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On uniformly univalent functions with respect to symmetrical points

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Abstract

In this paper, we define and study some new subclasses of starlike and close-to-convex functions with respect to symmetrical points. These functions map the open unit disc onto certain conic regions in the right half plane. Some basic properties, a necessary condition, and coefficient and arc length problems are investigated. The mapping properties of the functions in these classes are studied under a certain linear operator.

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1 Introduction

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$. Let S, K, S^*, and C be the subclasses of A which consist of univalent, close-to-convex, starlike (with respect to origin), and convex functions, respectively. For recent developments, extensions, and applications, see [1–25] and the references therein.

A function f in A is said to be uniformly convex in E if f is a univalent convex function along with the property that, for every circular arc γ contained in E, with center ξ also in E, the image curve $f(\gamma)$ is a convex arc. The class of uniformly convex functions is denoted by UCV. The corresponding class UST is defined by the relation that $f \in UCV$ if, and only if, $zf' \in UST$. It is well known [13] that $f \in UCV$ if, and only if

$$\left|\frac{zf''(z)}{f'(z)}\right| < \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \quad (z \in E).$$

Uniformly starlike and convex functions were first introduced by Goodman [3] and then studied by various other authors. If $f, g \in A$, we say f is subordinate to g in E, written as $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ such that $f(z) = g(w(z))$ for $z \in E$.

For $0 \leq \beta < 1$, the class $P(\beta)$ consists of functions $p(z)$ analytic in E with $p(0) = 1$ such that $\Re p(z) > \beta$ for $z \in E$, and, with $\beta = 0$, we obtain the well-known class P of Carathéodory functions with positive real part.

For $k \in [0, \infty)$, the conic regions Ω_k are defined as follows, see [5]:

$$\Omega_k = \{u + iv : u > k\sqrt{(u-1)^2 + v^2}\}.$$

For fixed k, Ω_k represents the conic regions bounded, successively, by the imaginary axis ($k = 0$), the right branch of a hyperbolic ($0 < k < 1$) and a parabola $v^2 = 2u - 1$ ($k = 1$). When $k > 1$, the domain becomes a bounded domain being the interior of the ellipse.

We shall consider the case when $k \in [0,1]$. Related to the domain Ω_k, the following functions $p_k(z)$, $k \in [0,1]$, play the role of extremal functions mapping in E onto Ω_k:

$$p_k(z) = \begin{cases} \frac{1+z}{1-z} & (k = 0), \\ 1 + \frac{2}{\pi^2}(\log \frac{1+\sqrt{z}}{1-\sqrt{z}})^2 & (k = 1), \\ 1 + \frac{2}{1-k^2} \sinh^2[(\frac{2}{\pi} \arccos k) \operatorname{arctanh} \sqrt{z}] & (0 < k < 1). \end{cases} \tag{1.2}$$

These functions are univalent in E and belong to the class P. Using the subordination concept, we define the class $P(p_k)$ as follows.

Let $p(z)$ be analytic in E with $p(0) = 1$. Then $p \in P(p_k)$ if, and only if, $p \prec p_k$ in E and $p_k(z)$ are given by (1.2).

The conic domains Ω_k can be generalized as given by

$$\Omega_{k,\beta} = (1-\beta)\Omega_k + \beta,$$

with the corresponding extremal function

$$p_{k,\beta}(z) = (1-\beta)p_k + \beta \quad \bigl(0 \leq \beta < 1, k \in [0,1]\bigr).$$

It can easily be seen that the analytic function $p(z)$, with $p(0) = 1$, belongs to the class $P(p_{k,\beta})$ if $p(z) \prec p_{k,\beta}(z)$ in E.

It is easy to verify that $P(p_{k,\beta})$ is a convex set. It is known [6] that

$$P(p_k) \subset P\left(\frac{k}{k+1}\right) \subset P,$$

and, for $p \in P(p_k)$, we have

$$\left|\arg p(z)\right| \leq \sigma \frac{\pi}{2},$$

where

$$\sigma = \frac{2}{\pi} \arctan \frac{1}{k}. \tag{1.3}$$

So we can write $p(z) = h^\sigma(z)$, $h \in P$.

