Charoensawan and Thangthong *Journal of Inequalities and Applications* 2014, **2014**:245
http://www.journalofinequalitiesandapplications.com/content/2014/1/245

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

(G,F)-Closed set and tripled point of coincidence theorems for generalized compatibility in partially metric spaces

Phakdi Charoensawan* and Chaiporn Thangthong

*Correspondence:
phakdi@hotmail.com
Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand

Abstract

In this work, we prove the existence of a tripled point of coincidence theorem for a pair $\{F,G\}$ of mappings $F,G:X\times X\times X\to X$ with φ-contraction mappings in partially ordered metric spaces without G-increasing property of F and mixed monotone property of G, using the concept of a (G,F)-closed set. We give some examples of a nonlinear contraction mapping, which is not applied to the existence of tripled coincidence point by G using the mixed monotone property. We also show the uniqueness of a tripled point of coincidence of the given mapping. Further, we apply our results to the existence and uniqueness of a tripled point of coincidence of the given mapping with G-increasing property of F and mixed monotone property of G in partially ordered metric spaces.

Keywords: tripled fixed point; tripled coincidence point; tripled point of coincidence; generalized compatible; invariant set; mixed g-monotone; partially ordered set; closed set

1 Introduction

The existence of a fixed point for the contraction type of mappings in partially ordered metric spaces has been studied by Ran and Reurings [1] and they established some new results for contractions in partially ordered metric spaces and presented applications to matrix equations. Following this line of research, Nieto and Rodriguez-Lopez [2, 3] extended the results in [1]. Later, Agarwal *et al.* [4] presented some new results for contractions in partially ordered metric spaces.

In 1987, Guo and Lakshmikantham [5] introduced the concept of a coupled fixed point. Later, Bhaskar and Lakshmikantham [6] introduced the concept of the mixed monotone property for contractive operators in partially ordered metric spaces. They also give some applications on the existence and uniqueness of the coupled fixed point theorems for mappings which satisfy the mixed monotone property. Lakshimikantham and Ćirić [7] extended the results in [6] by defining the mixed g-monotonicity and proved the existence and uniqueness of coupled coincidence point for such a mapping which satisfy the mixed monotone property in partially ordered metric spaces. As a continuation of this work, many authors conducted research on the coupled fixed point theory and coupled coincidence point theory in partially ordered metric spaces and different spaces. For example, see [7–29].

One of the interesting ways to developed coupled fixed point theory in partially ordered metric spaces is to consider the mapping $F : X \times X \to X$ without the mixed monotone property. Recently, Sintunavarat *et al.* [18, 19] proved some coupled fixed point theorems for nonlinear contractions without mixed monotone property and extended some coupled fixed point theorems of Bhaskar and Lakshmikantham [6] by using the concept of an F-invariant set due to Samet and Vetro [12]. Later, Kutbi *et al.* [23] introduced the concept of an F-closed set which is weaker than the concept of an F-invariant set and proved some coupled fixed point theorems without the condition of mixed monotone property.

In 2014, Hussain *et al.* [15] presented the new concept of generalized compatibility of a pair $\{F, G\}$ of mappings $F, G : X \times X \to X$ and proved some coupled coincidence point results of such a mapping without the mixed G-monotone property of F, which generalized some recent comparable results in the literature. They also showed some examples and an application to integral equations to support the result.

The notion of a tripled fixed point which is a fixed point of order $N = 3$ was introduced by Samet and Vetro [12]. Later, in 2011, Berinde and Borcut [30] defined the concept of a tripled fixed point in the case of ordered sets in order to keep the mixed monotone property for nonlinear mappings in partially ordered complete metric spaces and proved existence and uniqueness theorems for contractive type mappings. In 2012, Berinde and Borcut [31] introduced the concept of a tripled coincidence point for a pair of nonlinear contractive mappings $F : X^3 \to X$ and $g : X \to X$ and obtained tripled coincidence point theorems which generalized the results of [30]. Recently, Aydi *et al.* [32] introduced the concept of W-compatibility for mappings $F : X^3 \to X$ and $g : X \to X$ in an abstract metric space and defined the notion of a tripled point of coincidence. They also established tripled and common point of coincidence theorems in an abstract metric space.

A wide discussion on a tripled coincidence point in partially ordered metric spaces, using mixed the g-monotone property, has been dedicated to the improvement and generalization. Borcut [33] established tripled coincidence point theorems for a pair of mappings $F : X^3 \to X$ and $g : X \to X$ satisfying a nonlinear contractive condition and mixed g-monotone property in partially ordered metric spaces. The presented theorems extended existing results in literature. Recently, Choudhury *et al.* [34] established some tripled coincidence point results in partially ordered metric spaces depended on another contractions. Very recently, Aydi *et al.* [35] established tripled coincidence point theorems for a pair of mappings $F : X^3 \to X$ and $g : X \to X$ satisfying weak φ-contractions in partially ordered metric spaces. The results unified, generalized, and complemented various known comparable results by Berinde and Borcut [31]. After the publication of this work, some authors have studied tripled fixed point and tripled coincidence point theory in different directions in several spaces with applications (see [13, 14, 32–48]).

In 2013, Charoensawan [44] introduced the concept of an (F, g)-invariant set and proved the existence of a tripled coincidence point theorem and a tripled common fixed point theorem for a ϕ-contractive mapping in a complete metric space without the mixed g-monotone property. Very recently, Karapınar *et al.* [49] showed that the notion of a transitive F-closed (or F-invariant) set is equivalent to the concept of a preordered set, and then some recent multidimensional results using F-invariant sets can be reduced to well-known results on partially ordered metric spaces.

In this work, we generalize and extend a tripled point of coincidence theorem for a pair $\{F, G\}$ of mappings $F, G : X \times X \times X \to X$ with φ-contraction mappings in partially ordered metric spaces without the G-increasing property of F and the mixed monotone property of G by using the concept of a (G,F)-closed set.

2 Preliminaries

In this section, we give some definitions, propositions, examples, and remarks which are useful for the main results in this paper. Throughout this paper, $(X,\preceq)$ denotes a partially ordered set with the partial order $\preceq$. By $x \preceq y$, we mean $y \succeq x$. Let $(x,\preceq)$ be a partially ordered set, the partial order $\preceq_3$ for the product set $X \times X \times X$ defined in the following way: for all $(x,y,z),(u,v,w) \in X \times X \times X$,

$$(x,y,z) \preceq_3 (u,v,w) \quad \text{if and only if}$$
$$G(x,y,z) \preceq G(u,v,w), \qquad G(v,w,u) \preceq G(y,z,x) \quad \text{and} \quad G(w,u,v) \preceq G(z,x,y),$$
$$\text{where } G : X \times X \times X \to X \text{ is one-one.}$$

We say that (x,y,z) is comparable to (u,v,w) if either $(x,y,z) \preceq_3 (u,v,w)$ or $(u,v,w) \preceq_3 (x,y,z)$.

Guo and Lakshmikantham [5] introduced the concept of a coupled fixed point as follows.

Definition 2.1 [5] An element $(x,y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.

The concept of a mixed monotone property has been introduced by Bhaskar and Lakshmikantham in [6].

Definition 2.2 [6] Let $(X,\preceq)$ be a partially ordered set and $F : X \times X \to X$. We say F has the mixed monotone property if, for any $x,y \in X$,

$$x_1,x_2 \in X, \quad x_1 \preceq x_2 \quad \text{implies} \quad F(x_1,y) \preceq F(x_2,y)$$

and

$$y_1,y_2 \in X, \quad y_1 \preceq y_2 \quad \text{implies} \quad F(x,y_1) \succeq F(x,y_2).$$

In 2009, Lakshmikantham and Ćirić in [7] introduced the concept of a mixed g-monotone mapping and a coupled coincidence point as follows.

Definition 2.3 [7] Let $(X,\preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say F has the mixed g-monotone property if, for any $x,y \in X$,

$$x_1,x_2 \in X, \quad gx_1 \preceq gx_2 \quad \text{implies} \quad F(x_1,y) \preceq F(x_2,y)$$

and

$$y_1,y_2 \in X, \quad gy_1 \preceq gy_2 \quad \text{implies} \quad F(x,y_1) \succeq F(x,y_2).$$

Definition 2.4 [7] An element $(x,y) \in X \times X$ is called a coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x,y) = gx$ and $F(y,x) = gy$.

