^f_{C_0} x$. This completes the proof. □

A Banach space E is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x_0$ and $\|x_n\| \to \|x_0\|$ converges strongly to x_0. It is known that E^* has a Fréchet differentiable norm if and only if E is reflexive, strictly convex, and has the Kadec-Klee property; see, for example, [10].

Theorem 3.2 *Let E be a smooth Banach space such that E^* has a Fréchet differentiable norm. Let C be a nonempty closed convex subset of E. Let $C_1, C_2, C_3, \ldots$ be nonempty closed convex subsets of C, $f : E \to R$ be a convex and weakly continuous mapping with $C \subset \operatorname{int}(D(f))$. If $C_0 = M - \lim_{n\to\infty} C_n$ exists and is nonempty, then C_0 is a closed convex subset of C and, for each $x \in C$, $\{\Pi^f_{C_n} x\}$ converges strongly to $\Pi^f_{C_0} x$.*

Proof Fix $x \in C$ arbitrarily. We write $x_n = \Pi^f_{C_n} x$ and $x_0 = \Pi^f_{C_0} x$. By Theorem 3.1, we obtain $x_n \rightharpoonup x_0$. Since E^* has a Fréchet differentiable norm, E has the Kadec-Klee property. Therefore, it is sufficient to prove that $\|x_n\| \to \|x_0\|$ as $n \to \infty$. Since $x_0 \in C_0$, there exists a sequence $\{y_n\} \subset C$ such that $y_n \to x_0$ as $n \to \infty$ and $y_n \in C_n$ for each $n \in N$. It follows that

$$\begin{aligned} G(x_0, Jx) &\le \liminf_{n\to\infty} G(x_n, Jx) \\ &\le \limsup_{n\to\infty} G(x_n, Jx) \\ &\le \limsup_{n\to\infty} G(y_n, Jx) \\ &= G(x_0, Jx). \end{aligned}$$

Hence we obtain $G(x_0, Jx) = \lim_{n\to\infty} G(x_n, Jx)$. Since $\langle x_n, J(x)\rangle$ converges to $\langle x_0, J(x)\rangle$ and f is weakly continuous, we get

$$\lim_{n\to\infty} \|x_n\| = \|x_0\|.$$

Using the Kadec-Klee property of E, we obtain that $\{x_n\}$ converges strongly to x_0. This completes the proof. □

Definition 3.3 ([18]) Let C be a closed convex subset of a Banach space E, let $\{T_n\}_{n=1}^{\infty}$ be a countable family of mappings of C into itself with the nonempty common fixed point set F. The $\{T_n\}_{n=1}^{\infty}$ is said to be uniformly closed if $x_n \to x$ and $\|x_n - T_n x_n\| \to 0$ as $n \to \infty$ implies $x \in F$.

3.2 Strong convergence theorems

Lemma 3.4 (see Lemma 3.3 of Klin-eam and Suantai [5]) *Let C be a closed convex subset of a Banach space E and for all $n \in N$, let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ such that $\alpha_i \ge 0$, $i = 1, 2, \dots, n$, $\alpha_1 > 0$, and $\sum_{i=1}^{n} \alpha_i = 1$. Let T be an α-nonexpansive mapping from C into itself. If $\alpha_1 > \frac{1}{n-\sqrt[n-1]{2}}$, let $\{x_m\}$ be a bounded sequence in C, then $\|x_m - Tx_m\| \to 0$ if and only if $\|x_m - T_\alpha x_m\| \to 0$ as $m \to \infty$.*

Lemma 3.5 ([6]) *Let C be a closed convex subset of a Banach space E, and for all $n \in N$, let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ such that $\alpha_i \ge 0$, $i = 1, 2, \dots, n$, $\alpha_1 > 0$, and $\sum_{i=1}^{n} \alpha_i = 1$. Let T be an α-nonexpansive mapping from C into itself. If $\alpha_1 > \frac{1}{n-\sqrt[n-1]{2}}$, let $\{x_m\} \subset C$ converge strongly to x and $\|x_m - Tx_m\| \to 0$ converge strongly to 0 as $m \to \infty$, then $x \in F(T)$.*

Lemma 3.6 ([6]) *Let C be a closed convex subset of a uniformly convex and smooth Banach space E, and for all $n \in N$, let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ such that $\alpha_i \ge 0$, $i = 1, 2, \dots, n$, $\alpha_1 > 0$, and $\sum_{i=1}^{n} \alpha_i = 1$. Let T be an α-nonexpansive mapping from C into itself. If $\alpha_1 > \frac{1}{n-\sqrt[n-1]{2}}$, then $F(T)$ is closed and convex.*

Theorem 3.7 *Let C be a closed convex subset of a uniformly convex and smooth Banach space E, let $\{T_n\}_{n=1}^{\infty}$ be a uniformly closed countable family of α_n-nonexpansive mappings of C into itself such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, let $\alpha_n = (\alpha_{n1}, \alpha_{n2}, \dots, \alpha_{nN_0})$ such that $\alpha_{ni} \ge 0$, $i = 1, 2, \dots, N_0$, $\alpha_{n1} > 0$, and $\sum_{i=1}^{N_0} \alpha_{ni} = 1$. Let $f : E \to R$ be a convex and weakly continuous mapping with $C \subset \operatorname{int}(D(f))$. For any given Gauss $x_0 \in E$, $C_1 = C$, and $x_1 = \Pi_{C_1}^{f} x_0$, define a* *sequence* $\{x_n\}$ *in* C *by the following algorithm*:

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, \quad n \in N, \end{cases} \tag{3.1}$$

where $0 < a \le \beta_n \le 1$ *for all* $n \in N$. *If* $\alpha_{n1} > \frac{1}{N_0 - \sqrt[1]{2}}$, *then* $\{x_n\}$ *converges strongly to* $x^* = \Pi^f_F x_0$.

Proof Step 1. We show that C_n is closed and convex for each $n \ge 0$.

From the definitions of C_n, it is obvious that C_n is closed for each $n \ge 0$. Moreover, since $\|y_n - z\| \le \|x_n - z\|$ is equivalent to

$$\|y_n - x_n\|^2 + 2\langle y_n - x_n, Jx_n - Jz\rangle \le 0,$$

so C_n is convex for each $n \ge 0$.

Step 2. We show that $F \subset C_n$ for all $n \ge 0$. For all $p \in F$, we have that

$$\begin{aligned} \|y_n - p\| &= \big\|(1-\beta_n)x_n + \beta_n T_n x_n - p\big\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n \|T_n x_n - p\| \\ &= (1-\beta_n)\|x_n - p\| + \beta_n \big\|\alpha_{n1}(T_n x_n - T_n p) \\ &\quad + \alpha_{n2}\big(T_n x_n - T_n^2 p\big) + \cdots + \alpha_{nN_0}\big(T_n x_n - T_n^{N_0} p\big)\big\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n \frac{1-\alpha_{n1}^{N_0-1}}{\alpha_{n1}^{N_0-1}}\|x_n - T_n p\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned}$$

It implies that $p \in C_n$ for all $n \ge 0$. So, we have $F \subset C_n$ for all $n \ge 0$.

Step 3. We show that $\lim_{n\to\infty} x_n = x^* = \Pi^f_{\bar{C}} x_0$ and $x^* \in F$, where $\bar{C} = \bigcap_{n=1}^\infty C_n$. Indeed, since $\{C_n\}$ is a decreasing sequence of closed convex subsets of E such that $\bar{C} = \bigcap_{n=1}^\infty C_n$ is nonempty, it follows that

$$M - \lim_{n\to\infty} C_n = \bar{C} = \bigcap_{n=1}^\infty C_n \ne \emptyset.$$

By Theorem 3.2, we get

$$x_n \to x^* \quad \text{as } n \to \infty. \tag{3.2}$$

Noticing that $x_{n+1} = \Pi^f_{C_{n+1}} x_0 \in C_{n+1}$, we obtain that

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|.$$

In view of (3.2), we have that

$$\|y_n - x_{n+1}\| \to 0 \quad \text{as } n \to \infty$$

and

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \quad \text{as } n \to \infty.$$

From $y_n = (1 - \beta_n)x_n + \beta_n T_n x_n$, we have

$$\|x_n - T_n x_n\| = \frac{1}{\beta_n}\|y_n - x_n\|.$$

Because of the assumption that $0 < a \leq \beta_n \leq 1$, we have

$$\lim_{n\to\infty}\|x_n - T_n x_n\| = 0.$$

Since $\{x_n\}$ is uniformly closed, then $x^* \in F$.

Step 4. We show that $x^* = \Pi_F^f x_0$. Since $x^* = \Pi_{\bar{C}}^f x_0 \in F$ and F is a nonempty closed convex subset of $\bar{C} = \bigcap_{n=1}^{\infty} C_n$, we conclude that $x^* = \Pi_F^f x_0$. This completes the proof. □

Corollary 3.8 ([6]) *Let C be a closed convex subset of a uniformly convex and smooth Banach space E, let T be an α-nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$, let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{N_0})$ such that $\alpha_i \geq 0$, $i = 1, 2, \ldots, N_0$, $\alpha_1 > 0$, and $\sum_{i=1}^{N_0} \alpha_i = 1$. For any given Gauss $x_0 \in E$, $C_1 = C$, and $x_1 = \Pi_{C_1} x_0$, define a sequence $\{x_n\}$ in C by the following algorithm*:

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \in N, \end{cases} \tag{3.3}$$

where $0 < a \leq \beta_n \leq 1$ for all $n \in N$. If $\alpha_1 > \frac{1}{N_0 - \sqrt[1]{2}}$, then $\{x_n\}$ converges strongly to $x^* = \Pi_F x_0$.

Proof Substituting T to T_n in the proof of Theorem 3.7 and putting $f(x) \equiv 0$, we can draw from Theorem 3.7 the desired conclusion immediately. □

Remark 3.9 Theorem 3.7 extends the main results of [6] from a single mapping to a countable family of mappings and from the generalized projection operator to the modified generalized f-projection operator by a new method.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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