Also

$$P(p_{k,\beta}) \subset P\left(\frac{k+\beta}{k+1}\right) \subset P.$$

Sakaguchi [24] introduced and studied the class S_s^* of starlike functions with respect to symmetrical points. The class S_s^* includes the classes of convex and odd starlike functions with respect to the origin. It was shown [24] that a necessary and sufficient condition for $f \in S_s^*$ to be univalent and starlike with respect to symmetrical points in E is that

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right) \in P, \quad z \in E.$$

Das and Singh [2] defined the classes C_s of convex functions with respect to symmetrical points and showed that a necessary and sufficient condition for $f \in C_s$ is that

$$\frac{2(zf'(z))'}{(f(z)-f(-z))'} \in P, \quad z \in E.$$

It is also well known [2] that $f \in C_s$ if, and only if, $zf' \in S_s^*$.

We now define the following.

Definition 1.1 Let $f \in A$. The f is said to be in the class $k - ST_s(\beta)$ if, and only if,

$$\frac{2zf'(z)}{(f(z)-f(-z))} \in P(p_{k,\beta}), \quad z \in E.$$

It can easily be seen that

$$k - ST_s(\beta) \subset S_s^* \subset S_s^*, \qquad \beta_1 = \frac{k+\beta}{k+1}.$$

Also, for $\beta = 0 = k$, the class $k - ST_s(\beta)$ reduces to S_s^*.

The class $k - UCV_s(\beta)$ is defined as follows.

Definition 1.2 Let $f \in A$. Then $f \in k - UCV_s(\beta)$ if, and only if $zf' \in k - ST_s(\beta)$ for $z \in E$.

We note that

$$k - UCV_s(\beta) \subset C_s(\beta_1) \subset C_s, \qquad \beta_1 = \frac{k+\beta}{k+1}.$$

Definition 1.3 Let $f \in A$. Then $f \in k - UK_s(\beta)$ if, and only if, there exists $g \in k - ST_s(\beta)$ such that

$$\left(\frac{2zf'(z)}{g(z)-g(-z)}\right) \in P(p_{k,\beta}), \quad z \in E.$$

Since $P(p_{k,\beta}) \subset P(\beta_1) \subset P$, $\beta_1 = \frac{k+\beta}{k+1}$, and $k - ST_s(\beta) \subset S_s^*$, we note that

$$k - UK_s(\beta) \subset K_s \subset K,$$

where k_S consists of close-to-convex functions with respect to symmetrical starlike functions.

From the definition, it is clear that $k - UK_s(\beta)$ consists of univalent functions.

For $k = 0$, $\beta = 0$ and $f(z) = g(z)$, $k - UK_s(\beta)$ reduces to the class S_s^*.

2 Preliminary results

We shall need the following lemmas to prove our main results.

Lemma 2.1 [15] *Let $q(z)$ be a convex function in E with $q(0) = 1$ and let another function $h : E \to \mathbb{C}$ be with $\Re h(z) > 0$. Let $p(z)$ be analytic in E with $p(0) = 1$ such that*

$$\big(p(z) + h(z)zp'(z)\big) \prec q(z), \quad z \in E.$$

Then $p(z) \prec q(z)$, $z \in E$.

Lemma 2.2 *Let $N(z)$, $D(z)$ be analytic in E with*

$$N(0) = 0 = D(z)$$

and let $D \in S^*$ for $z \in E$. Then $\frac{N'(z)}{D'(z)} \in P(p_{k,\beta})$ implies that $\frac{N(z)}{D(z)} \in P(p_{k,\beta})$ for $z \in E$.

Proof Let

$$\frac{N(z)}{D(z)} = p(z).$$

Then

$$\frac{N'(z)}{D'(z)} = p(z) + h(z)\big(zp'(z)\big), \quad h(z) = \frac{1}{h_0(z)},$$

where

$$h_0(z) = \frac{zD'(z)}{D(z)} \in P.$$

Since $\frac{N'(z)}{D'(z)} \in P(p_{k,\beta})$, we have

$$\frac{N'(z)}{D'(z)} = \big(p(z) + h(z)\big(zp'(z)\big)\big) \prec p_{k,\beta}(z), \quad z \in E.$$

We now use Lemma 2.1 and this implies that

$$\frac{N(z)}{D(z)} = p(z) \prec p_{k,\beta}(z) \quad \text{in } E.$$

This proves that $\frac{N(z)}{D(z)} \in P(p_{k,\beta})$ for $z \in E$. □

The following lemma is an easy extension of a result proved in [5].