Definition 2.5 [7] Let X be a non-empty set and $F : X \times X \to X$ and $g : X \to X$. We say F and g are commutative if $gF(x,y) = F(gx,gy)$ for all $x, y \in X$.

Hussain *et al.* [15] introduced the concept of G-increasing and $\{F,G\}$ generalized compatibility and proved the coupled coincidence point for such mappings involving the (ψ,ϕ)-contractive condition as follows.

Definition 2.6 [15] Suppose that $F, G : X \times X \to X$ are two mappings. F is said to be G-increasing with respect to $\preceq$ if, for all $x, y, u, v \in X$, with $G(x,y) \preceq G(u,v)$ we have $F(x,y) \preceq F(u,v)$.

Definition 2.7 [15] An element $(x,y) \in X \times X$ is called a coupled coincidence point of mappings $F, G : X \times X \to X$ if $F(x,y) = G(x,y)$ and $F(y,x) = G(y,x)$.

Definition 2.8 [15] Let $F, G : X \times X \to X$. We say that the pair $\{F,G\}$ is generalized compatible if

$$\begin{cases} d(F(G(x_n,y_n),G(y_n,x_n)),G(F(x_n,y_n),F(y_n,x_n))) \to 0 & \text{as } n \to +\infty, \\ d(F(G(y_n,x_n),G(x_n,y_n)),G(F(y_n,x_n),F(x_n,y_n))) \to 0 & \text{as } n \to +\infty, \end{cases}$$

whenever (x_n) and (y_n) are sequences in X such that

$$\begin{cases} \lim_{n\to\infty} F(x_n,y_n) = \lim_{n\to\infty} G(x_n,y_n) = t_1, \\ \lim_{n\to\infty} F(y_n,x_n) = \lim_{n\to\infty} G(y_n,x_n) = t_2. \end{cases}$$

Definition 2.9 [15] Let $F, G : X \times X \to X$ be two maps. We say that the pair $\{F,G\}$ is commuting if

$$F\big(G(x,y),G(y,x)\big) = G\big(F(x,y),F(y,x)\big) \quad \text{for all } x, y \in X.$$

Let Φ denote the set of all functions $\phi : [0,\infty) \to [0,\infty)$ such that:

(i) ϕ is continuous and increasing,
(ii) $\phi(t) = 0$ if and only if $t = 0$,
(iii) $\phi(t+s) \le \phi(t) + \phi(s)$, for all $t, s \in [0,\infty)$.

Let Ψ be the set of all functions $\phi : [0,\infty) \to [0,\infty)$ such that $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0^+} \psi(t) = 0$.

Theorem 2.10 [15] *Let $(X,\preceq)$ be a partially ordered set and M be a non-empty subset of X^4 and let d be a metric on X such that (X,d) is a complete metric space. Assume that $F, G : X \times X \to X$ are two generalized compatible mappings such that F is G-increasing with respect to $\preceq$, G is continuous and has the mixed monotone property. Suppose that for any $x, y \in X$, there exist $u, v \in X$ such that $F(x,y) = G(u,v)$ and $F(y,x) = G(v,u)$. Suppose that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that the following holds*:

$$\phi\big(d\big(F(x,y),F(u,v)\big)\big) \le \frac{1}{2}\phi\big(d\big(G(x,y),G(u,v)\big) + d\big(G(y,x),G(v,u)\big)\big) - \psi\left(\frac{d(G(x,y),G(u,v)) + d(G(y,x),G(v,u))}{2}\right)$$

for all $x, y, u, v \in X$ with $G(x,y) \preceq G(u,v)$ and $G(y,x) \succeq G(v,u)$.

Also suppose that either

(a) *F is continuous or*

(b) *X has the following properties: for any two sequences $\{x_n\}$ and $\{y_n\}$ with*

  (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*

  (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.*

If there exist $(x_0, y_0) \in X \times X$ with

$$G(x_0, y_0) \preceq F(x_0, y_0) \quad \text{and} \quad G(y_0, x_0) \succeq F(y_0, x_0),$$

then there exist $(x, y) \in X \times X$ such that $G(x, y) = F(x, y)$ and $G(y, x) = F(y, x)$, that is, F and G have a coupled coincidence point.

Kutbi *et al.* [23] introduced the notion of an F-closed set which extended the notion of an F-invariant set as follows.

Definition 2.11 [23] Let $F : X \times X \to X$ be a mapping, and let M be a subset of X^4. We say that M is an F-closed subset of X^4 if, for all $x, y, u, v \in X$,

$$(x, y, u, v) \in M \quad \Rightarrow \quad \big(F(x, y), F(y, x), F(u, v), F(v, u)\big) \in M.$$

In 2010, Samet and Vetro [12] gave the notion of a fixed point of order $N = 3$ as follows.

Definition 2.12 [12] An element $(x, y, z) \in X \times X \times X$ is called a *tripled point of coincidence* of mappings F and g if $F(x, y, z) = x$, $F(y, z, x) = y$ and $F(z, x, y) = z$.

In 2012, Berinde and Borcut [31] introduced the concept of a tripled coincidence point and mixed g-monotonicity as follows.

Definition 2.13 [31] Let $(X, \preceq)$ be a partially ordered set and two mappings $F : X \times X \times X \to X$, $g : X \to X$. We say that F has the mixed g-monotone property if, for any $x, y, z \in X$,

$$x_1, x_2 \in X, \quad g(x_1) \preceq g(x_2) \quad \text{implies} \quad F(x_1, y, z) \preceq F(x_2, y, z), \tag{1}$$

$$y_1, y_2 \in X, \quad g(y_1) \preceq g(y_2) \quad \text{implies} \quad F(x, y_1, z) \succeq F(x, y_2, z) \tag{2}$$

and

$$z_1, z_2 \in X, \quad g(z_1) \preceq g(z_2) \quad \text{implies} \quad F(x, y, z_1) \preceq F(x, y, z_2). \tag{3}$$

Definition 2.14 [31] An element $(x, y, z) \in X \times X \times X$ is called a *tripled coincidence point* of mappings F and g if $F(x, y, z) = g(x)$, $F(y, x, y) = g(y)$ and $F(z, y, x) = g(z)$.

Aydi *et al.* [35] extended the tripled coincidence point theorems for mixed g-monotone operator obtained by Berinde and Borcut [31]. For the sake of completeness, we recollect the main results of Aydi *et al.* [35] here.

Let the set of functions $\Phi = \{\varphi : [0, +\infty) \to [0, +\infty) : \varphi(t) < t \text{ and } \lim_{r \to t^+} \varphi(r) < t, t > 0\}$.

Theorem 2.15 [35] *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$ and $g : X \to X$ be such that F has the mixed g-monotone property and $F(X^3) \subseteq g(X)$. Assume there is a function $\varphi \in \Phi$ such that*

$$\begin{aligned} &d\big(F(x,y,z),F(u,v,w)\big) + d\big(F(y,x,y),F(v,u,v)\big) + d\big(F(z,y,x),F(w,v,u)\big) \\ &\quad \le 3\varphi\left(\frac{d(g(x),g(u)) + d(g(y),g(v)) + d(g(z),g(w))}{3}\right) \end{aligned} \tag{4}$$

for all $x, y, z, u, v, w \in X$ with $g(x) \succeq g(u)$, $g(y) \preceq g(v)$ and $g(z) \succeq g(w)$. Assume that F is continuous, g is continuous and commutes with F.

If there exist $x_0, y_0, z_0 \in X$ such that

$$g(x_0) \preceq F(x_0,y_0,z_0), \qquad g(y_0) \succeq F(y_0,x_0,y_0) \quad \text{and} \quad g(z_0) \preceq F(z_0,y_0,x_0),$$

then there exist $x, y, z \in X$ such that

$$g(x) = F(x,y,z), \qquad g(y) = F(y,x,y) \quad \text{and} \quad g(z) = F(z,y,x).$$

Definition 2.16 [35] Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. We say that $(X, d, \preceq)$ is regular if the following conditions hold:

(i) if a non-decreasing sequence $\{x_n\} \to x$ in X, then $x_n \preceq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\} \to y$ in X, then $y \preceq y_n$ for all n.