Lemma 2.3 *Let $k \in [0,\infty)$ and γ_1, δ_1 be any complex numbers with $\gamma_1 \neq 0$ and let $\Re\{\frac{\gamma_1 k}{k+1} + \delta_1\} > \beta$. If $h(z)$ is analytic in E, $h(0) = 1$ and it satisfies*

$$\left(h(z) + \frac{zh'(z)}{\gamma_1 h(z) + \delta_1}\right) \prec p_{k,\beta}(z), \tag{2.1}$$

and $q_{k,\beta}(z)$ is an analytic solution of

$$\left(q_{k,\beta}(z) + \frac{zq'_{k,\beta}(z)}{\gamma_1 q_{k,\beta}(z) + \delta_1}\right) = p_{k,\beta}(z),$$

then $q_{k,\beta}$ is univalent and

$$h(z) \prec q_{k,\beta}(z) \prec p_{k,\beta}(z),$$

and $q_{k,\beta}(z)$ is the best dominant of (2.1).

3 The class $k - ST_s(\beta)$

In this section, we shall study some basic properties of the class $k - ST_s(\beta)$.

Theorem 3.1 *Let $f \in k - ST_s(\beta)$. Then the odd function*

$$\Psi(z) = \frac{1}{2}\big[f(z) - f(-z)\big], \tag{3.1}$$

belongs to $k - ST(\beta)$ in E.

In particular $\Psi(z)$ is an odd starlike function of order $\beta_1 = \frac{k+\beta}{k+1}$ in E.

Proof Logarithmic differentiation of (3.1) and simple computation yield

$$\begin{aligned}\frac{z\Psi'(z)}{\Psi(z)} &= \frac{1}{2}\left[\frac{2zf'(z)}{f(z) - f(-z)} + \frac{2(-z)f'(-z)}{f(-z) - f(z)}\right] \\ &= \frac{1}{2}\big[p_1(z) + p_2(z)\big], \quad \text{for } z \in E, p_1, p_2 \in P(p_{k,\beta}).\end{aligned}$$

Since $P(p_{k,\beta})$ is a convex set, it follows that $\frac{z\Psi'(z)}{\Psi(z)} \in P(p_{k,\beta})$ and thus $\Psi \in k - ST(\beta)$ in E. □

As a special case, we note that, for $k = 0 = \beta$, $\frac{1}{2}[f(z) - f(-z)] = \Psi(z) \in S^*$ in E, and hence $\frac{zf'}{\Psi} \in P$. We now discuss a geometric property for $f \in k - ST_s(\beta)$. Here we investigate the behavior of the inclusion of the tangent at a point $w(\theta) = f(re^{i\theta})$ to the image Γ_r of the circle $C_r = \{z : |z| = r\}$, $0 \leq r < 1$, $\theta \in [0, 2\pi]$, under the mapping by means of a function f from the class $f \in k - ST_s(\beta)$.

Let

$$\Phi(\theta) = \frac{\pi}{2} + \theta + \arg f'\big(re^{i\theta}\big) = \arg \frac{\partial}{\partial\theta} f\big(re^{i\theta}\big),$$

and, for $\theta_2 > \theta_1$, $\theta_1, \theta_2 \in [0, 2\pi]$,

$$\Phi(\theta_2) - \Phi(\theta_1) = \theta_2 + \arg f'(re^{i\theta_2}) - \theta_1 - \arg f'(re^{i\theta_1}).$$

Now, since

$$\theta + \arg f'(re^{i\theta}) = \theta + \Re\{-i \ln f'(re^{i\theta})\},$$

then

$$\frac{\partial}{\partial\theta}\left(\theta + \arg f'(re^{i\theta})\right) = \Re\left\{1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\}.$$

Hence

$$\int_{\theta_1}^{\theta_2} \frac{\partial}{\partial\theta}\left(\theta + \arg f'(re^{i\theta})\right) d\theta = \int_{\theta_1}^{\theta_2} \Re\left\{1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\} d\theta.$$

Also, on the other hand,

$$\begin{aligned}\int_{\theta_1}^{\theta_2} \frac{\partial}{\partial\theta}\left(\theta + \arg f'(re^{i\theta})\right) d\theta &= \theta_2 + \arg f'(re^{i\theta_2}) - \theta_1 - \arg f'(re^{i\theta_1}) \\ &= \Phi(\theta_2) - \Phi(\theta_1).\end{aligned}$$

So, the integral on the left side of the last inequality characterizes the increment of the angle of the inclination of the tangent to the curve Γ_r between the points $w(\theta_2)$ and $w(\theta_1)$ for $\theta_2 > \theta_1$.

We have the following necessary condition for $f \in k - ST_s(\beta)$.

Theorem 3.2 *Let $f \in k - ST_s(\beta)$. Then, with $z = re^{i\theta}$ and $0 \le \theta_1 < \theta_2 \le 2\pi$, $0 \le \beta < 1$ and $0 \le k \le 1$, we have*

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta > -\sigma\pi + 2\cos^{-1}\left\{\frac{2(1-\beta)}{1-(1-2\beta)r^2}\right\} + \beta_1(\theta_2 - \theta_1),$$

where σ is given by (1.3) *and $\beta_1 = \frac{k+\beta}{k+1}$.*

Proof Since $\frac{f'(z)}{\Psi'(z)} \in P(p_{k,\beta})$, $\Psi(z) = \frac{1}{2}[f(z) - f(-z)]$ and $\Psi \in k - UCV(\beta) \subset C(\beta)$.

We can write

$$f'(z) = \left(\Psi_1'(z)\right)^{1-\beta_1} h^\sigma(z), \quad \Psi_1 \in C, h \in P(\beta),$$

and this gives us, with $z = re^{i\theta}$, $0 \le r < 1$, $0 \le \theta_1 < \theta_2 \le 2\pi$,

$$\begin{aligned}\int_{\theta_1}^{\theta_2} \Re\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta &= (1-\beta_1)\int_{\theta_1}^{\theta_2} \Re\left\{\frac{(z\Psi_1'(z))'}{\Psi_1'(z)}\right\} d\theta \\ &\quad + \sigma\int_{\theta_1}^{\theta_2} \Re\frac{2h'(z)}{h(z)}\, d\theta + \beta_1(\theta_2 - \theta_1). \qquad (3.2)\end{aligned}$$

For $h \in P(\beta)$, we observe that

$$\begin{aligned}\frac{\partial}{\partial\theta}\arg h\big(re^{i\theta}\big) &= \frac{\partial}{\partial\theta}\Re\big\{-i\ln h\big(re^{i\theta}\big)\big\}\\ &= \Re\left\{re^{i\theta}\frac{h'(re^{i\theta})}{h(re^{i\theta})}\right\}.\end{aligned}$$

Therefore

$$\int_{\theta_1}^{\theta_2}\Re\left\{\frac{re^{i\theta}h'(re^{i\theta})}{h(re^{i\theta})}\right\}d\theta = \arg h\big(re^{i\theta_2}\big) - \arg h\big(re^{i\theta_1}\big),$$

and

$$\max_{h\in P(\beta)}\left|\int_{\theta_1}^{\theta_2}\Re\left\{\frac{re^{i\theta}h'(re^{i\theta})}{h(re^{i\theta})}\right\}d\theta\right| = \max_{h\in P(\beta)}\left|\arg h\big(re^{i\theta_2}\big) - \arg h\big(re^{i\theta_1}\big)\right|.$$

We can write

$$\frac{1}{1-\beta}\big[h(z)-\beta\big] = p(z), \quad p \in P,$$

and for $|z| = r < 1$, it is well known that

$$\left|p(z) - \frac{1+r^2}{1-r^2}\right| \le \frac{2r}{1-r^2}.$$

From this, we have

$$\left|h(z) - \frac{1+(1-2\beta)r^2}{1-r^2}\right| \le \frac{2(1-\beta)r}{1-r^2}.$$

Thus the values of h are contained in the circle of Apollonius whose diameter is the line segment from $\frac{1-(1-2\beta)r}{1+r}$ to $\frac{1+(1-2\beta)r}{1-r}$ and has the radius $\frac{2(1-\beta)r}{1-r^2}$. So $|\arg h(z)|$ attains its maximum at points where a ray from origin is tangent to the circle, that is, when

$$\arg h(z) = \pm\sin^{-1}\left(\frac{2(1-\beta)r}{1-(1-2\beta)r^2}\right). \tag{3.3}$$

From (3.3), we observe that

$$\begin{aligned}\max_{h\in P(\beta)}\left|\int_{\theta_1}^{\theta_2}\Re\left\{re^{i\theta}\frac{h'(re^{i\theta})}{h(re^{i\theta})}\right\}d\theta\right| &\le 2\sin^{-1}\left(\frac{2(1-\beta)r}{1-(1-2\beta)r^2}\right)\\ &= \pi - 2\cos^{-1}\left(\frac{2(1-\beta)r}{1-(1-2\beta)r^2}\right).\end{aligned} \tag{3.4}$$

Also, for $\Psi_1 \in C$,

$$\int_{\theta_1}^{\theta_2}\Re\left\{1 + re^{i\theta}\frac{\Psi_1''(re^{i\theta})}{\Psi_1'(re^{i\theta})}\right\}d\theta \ge 0. \tag{3.5}$$