Theorem 2.17 [35] *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that $(X, d, \preceq)$ is regular. Suppose that there exist $\varphi \in \Phi$ and mappings $F : X \times X \times X \to X$ and $g : X \to X$ are such that (4) hold for any $x, y, z, u, v, w \in X$ with $g(x) \succeq g(u)$, $g(y) \preceq g(v)$ and $g(z) \succeq g(w)$. Suppose also that $(g(X), d)$ is complete, F has the mixed g-monotone property, and $F(X^3) \subseteq g(X)$.*

If there exist $x_0, y_0, z_0 \in X$ such that

$$g(x_0) \preceq F(x_0,y_0,z_0), \qquad g(y_0) \succeq F(y_0,x_0,y_0) \quad \text{and} \quad g(z_0) \preceq F(z_0,y_0,x_0),$$

then there exist $x, y, z \in X$ such that

$$g(x) = F(x,y,z), \qquad g(y) = F(y,x,y) \quad \text{and} \quad g(z) = F(z,y,x).$$

Now, we give the notion of a (G, F)-closed set which is useful for our main results.

Definition 2.18 Suppose that $F, G : X \times X \times X \to X$ are two mapping. F is said to be G-increasing with respect to $\preceq$ if, for all $x, y, z, u, v, w \in X$, with $G(x,y,z) \preceq G(u,v,w)$ we have $F(x,y,z) \preceq F(u,v,w)$.

Definition 2.19 An element $(x, y, z) \in X \times X \times X$ is called a tripled point of coincidence of mappings $F, G : X \times X \times X \to X$ if $F(x,y,z) = G(x,y,z)$, $F(y,z,x) = G(y,z,x)$ and $F(z,x,y) = G(z,x,y)$.

Definition 2.20 Let $F, G : X \times X \to X$. We say that the pair $\{F, G\}$ is generalized compatible if

$$\begin{aligned}
&\lim_{n\to\infty} d\bigl(F\bigl(G(x_n,y_n,z_n),G(y_n,z_n,x_n),G(z_n,x_n,y_n)\bigr),\\
&\quad G\bigl(F(x_n,y_n,z_n),F(y_n,z_n,x_n),F(z_n,x_n,y_n)\bigr)\bigr) = 0,\\
&\lim_{n\to\infty} d\bigl(F\bigl(G(y_n,z_n,x_n),G(z_n,x_n,y_n),G(x_n,y_n,z_n)\bigr),\\
&\quad G\bigl(F(y_n,z_n,x_n),F(z_n,x_n,y_n),F(x_n,y_n,z_n)\bigr)\bigr) = 0,\\
&\lim_{n\to\infty} d\bigl(F\bigl(G(z_n,x_n,y_n),G(x_n,y_n,z_n),G(y_n,z_n,x_n)\bigr),\\
&\quad G\bigl(F(z_n,x_n,y_n),F(x_n,y_n,z_n),F(y_n,z_n,x_n)\bigr)\bigr) = 0,
\end{aligned}$$

whenever (x_n), (y_n), and (z_n) are sequences in X such that

$$\begin{cases}
\lim_{n\to\infty} F(x_n,y_n,z_n) = \lim_{n\to\infty} G(x_n,y_n,z_n) = t_1,\\
\lim_{n\to\infty} F(y_n,z_n,x_n) = \lim_{n\to\infty} G(y_n,z_n,x_n) = t_2,\\
\lim_{n\to\infty} F(z_n,x_n,y_n) = \lim_{n\to\infty} G(z_n,x_n,y_n) = t_3.
\end{cases}$$

Definition 2.21 Let $F, G : X \times X \to X$ be two maps. We say that the pair $\{F, G\}$ is commuting if

$$F\bigl(G(x,y,z),G(y,z,x),G(z,x,y)\bigr) = G\bigl(F(x,y,z),F(y,z,x),F(z,x,y)\bigr) \quad \text{for all } x,y,z \in X.$$

Definition 2.22 Let $F, G : X \times X \to X$ be two mapping, and let M be a subset of X^6. We say that M is an (G, F)-closed subset of X^6 if, for all $x, y, z, u, v, w \in X$,

$$\begin{aligned}
&\bigl(G(x,y,z),G(y,z,x),G(z,x,y),G(u,v,w),G(v,w,u),G(w,u,v)\bigr) \in M\\
&\quad\Rightarrow\quad \bigl(F(x,y,z),F(y,z,x),F(z,x,y),F(u,v,w),F(v,w,u),F(w,u,v)\bigr) \in M.
\end{aligned}$$

Definition 2.23 Let $(X, \preceq)$ be a metric space and M be a subset of X^6. We say that M satisfies *the transitive property* if and only if, for all $x, y, z, u, v, w, a, b, c \in X$,

$$\begin{aligned}
&\bigl(G(x,y,z),G(y,z,x),G(z,x,y),G(u,v,w),G(v,w,u),G(w,u,v)\bigr) \in M \quad \text{and}\\
&\bigl(G(u,v,w),G(v,w,u),G(w,u,v),G(a,b,c),G(b,c,a),G(c,a,b)\bigr) \in M\\
&\quad\Rightarrow\quad \bigl(G(x,y,z),G(y,z,x),G(z,x,y),G(a,b,c),G(b,c,a),G(c,a,b)\bigr) \in M.
\end{aligned}$$

Remark The set $M = X^6$ is a trivially (G, F)-closed set, which satisfies the transitive property.

Example 2.24 Let (X, d) be a metric space endowed with a partial order $\preceq$. Let $F, G : X \times X \times X \to X$ be two generalized compatible mappings such that F is G-increasing with respect to $\preceq$, G is continuous and has the mixed monotone property. Define a subset $M \subseteq X^6$ by

$$M = \bigl\{(x,y,z,u,v,w) \in X^6 : x \preceq u, y \succeq v \text{ and } z \preceq w\bigr\}.$$

Let $(G(x,y,z),G(y,z,x),G(z,x,y),G(u,v,w),G(v,w,u),G(w,u,v))\in M$. It is easy to see that, since F is G-increasing with respect to $\preceq$, we have

$$F(x,y,z)\preceq F(u,v,w),\qquad F(y,z,x)\succeq F(v,w,u)\quad\text{and}\quad F(z,x,y)\preceq F(w,u,v),$$

and this implies that

$$\bigl(F(x,y,z),F(y,z,x),F(z,x,y),F(u,v,w),F(v,w,u),F(w,u,v)\bigr)\in M.$$

Then M is (G,F)-closed subset of X^6, which satisfies the transitive property.

3 Main results

Let Φ denote the set of functions $\varphi:[0,\infty)\to[0,\infty)$ satisfying

1. $\varphi(t)<t$ for all $t>0$,
2. $\lim_{r\to t^+}\varphi(r)<t$ for all $t>0$.

Theorem 3.1 *Let $(X,\preceq)$ be a partially ordered set and M be a non-empty subset of X^6 and let d be a metric on X such that (X,d) is a complete metric space. Assume that $F,G:X\times X\times X\to X$ are two generalized compatible mappings such that G is continuous and for any $x,y,z\in X$, there exist $u,v,w\in X$ such that $F(x,y,z)=G(u,v,w)$, $F(y,z,x)=G(v,w,u)$, and $F(z,x,y)=G(w,u,v)$. Suppose that there exists $\varphi\in\Phi$ such that the following holds*:

$$\begin{aligned}&d\bigl(F(x,y,z),F(u,v,w)\bigr)+d\bigl(F(y,z,x),F(v,w,u)\bigr)+d\bigl(F(z,x,y),F(w,u,v)\bigr)\\&\quad\le\varphi\Bigl(d\bigl(G(x,y,z),G(u,v,w)\bigr)+d\bigl(G(y,z,x),G(v,w,u)\bigr)+d\bigl(G(z,x,y),G(w,u,v)\bigr)\Bigr)\end{aligned}\tag{5}$$

for all $x,y,z,u,v,w\in X$ with

$$\bigl(G(x,y,z),G(y,z,x),G(z,x,y),G(u,v,w),G(v,w,u),G(w,u,v)\bigr)\in M.$$

Also suppose that either

(a) *F is continuous or*

(b) *for any three sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ with*

$$\begin{aligned}&\bigl(G(x_n,y_n,z_n),G(y_n,z_n,x_n),G(z_n,x_n,y_n),\\&\quad G(x_{n+1},y_{n+1},z_{n+1}),G(y_{n+1},z_{n+1},x_{n+1}),G(z_{n+1},x_{n+1},y_{n+1})\bigr)\in M\end{aligned}$$

and

$$\bigl\{G(x_n,y_n,z_n)\bigr\}\to x,\qquad\bigl\{G(y_n,z_n,x_n)\bigr\}\to y,\qquad\bigl\{G(z_n,x_n,y_n)\bigr\}\to z$$

for all $n\ge1$ implies

$$\bigl(G(x_n,y_n,z_n),G(y_n,z_n,x_n),G(z_n,x_n,y_n),x,y,z\bigr)\in M\quad\textit{for all } n\ge1.$$

If there exist $x_0,y_0,z_0\in X\times X$ such that

$$\bigl(G(x_0,y_0,z_0),G(y_0,z_0,x_0),G(z_0,x_0,y_0),F(x_0,y_0,z_0),F(y_0,z_0,x_0),F(z_0,x_0,y_0)\bigr)\in M$$

and M is an (G,F)-closed, then there exist $(x,y,z) \in X \times X \times X$ such that $G(x,y,z) = F(x,y,z)$, $G(y,z,x) = F(y,z,x)$, and $G(z,x,y) = F(z,x,y)$, that is, F and G have a tripled point of coincidence.