Using (3.4) and (3.5) in (3.2), we obtain the required result. □

We note the following special cases:

1. For $k = 0$, $0 \leq \theta_1 < \theta_2 \leq 2\pi$, $z = re^{i\theta}$, it follows from Theorem 3.2 that

$$\int_{\theta_1}^{\theta_2} \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta > -\pi \quad (z \in E).$$

   This is a necessary and sufficient condition for f to be close-to-convex (hence univalent) in E; see [7]. This also shows that $ST_s(\beta) \subset K$.
2. For $k = 1$ $\int_{\theta_1}^{\theta_2} \Re\{1 + \frac{zf''(z)}{f'(z)}\} d\theta > -\frac{\pi}{2}$.
3. When $k \in [0,1]$, it is obvious that $\sigma \in (0,1]$. In this case, the class $k - ST_s(\beta)$ consists of strongly close-to-convex functions of order σ in the sense of Pommerenke [20, 21].

Theorem 3.3 (Integral representation) *Let $f \in k - ST_s(\beta)$. Then*

$$f'(z) = \frac{1}{2}p(z)\exp\int_0^z \frac{1}{t}\bigl[p(t) + p(-t) - 2\bigr]\,dt,$$

where $p \in P(p_{k,\beta})$, $z \in E$.

Proof Since $f \in k - ST_s(\beta)$, we can write

$$\frac{2zf'(z)}{f(z) - f(-z)} = p(z), \quad p \in P(p_{k,\beta}).$$

This gives us

$$\frac{2[f(z) - f(-z)]'}{f(z) - f(-z)} - \frac{1}{z} = \frac{1}{2}\bigl[p(z) - p(-z) - 2\bigr]$$

and the result follows when we integrate. □

When $k = 0$, $\beta = 0$, we obtain the result for the class S_s^* given in [5].

We now study the class $k - ST_s(\beta)$ under a certain integral operator.

Theorem 3.4 *Let $g \in k - ST_s(\beta)$ and let for $m = 1, 2, 3, \ldots$, G be defined by*

$$G(z) = \frac{m+1}{2z^m}\int_0^z t^{m-1}\bigl\{g(t) - g(-t)\bigr\}\,dt. \tag{3.6}$$

Then $G(z)$ belongs to $k - ST_s(\beta)$ in E.

Proof Let

$$J(z) = \int_0^z t^{m-1}\frac{g(t) - g(-t)}{2}\,dt.$$

Since $g \in k - ST_s(\beta)$, $\frac{1}{2}\{g(z) - g(-z)\} \in k - ST(\beta) \subset S^*(\beta_1) \subset S^*$, and $\beta_1 = \frac{k+\beta}{k+1}$. Therefore it can easily be verified that $J(z)$ is $(m+1)$-valently starlike in E.

We can write (3.6) as

$$z^m G(z) = (m+1)J(z),$$

and, differentiating logarithmically, we have

$$\frac{zG'(z)}{G(z)} = \frac{zJ'(z) - mJ(z)}{J(z)} = \frac{N(z)}{D(z)},$$

say, where $N(0) = D(0) = 0$ and D is $(m+1)$-valently starlike.

Let

$$\frac{N(z)}{D(z)} = h(z).$$

Then

$$\begin{aligned} \frac{N'(z)}{D'(z)} &= h(z) + \frac{zh'(z)}{h_0(z)}, \quad h_0(z) = \frac{zD'(z)}{D(z)} \in P \\ &= h(z) + H_0(z)\big(zh'(z)\big), \quad H_0 = \frac{1}{h_0} \in P. \end{aligned} \tag{3.7}$$

Since

$$\begin{aligned} \frac{N'(z)}{D'(z)} &= \frac{(zh'(z))' - mJ'(z)}{J'(z)} \\ &= \left\{ \frac{(zJ'(z))'}{J'(z)} - m \right\} \in P(p_{k,\beta}). \end{aligned}$$

We now apply Lemma 2.2 to obtain

$$\frac{N(z)}{D(z)} = \frac{zG'(z)}{G(z)} \in P(p_{k,\beta}), \quad z \in E.$$

This proves that $G \in k - ST(\beta)$ in E. □

Theorem 3.5 *Let $f, g \in k - ST_s(\beta)$ and let F be defined by the following integral operator*:

$$F(z) = \left(\gamma + \frac{1}{\delta}\right) z^{1-\frac{1}{\delta}} \int_0^z t^{\frac{1}{\delta}-2} \left[\frac{f(t) - f(-t)}{2}\right]^{\frac{1}{1+\gamma}} \left[\frac{g(t) - g(-t)}{2}\right] dt, \tag{3.8}$$

where $z \in E$, $\delta > 0$, $\gamma \geq 0$ and $[\frac{k(1+\gamma)}{k+1} + (\frac{1}{\delta} - 1)] > \beta$. Then $F(z)$ belongs to $k - ST(\beta)$ for $z \in E$.

When $g(z) = z$, $\gamma = 0$, we obtain a generalized form of the Bernardi operator; see [1]. Also for $g(z) = z$, $\gamma = 0$, and $\delta = \frac{1}{2}$, we have the well-known integral operator studied by Libera [11] who showed that it preserves the geometric properties of convexity, starlikeness, and close-to-convexity.

Proof Let $\frac{f(z)-f(-z)}{2} = \Psi_1(z)$, $\frac{g(z)-g(-z)}{2} = \Psi_2(z)$. Then $\Psi_1, \Psi_2 \in k - ST(\beta)$ in E. We can write (3.8) as

$$F(z) = \left(\gamma + \frac{1}{\delta}\right) z^{1-\frac{1}{\delta}} \int_0^z t^{\frac{1}{\delta}-2} \big(\Psi_1(t)\big)^{\frac{1}{1+\gamma}} \big(\Psi_2(t)\big)\, dt. \tag{3.9}$$

Differentiating (3.9) logarithmically, and with $p(z) = \frac{zF'(z)}{F(z)}$, we have

$$\frac{\gamma}{1+\gamma}\frac{z\Psi_1'}{\Psi_1(z)} + \frac{1}{1+\gamma}\frac{z\Psi_2'}{\Psi_2(z)} = p(z) + \frac{zp'(z)}{(1+\gamma)p(z) + (\frac{1}{\delta} - 1)}. \tag{3.10}$$

Since, for $i = 1, 2$, $\Psi_i \in k - ST(\beta)$, $\frac{z\Psi_1'(z)}{\Psi_1} = h_1(z)$, $\frac{z\Psi_2'(z)}{\Psi_2} = h_2(z)$ both belong to $P(p_{k,\beta})$ in E, and $P(p_{k,\beta})$ is a convex set. Therefore

$$\left(\frac{\gamma}{1+\gamma}h_1(z) + \frac{1}{1+\gamma}h_2(z)\right) \in P(p_{k,\beta}), \quad z \in E. \tag{3.11}$$

From (3.10) and (3.11), it follows that

$$\left(p(z) + \frac{zp'(z)}{(1+\gamma)p(z) + (\frac{1}{\delta} - 1)}\right) \prec p_{k,\beta}(z).$$

We now apply Lemma 2.3 which gives us

$$p(z) \prec q_{k,\beta}(z) \prec p_{k,\beta}(z).$$

Thus $F \in k - ST(\beta)$ and the proof is complete. □

4 The class $k - UK_s(\beta)$

Here we shall study some properties of the class $k - UK_s(\beta)$ which consists of k-uniformly close-to-convex functions.

Let $L(r,f)$ denote the length of the image of the circle $|z| = r$ under f. We prove the following.

Theorem 4.1 *Let $f \in k - UK_s(\beta)$. Then, for $0 < r < 1$, $k \in [0,1]$,*

$$L(r,f) = O(1)\left(\frac{1}{1-r}\right)^{\sigma-\beta_1}, \quad \beta_1 < \frac{\sigma}{2},$$

where $\beta_1 = \frac{k+\beta}{k+1}$ and σ is given by (1.3), *and $O(1)$ is a constant depending only on k, β.*

Proof For $f \in k - UK_s(\beta)$, we can write

$$zf'(z) = \Psi(z)h^{\sigma}(z), \quad h \in P, \Psi \in S^*(\beta_1), \tag{4.1}$$

and $\Psi(z) = \{g(z) - g(-z)\}$, $g \in k - ST_s(\beta)$.