Proof Let $x_0, y_0, z_0 \in X$ be such that

$$\bigl(G(x_0,y_0,z_0), G(y_0,z_0,x_0), G(z_0,x_0,y_0), F(x_0,y_0,z_0), F(y_0,z_0,x_0), F(z_0,x_0,y_0)\bigr) \in M.$$

From the assumption, there exist $(x_1,y_1,z_1) \in X \times X \times X$ such that

$$F(x_0,y_0,z_0) = G(x_1,y_1,z_1), \qquad F(y_0,z_0,x_0) = G(y_1,z_1,x_1) \quad \text{and}$$
$$F(z_0,x_0,y_0) = G(z_1,x_1,y_1).$$

Again from assumption, we can choose $x_2, y_2, z_2 \in X$ such that

$$F(x_1,y_1,z_1) = G(x_2,y_2,z_2), \qquad F(y_1,z_1,x_1) = G(y_2,z_2,x_2) \quad \text{and}$$
$$F(z_1,x_1,y_1) = G(z_2,x_2,y_2).$$

By repeating this argument, we can construct three sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$ in X such that

$$\begin{aligned} &F(x_n,y_n,z_n) = G(x_{n+1},y_{n+1},z_{n+1}), \qquad F(y_n,z_n,x_n) = G(y_{n+1},z_{n+1},x_{n+1}) \quad \text{and} \\ &F(z_n,x_n,y_n) = G(z_{n+1},x_{n+1},y_{n+1}) \quad \text{for all } n \geq 1. \end{aligned} \tag{6}$$

Since

$$\bigl(G(x_0,y_0,z_0), G(y_0,z_0,x_0), G(z_0,x_0,y_0), F(x_0,y_0,z_0), F(y_0,z_0,x_0), F(z_0,x_0,y_0)\bigr) \in M$$

and M is (G,F)-closed, we get

$$\begin{aligned} &\bigl(G(x_0,y_0,z_0), G(y_0,z_0,x_0), G(z_0,x_0,y_0), F(x_0,y_0,z_0), F(y_0,z_0,x_0), F(z_0,x_0,y_0)\bigr) \\ &\quad = \bigl(G(x_0,y_0,z_0), G(y_0,z_0,x_0), G(z_0,x_0,y_0), G(x_1,y_1,z_1), G(y_1,z_1,x_1), G(z_1,x_1,y_1)\bigr) \in M \\ &\quad \Rightarrow \quad \bigl(F(x_0,y_0,z_0), F(y_0,z_0,x_0), F(z_0,x_0,y_0), F(x_1,y_1,z_1), F(y_1,z_1,x_1), F(z_1,x_1,y_1)\bigr) \\ &\qquad\quad = \bigl(G(x_1,y_1,z_1), G(y_1,z_1,x_1), G(z_1,x_1,y_1), \\ &\qquad\qquad G(x_2,y_2,z_2), G(y_2,z_2,x_2), G(z_2,x_2,y_2)\bigr) \in M. \end{aligned}$$

Again, using the fact that M is (G,F)-closed, we have

$$\begin{aligned} &\bigl(G(x_1,y_1,z_1), G(y_1,z_1,x_1), G(z_1,x_1,y_1), G(x_2,y_2,z_2), G(y_2,z_2,x_2), G(z_2,x_2,y_2)\bigr) \in M \\ &\quad \Rightarrow \quad \bigl(F(x_1,y_1,z_1), F(y_1,z_1,x_1), F(z_1,x_1,y_1), F(x_2,y_2,z_2), F(y_2,z_2,x_2), F(z_2,x_2,y_2)\bigr) \\ &\qquad\quad = \bigl(G(x_2,y_2,z_2), G(y_2,z_2,x_2), G(z_2,x_2,y_2), \\ &\qquad\qquad G(x_3,y_3,z_3), G(y_3,z_3,x_3), G(z_3,x_3,y_3)\bigr) \in M. \end{aligned}$$

Continuing this process, for all $n \geq 0$, we get

$$\begin{aligned} &\big(G(x_n, y_n, z_n), G(y_n, z_n, x_n), G(z_n, x_n, y_n), \\ &\quad G(x_{n+1}, y_{n+1}, z_{n+1}), G(y_{n+1}, z_{n+1}, x_{n+1}), G(z_{n+1}, x_{n+1}, y_{n+1})\big) \in M. \end{aligned} \tag{7}$$

For all $n \geq 0$, denote

$$\begin{aligned} \delta_n &= d\big(G(x_n, y_n, z_n), G(x_{n+1}, y_{n+1}, z_{n+1})\big) + d\big(G(y_n, z_n, x_n), G(y_{n+1}, z_{n+1}, x_{n+1})\big) \\ &\quad + d\big(G(z_n, x_n, y_n), G(z_{n+1}, x_{n+1}, y_{n+1})\big). \end{aligned} \tag{8}$$

We can suppose that $\delta_n > 0$ for all $n \geq 0$. If not, (x_n, y_n, z_n) will be a tripled point of coincidence and the proof is finished. From (5), (6), and (7), we have

$$\begin{aligned} &d\big(G(x_{n+1}, y_{n+1}, z_{n+1}), G(x_{n+2}, y_{n+2}, z_{n+2})\big) \\ &\qquad + d\big(G(y_{n+1}, z_{n+1}, x_{n+1}), G(y_{n+2}, z_{n+2}, x_{n+2})\big) \\ &\qquad + d\big(G(z_{n+1}, x_{n+1}, y_{n+1}), G(z_{n+2}, x_{n+2}, y_{n+2})\big) \\ &\quad = d\big(F(x_n, y_n, z_n), F(x_{n+1}, y_{n+1}, z_{n+1})\big) \\ &\qquad + d\big(F(y_n, z_n, x_n), F(y_{n+1}, z_{n+1}, x_{n+1})\big) \\ &\qquad + d\big(F(z_n, x_n, y_n), F(z_{n+1}, x_{n+1}, y_{n+1})\big) \\ &\quad \leq \varphi\big(d\big(G(x_n, y_n, z_n), G(x_{n+1}, y_{n+1}, z_{n+1})\big) \\ &\qquad + d\big(G(y_n, z_n, x_n), G(y_{n+1}, z_{n+1}, x_{n+1})\big) \\ &\qquad + d\big(G(z_n, x_n, y_n), G(z_{n+1}, x_{n+1}, y_{n+1})\big)\big) \\ &\quad = \varphi(\delta_n). \end{aligned} \tag{9}$$

Therefore, the sequence $\{\delta_n\}_{n=1}^{\infty}$ satisfies

$$\delta_{n+1} \leq \varphi(\delta_n) \quad \text{for all } n \geq 0. \tag{10}$$

Using property of φ it follows that the sequence $\{\delta_n\}_{n=1}^{\infty}$ is decreasing. Therefore, there exists some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \delta. \tag{11}$$