Since $\Psi \in S^*(\beta_1)$ and is odd, there exists an odd starlike function $\Psi_1(z)$ such that

$$\Psi(z) = z\left(\frac{\Psi_1(z)}{z}\right)^{1-\beta_1} = z\left(\frac{\Psi_1(z)}{z}\right)^{\frac{1-\beta_1}{k+1}}.$$

Thus, with $z = re^{i\theta}$,

$$L(r,f) = \int_0^{2\pi} \left|zf'(z)\right| d\theta = \int_0^{2\pi} \left|z^{\beta_1}\left(\Psi_1(z)\right)^{1-\beta_1} h^{\sigma}(z)\right| d\theta,$$

and using Hölder's inequality, we have

$$L(r,f) \leq 2\pi r^{\beta_1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \Psi_1(z) \right|^{(1-\beta)(\frac{z}{z-\sigma})} \, d\theta \right)^{\frac{2-\sigma}{z}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| h(z) \right|^2 \, d\theta \right)^{\frac{\sigma}{2}}. \tag{4.2}$$

For $h \in P$, it is well known [20] that

$$\frac{1}{2\pi} \int_0^{2\pi} \left| h(z) \right|^2 \, d\theta \leq \frac{1+3r^2}{1-r^2}. \tag{4.3}$$

Using (4.3) and subordination for odd starlike functions in (4.2), it follows that

$$\begin{aligned} L(r,f) &\leq C(\beta_1,\sigma) \left(\frac{1}{1-r^2} \right)^{[(1-\beta_1)(\frac{2}{2-\sigma})-1][\frac{1+3r^2}{1-r}]^{\frac{\sigma}{2}}} \\ &= O(1) \left(\frac{1}{1-r} \right)^{\sigma-\beta_1}, \end{aligned}$$

where C and $O(1)$ are constants depending only on β_1 and σ. This completes the proof. □

We now discuss the growth rate of coefficients of $f \in k - UK_s(\beta)$.

Theorem 4.2 *Let $f \in k - UK_s(\beta)$ and be given by* (1.1). *Then*

$$a_n = O(1) n^{\sigma-\beta_1-1}, \quad n \geq 1, \beta_1 < \frac{\sigma}{2},$$

where $O(1)$ is a constant depending only on σ and β_1 and σ, β_1 are as given in Theorem 4.1.

Proof For $z = re^{i\theta}$, $n \geq 1$, Cauchy's Theorem gives us

$$\begin{aligned} n|a_n| &= \frac{1}{2\pi r^{n+1}} \left| \int_0^{2\pi} zf'(z) e^{-in\theta} \, d\theta \right| \\ &\leq \frac{1}{2\pi r^{n+1}} \int_0^{2\pi} \left| zf'(z) \right| d\theta \\ &= \frac{1}{2\pi r^n} L(r,f). \end{aligned}$$

With $r = (1 - \frac{1}{n})$, we use Theorem 4.1 and obtain the required result. □

Theorem 4.3 *Let $f \in k - UK_s(\beta)$ and let F be defined by*

$$F(z) = \frac{m+1}{2z^m} \int_0^z t^{m-1} \left\{ f(t) - f(-t) \right\} dt. \tag{4.4}$$

Then $F \in k - UK_s(\beta)$ in E. That is, the class $k - UK_s(\beta)$ is preserved under the integral operator (4.4).

Proof Since $f \in k - UK_s(\beta)$, we can write

$$\left\{ \frac{2zf'(z)}{g(z) - g(-z)} \right\} \in P(p_{k,\beta}), \quad g \in k - ST_s(\beta) \subset S_S^*(\beta_1).$$

Let $G(z) = \frac{1}{2}\{g_1(z) - g_1(-z)\}$ and be defined by (3.5). By Theorem 3.4, $g_1 \in k - ST(\beta)$ and $G \in k - S_sT(\beta) \subset S_s^*(\beta_1)$. Let $G = zG_1'$. Then we can write

$$G_1'(z) = \frac{1}{2}\big[zg_1(z) - g_1(-z)\big]', \quad G_1 \in k - UCV_s(\beta).$$

Thus, from (4.4) and $g = zg_1'$, $g_1 \in C_s(\beta_1)$, we have

$$\begin{aligned}\frac{2F'(z)}{[g_1(z) - g_1(-z)]'} &= \frac{z^m\{f(z) - f(-z)\} - m\int_0^z t^{m-1}\{f(t) - f(-t)\}\,dt}{z^m\{g_1(z) - g_1(-z)\} - m\int_0^z t^{m-1}\{g_1(t) - g_1(-t)\}\,dt} \\ &= \frac{N(z)}{D(z)},\end{aligned}$$

say. We note that $N(0) = D(0) = 0$, and for $g_1 \in C_S(\beta_1)$,

$$\begin{aligned}\frac{(zD'(z))'}{D'(z)} &= m + \frac{\{z[g_1(z) - g_1(-z)]'\}'}{\{g_1(z) - g_1(-z)\}'} \\ &= m + h_1(z), \quad h_1 \in P(\beta_1).\end{aligned}$$

Since $P(\beta_1)$ is a convex set, $D \in C_s(\beta_1) \subset S^*$ in E. We thus have

$$\frac{N'(z)}{D'(z)} = \frac{1}{2}\left[\frac{2zf'(z)}{[g_1(z) - g_1(-z)]'} + \frac{2(-z)f'(-z)}{[g_1(-z) - g_1(z)]'}\right] \in P(p_{k,\beta}).$$

Now, using Lemma 2.2, it follows that

$$\frac{N(z)}{D(z)} = \frac{2F'(z)}{(g_1(z) - g_1(-z))'} \in P(p_{k,\beta}) \quad \text{for } z \in E.$$

This proves that $F \in k - UK_S(\beta)$ in E. □

We study a partial converse of the above result as follows.

Theorem 4.4 *Let* $(\frac{2zf'(z)}{g(z)-g(-z)}) \prec p_k(z)$ *in* E *and let*

$$F_1(z) = \frac{1}{1+m} z^{1-m}\big(z^m f(z)\big)', \quad m = 1, 2, 3, \ldots. \tag{4.5}$$

Then $F_1 \in K_s$ *for* $|z| < r_1$, *where*

$$r_1 = \left\{\frac{m+1}{(2-\beta_1) + \sqrt{(z-\beta_1)^2 + (m+1)(m-1+2\beta_1)}}\right\}, \quad \beta_1 = \frac{k+\beta}{k+1}. \tag{4.6}$$

Proof We shall need the following well-known results for $p \in P(\alpha)$, $0 \le \alpha < 1$; see [4]:

$$\frac{1-(1-2\alpha)r}{1+r} \le \big|p(z)\big| \le \frac{1+(1-2\alpha)r}{1-r}, \tag{4.7}$$

$$\big|p'(z)\big| \le \frac{2[\Re p(z) - \alpha]}{1-r^2}. \tag{4.8}$$

Since $f \in k - UK_s(\beta)$, there exists $g \in S_s^*(\beta_1)$ such that, for $z \in E$.

$$\left(\frac{2zf'(z)}{g(z) - g(-z)}\right) = p(z), \quad p \in P(p_k) \subset P(\alpha), \alpha = \frac{k}{k+1}.$$

From (4.5), we have

$$F_1(z) = \frac{1}{1+m}\left[mf(z) + zf'(z)\right],$$

and this gives us

$$\begin{aligned}\frac{2zF_1'(z)}{g(z) - g(-z)} &= \frac{1}{m+1}\left[\frac{2mf'(z)}{g(z) - g(-z)} + \frac{2z(zf'(z))'}{g(z) - g(-z)}\right] \\ &= \frac{1}{m+1}\left[mp(z) + zp'(z) + p(z)h(z)\right],\end{aligned}$$

where

$$h(z) = \frac{z\Psi'(z)}{\Psi(z)} \in P(\beta_1), \quad \Psi(z) = g(z) - g(-z).$$

Now, using (4.7) and (4.8), we have

$$\begin{aligned}\Re\left\{\frac{2zF_1'(z)}{g(z) - g(-z)}\right\} &\geq \frac{(\Re p(z) - \alpha)}{1+m}\left\{m + \frac{1 - (1 - 2\beta_1)r}{1+r} - \frac{2r}{1-r^2}\right\} \\ &= \frac{\Re p(z) - \alpha}{1+m}\left[\frac{T(r)}{1-r^2}\right],\end{aligned} \tag{4.9}$$

where

$$T(r) = (m+1) - 2(2 - \beta_1)r + (-m - 2\beta_1 + 1)r^2.$$

We note that $T(0) = 1 + m > 0$ and $T(1) = -3 < 0$. So there exists $r_1 \in (0, 1)$. The right hand side of (4.9) is positive for $|z| < r_1$, where r_1 is given by (4.6). This implies that $F \in K_s$ for $|z| < r_1$ and the proof is complete. □

We have the following special cases.

1. For $k = 0 = \beta, f \in K_s$. Then F_1, defined by (4.5) belongs to K_s for $|z| < r_0 = \frac{1+m}{2+\sqrt{3+m^2}}$.
2. When $m = 1$ and $\beta_1 = 0$ (that is, $k = 0 = \beta$), then $F_1(z) = \frac{(zf(z))'}{2}$ belongs to the same class for $|z| < \frac{1}{2}$. This result has been proved by Livingston [12] for convex and starlike functions.

Competing interests
The author declares that she has no competing interest.

Author's contributions
The article is the work of one author, who read and approved the final manuscript.

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