We shall prove that $\delta = 0$. Assume, to the contrary, that $\delta > 0$. Then by letting $n \to \infty$ in (10) and using the property of φ, we have

$$\delta = \lim_{n\to\infty} \delta_{n+1} \leq \lim_{n\to\infty} \varphi(\delta_n) = \lim_{\delta_n \to \delta^+} \varphi(\delta_n) < \delta,$$

a contradiction. Thus $\delta = 0$ and hence

$$\lim_{n\to\infty} \delta_n = 0. \tag{12}$$

We now prove that $\{G(x_n,y_n,z_n)\}_{n=1}^{\infty}$, $\{G(y_n,z_n,x_n)\}_{n=1}^{\infty}$, and $\{G(z_n,x_n,y_n)\}_{n=1}^{\infty}$ are Cauchy sequences in (X,d). Suppose, to the contrary, that at least one of the sequences $\{G(x_n,y_n,z_n)\}_{n=1}^{\infty}$ or $\{G(y_n,z_n,x_n)\}_{n=1}^{\infty}$ or $\{G(z_n,x_n,y_n)\}_{n=1}^{\infty}$ is not a Cauchy sequence. Then exists an $\epsilon > 0$ for which we can find subsequences $\{G(x_{m(k)},y_{m(k)},z_{m(k)})\}$, $\{G(x_{n(k)},y_{n(k)},z_{n(k)})\}$ of $\{G(x_n,y_n,z_n)\}_{n=1}^{\infty}$, $\{G(y_{m(k)},z_{m(k)},x_{m(k)})\}$, $\{G(y_{n(k)},z_{n(k)},x_{n(k)})\}$ of $\{G(y_n,z_n,x_n)\}_{n=1}^{\infty}$, and $\{G(z_{m(k)},x_{m(k)},y_{m(k)})\}$, $\{G(z_{n(k)},x_{n(k)},y_{n(k)})\}$ of $\{G(z_n,x_n,y_n)\}_{n=1}^{\infty}$, respectively, with $n(k) > m(k) \geq k$ such that

$$\begin{aligned} \epsilon &< D_k \\ &= d\big(G(x_{m(k)},y_{m(k)},z_{m(k)}),G(x_{n(k)},y_{n(k)},z_{n(k)})\big) \\ &\quad + d\big(G(y_{m(k)},z_{m(k)},x_{m(k)}),G(y_{n(k)},z_{n(k)},x_{n(k)})\big) \\ &\quad + d\big(G(z_{m(k)},x_{m(k)},y_{m(k)}),G(z_{n(k)},x_{n(k)},y_{n(k)})\big). \end{aligned} \tag{13}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that is the smallest integer with $n(k) > m(k) \geq k$ and satisfying (13). Then

$$\begin{aligned} & d\big(G(x_{m(k)},y_{m(k)},z_{m(k)}),G(x_{n(k)-1},y_{n(k)-1},z_{n(k)-1})\big) \\ &\qquad + d\big(G(y_{m(k)},z_{m(k)},x_{m(k)}),G(y_{n(k)-1},z_{n(k)-1},x_{n(k)-1})\big) \\ &\qquad + d\big(G(z_{m(k)},x_{m(k)},y_{m(k)}),G(z_{n(k)-1},x_{n(k)-1},y_{n(k)-1})\big) \\ &\quad \leq \epsilon. \end{aligned} \tag{14}$$

Using (13), (14), and the triangle inequality, we have

$$\begin{aligned} \epsilon &< D_k \\ &\leq d\big(G(x_{m(k)},y_{m(k)},z_{m(k)}),G(x_{n(k)-1},y_{n(k)-1},z_{n(k)-1})\big) \\ &\quad + d\big(G(x_{n(k)-1},y_{n(k)-1},z_{n(k)-1}),G(x_{n(k)},y_{n(k)},z_{n(k)})\big) \\ &\quad + d\big(G(y_{m(k)},z_{m(k)},x_{m(k)}),G(y_{n(k)-1},z_{n(k)-1},x_{n(k)-1})\big) \\ &\quad + d\big(G(y_{n(k)-1},z_{n(k)-1},x_{n(k)-1}),G(y_{n(k)},z_{n(k)},x_{n(k)})\big) \\ &\quad + d\big(G(z_{m(k)},x_{m(k)},y_{m(k)}),G(z_{n(k)-1},x_{n(k)-1},y_{n(k)-1})\big) \\ &\quad + d\big(G(z_{n(k)-1},x_{n(k)-1},y_{n(k)-1}),G(z_{n(k)},x_{n(k)},y_{n(k)})\big) \\ &\leq \epsilon + \delta_{n(k)-1}. \end{aligned} \tag{15}$$

Letting $k \to \infty$ in (15) and using (12), we get

$$\lim_{n\to\infty} D_k = \epsilon. \tag{16}$$

Again, for all $k \geq 0$, we have

$$\begin{aligned} D_k &= d\big(G(x_{m(k)},y_{m(k)},z_{m(k)}),G(x_{n(k)},y_{n(k)},z_{n(k)})\big) \\ &\quad + d\big(G(y_{m(k)},z_{m(k)},x_{m(k)}),G(y_{n(k)},z_{n(k)},x_{n(k)})\big) \end{aligned}$$

$$
\begin{aligned}
&+ d\bigl(G(z_{m(k)}, x_{m(k)}, y_{m(k)}), G(z_{n(k)}, x_{n(k)}, y_{n(k)})\bigr) \\
&\le d\bigl(G(x_{m(k)}, y_{m(k)}, z_{m(k)}), G(x_{m(k)+1}, y_{m(k)+1}, z_{m(k)+1})\bigr) \\
&\quad + d\bigl(G(x_{m(k)+1}, y_{m(k)+1}, z_{m(k)+1}), G(x_{n(k)+1}, y_{n(k)+1}, z_{n(k)+1})\bigr) \\
&\quad + d\bigl(G(x_{n(k)+1}, y_{n(k)+1}, z_{n(k)+1}), G(x_{n(k)}, y_{n(k)}, z_{n(k)})\bigr) \\
&\quad + d\bigl(G(y_{m(k)}, z_{m(k)}, x_{m(k)}), G(y_{m(k)+1}, z_{m(k)+1}, x_{m(k)+1})\bigr) \\
&\quad + d\bigl(G(y_{m(k)+1}, z_{m(k)+1}, x_{m(k)+1}), G(y_{n(k)+1}, z_{n(k)+1}, x_{n(k)+1})\bigr) \\
&\quad + d\bigl(G(y_{n(k)+1}, z_{n(k)+1}, x_{n(k)+1}), G(y_{n(k)}, z_{n(k)}, x_{n(k)})\bigr) \\
&\quad + d\bigl(G(z_{m(k)}, x_{m(k)}, y_{m(k)}), G(z_{m(k)+1}, x_{m(k)+1}, y_{m(k)+1})\bigr) \\
&\quad + d\bigl(G(z_{m(k)+1}, x_{m(k)+1}, y_{m(k)+1}), G(z_{n(k)+1}, x_{n(k)+1}, y_{n(k)+1})\bigr) \\
&\quad + d\bigl(G(z_{n(k)+1}, x_{n(k)+1}, y_{n(k)+1}), G(z_{n(k)}, x_{n(k)}, y_{n(k)})\bigr) \\
&\le \delta_{m(k)} + \delta_{n(k)} \\
&\quad + d\bigl(G(x_{m(k)+1}, y_{m(k)+1}, z_{m(k)+1}), G(x_{n(k)+1}, y_{n(k)+1}, z_{n(k)+1})\bigr) \\
&\quad + d\bigl(G(y_{m(k)+1}, z_{m(k)+1}, x_{m(k)+1}), G(y_{n(k)+1}, z_{n(k)+1}, x_{n(k)+1})\bigr) \\
&\quad + d\bigl(G(z_{m(k)+1}, x_{m(k)+1}, y_{m(k)+1}), G(z_{n(k)+1}, x_{n(k)+1}, y_{n(k)+1})\bigr). \qquad (17)
\end{aligned}
$$

From (7) and $n(k) > m(k)$ we have

$$
\begin{aligned}
&\bigl(G(x_{m(k)}, y_{m(k)}, z_{m(k)}), G(y_{m(k)}, x_{m(k)}, z_{m(k)}), \\
&\quad G(z_{m(k)}, x_{m(k)}, y_{m(k)}), G(x_{m(k)+1}, y_{m(k)+1}, z_{m(k)+1}), \\
&\quad G(y_{m(k)+1}, x_{m(k)+1}, z_{m(k)+1}), G(z_{m(k)+1}, x_{m(k)+1}, y_{m(k)+1})\bigr) \in M
\end{aligned}
$$

and

$$
\begin{aligned}
&\bigl(G(x_{m(k)+1}, y_{m(k)+1}, z_{m(k)+1}), G(y_{m(k)+1}, x_{m(k)+1}, z_{m(k)+1}), \\
&\quad G(z_{m(k)+1}, x_{m(k)+1}, y_{m(k)+1}), G(x_{m(k)+2}, y_{m(k)+2}, z_{m(k)+2}), \\
&\quad G(y_{m(k)+2}, x_{m(k)+2}, z_{m(k)+2}), G(z_{m(k)+2}, x_{m(k)+2}, y_{m(k)+2})\bigr) \in M.
\end{aligned}
$$

Using the transitive property of M, we get

$$
\begin{aligned}
&\bigl(G(x_{m(k)}, y_{m(k)}, z_{m(k)}), G(y_{m(k)}, x_{m(k)}, z_{m(k)}), \\
&\quad G(z_{m(k)}, x_{m(k)}, y_{m(k)}), G(x_{m(k)+2}, y_{m(k)+2}, z_{m(k)+2}), \\
&\quad G(y_{m(k)+2}, x_{m(k)+2}, z_{m(k)+2}), G(z_{m(k)+2}, x_{m(k)+2}, y_{m(k)+2})\bigr) \in M.
\end{aligned}
$$

Continuing this process, we have

$$
\begin{aligned}
&\bigl(G(x_{m(k)}, y_{m(k)}, z_{m(k)}), G(y_{m(k)}, x_{m(k)}, z_{m(k)}), G(z_{m(k)}, x_{m(k)}, y_{m(k)}), \\
&\quad G(x_{n(k)}, y_{n(k)}, z_{n(k)}), G(y_{n(k)}, x_{n(k)}, z_{n(k)}), G(z_{n(k)}, x_{n(k)}, y_{n(k)})\bigr) \in M. \qquad (18)
\end{aligned}
$$

From (5), (6), and (18), we have

$$\begin{aligned}
&d\big(G(x_{m(k)+1},y_{m(k)+1},z_{m(k)+1}),G(x_{n(k)+1},y_{n(k)+1},z_{n(k)+1})\big)\\
&\quad+d\big(G(y_{m(k)+1},z_{m(k)+1},x_{m(k)+1}),G(y_{n(k)+1},z_{n(k)+1},x_{n(k)+1})\big)\\
&\quad+d\big(G(z_{m(k)+1},x_{m(k)+1},y_{m(k)+1}),G(z_{n(k)+1},x_{n(k)+1},y_{n(k)+1})\big)\\
&=d\big(F(x_{m(k)},y_{m(k)},z_{m(k)}),F(x_{n(k)},y_{n(k)},z_{n(k)})\big)\\
&\quad+d\big(F(y_{m(k)},z_{m(k)},x_{m(k)}),F(y_{n(k)},z_{n(k)},x_{n(k)})\big)\\
&\quad+d\big(F(z_{m(k)},x_{m(k)},y_{m(k)}),F(z_{n(k)},x_{n(k)},y_{n(k)})\big)\\
&\le\varphi\big(d\big(G(x_{m(k)},y_{m(k)},z_{m(k)}),G(x_{n(k)},y_{n(k)},z_{n(k)})\big)\\
&\quad+d\big(G(y_{m(k)},z_{m(k)},x_{m(k)}),G(y_{n(k)},z_{n(k)},x_{n(k)})\big)\\
&\quad+d\big(G(z_{m(k)},x_{m(k)},y_{m(k)}),G(z_{n(k)},x_{n(k)},y_{n(k)})\big)\big)\\
&=\varphi(D_k),
\end{aligned}\tag{19}$$

which, by (17), yields

$$D_k\le\delta_{m(k)}+\delta_{n(k)}+\varphi(D_k).\tag{20}$$

Letting $k\to\infty$ in the above inequality and using (12) and (16) we get

$$\epsilon=\lim_{k\to\infty}D_k\le\lim_{k\to\infty}\big(\delta_{m(k)}+\delta_{n(k)}+\varphi(D_k)\big)=\lim_{D_k\to\epsilon^+}\varphi(D_k)<\epsilon,$$

a contradiction. Hence $\{G(x_n,y_n,z_n)\}_{n=1}^{\infty}$, $\{G(y_n,z_n,x_n)\}_{n=1}^{\infty}$, and $\{G(z_n,x_n,y_n)\}_{n=1}^{\infty}$ are Cauchy sequences in (X,d). Since (X,d) is complete and (6), there exist $x,y,z\in X$ such that

$$\begin{aligned}
&\lim_{n\to\infty}G(x_n,y_n,z_n)=\lim_{n\to\infty}F(x_n,y_n,z_n)=x,\\
&\lim_{n\to\infty}G(y_n,z_n,x_n)=\lim_{n\to\infty}F(y_n,z_n,x_n)=y\quad\text{and}\\
&\lim_{n\to\infty}G(z_n,x_n,y_n)=\lim_{n\to\infty}F(z_n,x_n,y_n)=z.
\end{aligned}\tag{21}$$

Since the pair $\{F,G\}$ satisfies the generalized compatibility, from (21), we have

$$\begin{aligned}
&\lim_{n\to\infty}d\big(F\big(G(x_n,y_n,z_n),G(y_n,z_n,x_n),G(z_n,x_n,y_n)\big),\\
&\quad G\big(F(x_n,y_n,z_n),F(y_n,z_n,x_n),F(z_n,x_n,y_n)\big)\big)=0,\\
&\lim_{n\to\infty}d\big(F\big(G(y_n,z_n,x_n),G(z_n,x_n,y_n),G(x_n,y_n,z_n)\big),\\
&\quad G\big(F(y_n,z_n,x_n),F(z_n,x_n,y_n),F(x_n,y_n,z_n)\big)\big)=0,\\
&\lim_{n\to\infty}d\big(F\big(G(z_n,x_n,y_n),G(x_n,y_n,z_n),G(y_n,z_n,x_n)\big),\\
&\quad G\big(F(z_n,x_n,y_n),F(x_n,y_n,z_n),F(y_n,z_n,x_n)\big)\big)=0.
\end{aligned}\tag{22}$$

Suppose that assumption (a) holds. For all $n \geq 0$, by the triangle inequality we have

$$\begin{aligned} & d\big(G(x,y,z), F\big(G(x_n,y_n,z_n), G(y_n,z_n,x_n), G(z_n,x_n,y_n)\big)\big) \\ & \quad \leq d\big(G(x,y,z), G\big(F(x_n,y_n,z_n), F(y_n,z_n,x_n), F(z_n,x_n,y_n)\big)\big) \\ & \qquad + d\big(G\big(F(x_n,y_n,z_n), F(y_n,z_n,x_n), F(z_n,x_n,y_n)\big), \\ & \qquad F\big(G(x_n,y_n,z_n), G(y_n,z_n,x_n), G(z_n,x_n,y_n)\big)\big), \end{aligned} \tag{23}$$

$$\begin{aligned} & d\big(G(y,z,x), F\big(G(y_n,z_n,x_n), G(z_n,x_n,y_n), G(x_n,y_n,z_n)\big)\big) \\ & \quad \leq d\big(G(y,z,x), G\big(F(y_n,z_n,x_n), F(z_n,x_n,y_n), F(x_n,y_n,z_n)\big)\big) \\ & \qquad + d\big(G\big(F(y_n,z_n,x_n), F(z_n,x_n,y_n), F(x_n,y_n,z_n)\big), \\ & \qquad F\big(G(y_n,z_n,x_n), G(z_n,x_n,y_n), G(x_n,y_n,z_n)\big)\big) \end{aligned} \tag{24}$$

and

$$\begin{aligned} & d\big(G(z,x,y), F\big(G(z_n,x_n,y_n), G(x_n,y_n,z_n), G(y_n,z_n,x_n)\big)\big) \\ & \quad \leq d\big(G(z,x,y), G\big(F(z_n,x_n,y_n), F(x_n,y_n,z_n), F(y_n,z_n,x_n)\big)\big) \\ & \qquad + d\big(G\big(F(z_n,x_n,y_n), F(x_n,y_n,z_n), F(y_n,z_n,x_n)\big), \\ & \qquad F\big(G(z_n,x_n,y_n), G(x_n,y_n,z_n), G(y_n,z_n,x_n)\big)\big). \end{aligned} \tag{25}$$

Taking the limit as $n \to \infty$ in (23), (24), and (25). Using (21), (22), and the fact that F and G are continuous, we have

$$G(x,y,z) = F(x,y,z), \qquad G(y,z,x) = F(y,z,x) \quad \text{and} \quad G(z,x,y) = F(z,x,y). \tag{26}$$

Therefore (x,y,z) is a tripled point of coincidence of F and G.

Suppose now assumption (b) holds. Since $\{G(x_n,y_n,z_n)\}_{n=1}^{\infty}$ converges to x, $\{G(y_n,z_n,x_n)\}_{n=1}^{\infty}$ converges to y and $\{G(z_n,x_n,y_n)\}_{n=1}^{\infty}$ converges to z. From (7) and assumption (b), for all $n \geq 1$, we have

$$\big(G(x_n,y_n,z_n), G(y_n,z_n,x_n), G(z_n,x_n,y_n), x, y, z\big) \in M. \tag{27}$$

Since the pair $\{F,G\}$ satisfies the generalized compatibility, G is continuous and by (21), we have

$$\begin{aligned} G(x,y,z) &= \lim_{n\to\infty} G\big(G(x_n,y_n,z_n), G(y_n,z_n,x_n), G(z_n,x_n,y_n)\big) \\ &= \lim_{n\to\infty} G\big(F(x_n,y_n,z_n), F(y_n,z_n,x_n), F(z_n,x_n,y_n)\big) \\ &= \lim_{n\to\infty} F\big(G(x_n,y_n,z_n), G(y_n,z_n,x_n), G(z_n,x_n,y_n)\big), \end{aligned} \tag{28}$$

$$\begin{aligned} G(y,z,x) &= \lim_{n\to\infty} G\big(G(y_n,z_n,x_n), G(z_n,x_n,y_n), G(x_n,y_n,z_n)\big) \\ &= \lim_{n\to\infty} G\big(F(y_n,z_n,x_n), F(z_n,x_n,y_n), F(x_n,y_n,z_n)\big) \\ &= \lim_{n\to\infty} F\big(G(y_n,z_n,x_n), G(z_n,x_n,y_n), G(x_n,y_n,z_n)\big) \end{aligned} \tag{29}$$

and

$$\begin{aligned} G(z,x,y) &= \lim_{n\to\infty} G\big(G(z_n,x_n,y_n),G(x_n,y_n,z_n),G(y_n,z_n,x_n)\big) \\ &= \lim_{n\to\infty} G\big(F(z_n,x_n,y_n),F(x_n,y_n,z_n),F(y_n,z_n,x_n)\big) \\ &= \lim_{n\to\infty} F\big(G(z_n,x_n,y_n),G(x_n,y_n,z_n),G(y_n,z_n,x_n)\big). \end{aligned} \tag{30}$$

Then, by (5), (6), (27), (28), (29), (30), and the triangle inequality, we have

$$\begin{aligned} &d\big(G(x,y,z),F(x,y,z)\big)+d\big(G(y,z,x),F(y,z,x)\big)+d\big(G(z,x,y),F(z,x,y)\big) \\ &\quad\le d\big(G(x,y,z),F\big(G(x_n,y_n,z_n),G(y_n,z_n,x_n),G(z_n,x_n,y_n)\big)\big) \\ &\qquad + d\big(F\big(G(x_n,y_n,z_n),G(y_n,z_n,x_n),G(z_n,x_n,y_n)\big),F(x,y,z)\big) \\ &\qquad + d\big(G(y,z,x),F\big(G(y_n,z_n,x_n),G(z_n,x_n,y_n),G(x_n,y_n,z_n)\big)\big) \\ &\qquad + d\big(F\big(G(y_n,z_n,x_n),G(z_n,x_n,y_n),G(x_n,y_n,z_n)\big),F(y,z,x)\big) \\ &\qquad + d\big(G(z,x,y),F\big(G(z_n,x_n,y_n),G(x_n,y_n,z_n),G(y_n,z_n,x_n)\big)\big) \\ &\qquad + d\big(F\big(G(z_n,x_n,y_n),G(x_n,y_n,z_n),G(y_n,z_n,x_n)\big),F(z,x,y)\big) \\ &\quad\le \varphi\big(d\big(G\big(G(x_n,y_n,z_n),G(y_n,z_n,x_n),G(z_n,x_n,y_n)\big),G(x,y,z)\big) \\ &\qquad + d\big(G\big(G(y_n,z_n,x_n),G(z_n,x_n,y_n),G(x_n,y_n,z_n)\big),G(y,z,x)\big) \\ &\qquad + d\big(G\big(G(z_n,x_n,y_n),G(x_n,y_n,z_n),G(y_n,z_n,x_n)\big),G(z,x,y)\big)\big) \\ &\qquad + d\big(G(x,y,z),F\big(G(x_n,y_n,z_n),G(y_n,z_n,x_n),G(z_n,x_n,y_n)\big)\big) \\ &\qquad + d\big(G(y,z,x),F\big(G(y_n,z_n,x_n),G(z_n,x_n,y_n),G(x_n,y_n,z_n)\big)\big) \\ &\qquad + d\big(G(z,x,y),F\big(G(z_n,x_n,y_n),G(x_n,y_n,z_n),G(y_n,z_n,x_n)\big)\big). \end{aligned}$$

Letting now $n \to \infty$ in the above inequality and using the property of φ that $\lim_{r\to 0^+}\varphi(r) = 0$, we have

$$d\big(G(x,y,z),F(x,y,z)\big)+d\big(G(y,z,x),F(y,z,x)\big)+d\big(G(z,x,y),F(z,x,y)\big)=0,$$

which implies that

$$G(x,y,z)=F(x,y,z), \qquad G(y,z,x)=F(y,z,x) \quad \text{and} \quad G(z,x,y)=F(z,x,y). \qquad \square$$

Next, we give an example to validate Theorem 3.1.

Example 3.2 Let $X = [0,1]$, $d(x,y) = |x-y|$ and $F,G: X\times X\times X \to X$ be defined by

$$F(x,y,z) = \frac{x^2+y^2+z^2}{8} \quad \text{and} \quad G(x,y,z) = x+y+z.$$

Clearly, G does not satisfy the mixed monotone property.

Now we prove that for any $x, y, z \in X$, there exist $u, v, w \in X$ such that $F(x,y,z) = G(u,v,w)$, $F(y,z,x) = G(v,w,u)$, and $F(z,x,y) = G(w,u,v)$. It is easy to see that there exist $u = x^2, v = y^2, w = z^2 \in X$ such that

$$F(x,y,z) = G(x^2,y^2,z^2) = G(u,v,w), \qquad F(y,z,x) = G(y^2,z^2,x^2) = G(v,w,u) \quad \text{and}$$
$$F(z,x,y) = G(z^2,x^2,y^2) = G(w,u,v).$$

Now, we prove that the pair $\{F, G\}$ satisfies the generalized compatibility hypothesis. Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be three sequences in X such that

$$\begin{cases} \lim_{n\to\infty} F(x_n,y_n,z_n) = \lim_{n\to\infty} G(x_n,y_n,z_n) = t_1, \\ \lim_{n\to\infty} F(y_n,z_n,x_n) = \lim_{n\to\infty} G(y_n,z_n,x_n) = t_2, \\ \lim_{n\to\infty} F(z_n,x_n,y_n) = \lim_{n\to\infty} G(z_n,x_n,y_n) = t_3. \end{cases}$$

Then we must have $t_1 = t_2 = t_3 = 0$ and it is easy to prove that

$$\lim_{n\to\infty} d\bigl(F\bigl(G(x_n,y_n,z_n), G(y_n,z_n,x_n), G(z_n,x_n,y_n)\bigr),$$
$$G\bigl(F(x_n,y_n,z_n), F(y_n,z_n,x_n), F(z_n,x_n,y_n)\bigr)\bigr) = 0,$$
$$\lim_{n\to\infty} d\bigl(F\bigl(G(y_n,z_n,x_n), G(z_n,x_n,y_n), G(x_n,y_n,z_n)\bigr),$$
$$G\bigl(F(y_n,z_n,x_n), F(z_n,x_n,y_n), F(x_n,y_n,z_n)\bigr)\bigr) = 0,$$
$$\lim_{n\to\infty} d\bigl(F\bigl(G(z_n,x_n,y_n), G(x_n,y_n,z_n), G(y_n,z_n,x_n)\bigr),$$
$$G\bigl(F(z_n,x_n,y_n), F(x_n,y_n,z_n), F(y_n,z_n,x_n)\bigr)\bigr) = 0.$$

Now, for all $x, y, z, u, v, w \in X$ with

$$\bigl(G(x,y,z), G(y,z,x), G(z,x,y), G(u,v,w), G(v,w,u), G(w,u,v)\bigr) \in M = X^6,$$

we let $\varphi : [0,+\infty) \to [0,+\infty)$ be a function defined by $\varphi(t) = \frac{t}{4}$, then we have

$$\begin{aligned}
&d\bigl(F(x,y,z), F(u,v,w)\bigr) + d\bigl(F(y,z,x), F(v,w,u)\bigr) + d\bigl(F(z,x,y), F(w,u,v)\bigr) \\
&\quad = \left|\frac{x^2+y^2+z^2}{8} - \frac{u^2+v^2+w^2}{8}\right| + \left|\frac{y^2+z^2+x^2}{8} - \frac{v^2+w^2+u^2}{8}\right| \\
&\qquad + \left|\frac{z^2+x^2+y^2}{8} - \frac{+w^2+u^2+v^2}{8}\right| \\
&\quad = 3\left|\frac{(x^2-u^2)+(y^2-v^2)+(z^2-w^2)}{8}\right| \\
&\quad = 3\left|\frac{(x-u)(x+u)}{8} + \frac{(y-v)(y+v)}{8} + \frac{(z-w)(z+w)}{8}\right| \\
&\quad \le \frac{3}{4}\bigl|(x+y+z)-(u+v+w)\bigr| \\
&\quad = \varphi\bigl(3\bigl|(x+y+z)-(u+v+w)\bigr|\bigr) \\
&\quad = \varphi\bigl(\bigl|(x+y+z)-(u+v+w)\bigr| + \bigl|(y+z+x)-(v+w+u)\bigr|
\end{aligned}$$

$$\begin{aligned}
&+ \left|(z + x + y) - (w + u + v)\right|\big) \\
&= \varphi\big(d\big(G(x,y,z), G(u,v,w)\big) + d\big(G(y,z,x), G(v,w,u)\big) \\
&\quad + d\big(G(z,x,y), G(w,u,v)\big)\big).
\end{aligned}$$

Therefore condition (5) is satisfied. Thus all the requirements of Theorem 3.1 are satisfied and $(0,0,0)$ is a tripled point of coincidence of F and G.

Next, we show the uniqueness of the tripled point of coincidence of F and G.

Theorem 3.3 *In addition to the hypotheses of Theorem* 3.1, *suppose that, for every* $(x,y,z), (x^*,y^*,z^*) \in X \times X \times X$, *there exist* $(u,v,w) \in X \times X \times X$ *such that*

$$\begin{aligned}
&\big(G(x,y,z), G(y,z,x), G(z,x,y), G(u,v,w), G(v,w,u), G(w,u,v)\big) \in M \quad \textit{and} \\
&\big(G\big(x^*,y^*,z^*\big), G\big(y^*,z^*,x^*\big), G\big(z^*,x^*,y^*\big), G(u,v,w), G(v,w,u), G(w,u,v)\big) \in M.
\end{aligned}$$

Then F and G have a unique tripled point of coincidence. Moreover, if the pair $\{F,G\}$ *is commuting, then F and G have a unique tripled fixed point, that is, there exist unique* $(a,b,c) \in X^3$ *such that*

$$\begin{aligned}
&a = G(a,b,c) = F(a,b,c), \qquad b = G(b,c,a) = f(b,c,a) \quad \textit{and} \\
&c = G(c,a,b) = f(c,a,b).
\end{aligned}$$

Proof From Theorem 3.1, we know that F and G have a tripled point of coincidence. Suppose that (x,y,z), (x^*,y^*,z^*) are tripled points of coincidence of F and G, that is,

$$\begin{aligned}
&F(x,y,z) = G(x,y,z), \qquad F(y,z,x) = G(y,z,x), \qquad F(z,x,y) = G(z,x,y) \quad \text{and} \\
&F\big(x^*,y^*,z^*\big) = G\big(x^*,y^*,z^*\big), \qquad F\big(y^*,z^*,x^*\big) = G\big(y^*,z^*,x^*\big), \\
&F\big(z^*,x^*,y^*\big) = G\big(z^*,x^*,y^*\big).
\end{aligned} \tag{31}$$

Now we show that $G(x,y,z) = G(x^*,y^*,z^*)$, $G(y,z,x) = G(y^*,z^*,x^*)$, and $G(z,x,y) = G(z^*,x^*,y^*)$. By the hypothesis there exist $(u,v,w) \in X^3$ such that

$$\begin{aligned}
&\big(G(x,y,z), G(y,z,x), G(z,x,y), G(u,v,w), G(v,w,u), G(w,u,v)\big) \in M \quad \text{and} \\
&\big(G\big(x^*,y^*,z^*\big), G\big(y^*,z^*,x^*\big), G\big(z^*,x^*,y^*\big), G(u,v,w), G(v,w,u), G(w,u,v)\big) \in M.
\end{aligned}$$

We put $u_0 = u$, $v_0 = v$ and $w_0 = w$ and define three sequences $\{G(u_n,v_n,w_n)\}_{n=1}^{\infty}$, $\{G(v_n,w_n,u_n)\}_{n=1}^{\infty}$ and $\{G(w_n,u_n,v_n)\}_{n=1}^{\infty}$ as follows:

$$\begin{aligned}
&F(u_n,v_n,w_n) = G(u_{n+1},v_{n+1},w_{n+1}), \qquad F(v_n,w_n,u_n) = G(v_{n+1},w_{n+1},u_{n+1}) \quad \text{and} \\
&F(w_n,u_n,v_n) = G(w_{n+1},u_{n+1},v_{n+1}) \quad \text{for all } n \geq 0.
\end{aligned}$$

Since M is (G,F)-closed and

$$\big(G(x,y,z), G(y,z,x), G(z,x,y), G(u,v,w), G(v,w,u), G(w,u,v)\big) \in M,$$

we have

$$\begin{aligned}
&\big(G(x,y,z),G(y,z,x),G(z,x,y),G(u,v,w),G(v,w,u),G(w,u,v)\big)\\
&\quad=\big(G(x,y,z),G(y,z,x),G(z,x,y),G(u_0,v_0,w_0),G(v_0,w_0,u_0),G(w_0,u_0,v_0)\big)\in M\\
&\quad\Rightarrow\quad \big(F(x,y,z),F(y,z,x),F(z,x,y),F(u_0,v_0,w_0),F(v_0,w_0,u_0),F(w_0,u_0,v_0)\big)\\
&\qquad\qquad=\big(G(x,y,z),G(y,z,x),G(z,x,y),G(u_1,v_1,w_1),\\
&\qquad\qquad\quad G(v_1,w_1,u_1),G(w_1,u_1,v_1)\big)\in M.
\end{aligned}$$

From

$$\big(G(x,y,z),G(y,z,x),G(z,x,y),G(u_1,v_1,w_1),G(v_1,w_1,u_1),G(w_1,u_1,v_1)\big)\in M,$$

if we use again the property of (G,F)-closedness, then

$$\begin{aligned}
&\big(G(x,y,z),G(y,z,x),G(z,x,y),G(u_1,v_1,w_1),G(v_1,w_1,u_1),G(w_1,u_1,v_1)\big)\in M\\
&\quad\Rightarrow\quad \big(F(x,y,z),F(y,z,x),F(z,x,y),F(u_1,v_1,w_1),F(v_1,w_1,u_1),F(w_1,u_1,v_1)\big)\\
&\qquad\qquad=\big(G(x,y,z),G(y,z,x),G(z,x,y),G(u_2,v_2,w_2),\\
&\qquad\qquad\quad G(v_2,w_2,u_2),G(w_2,u_2,v_2)\big)\in M.
\end{aligned}$$

By repeating this process, for all $n\geq 0$, we get

$$\big(G(x,y,z),G(y,z,x),G(z,x,y),G(u_n,v_n,w_n),G(v_n,w_n,u_n),G(w_n,u_n,v_n)\big)\in M. \tag{32}$$

Using (5), (31), and (32), for all n, we have

$$\begin{aligned}
&d\big(G(x,y,z),G(u_{n+1},v_{n+1},w_{n+1})\big)+d\big(G(y,z,x),G(v_{n+1},w_{n+1},u_{n+1})\big)\\
&\qquad+d\big(G(z,x,y),G(w_{n+1},u_{n+1},v_{n+1})\big)\\
&\quad=d\big(F(x,y,z),F(u_n,v_n,w_n)\big)+d\big(F(y,z,x),F(v_n,w_n,u_n)\big)\\
&\qquad+d\big(F(z,x,y),F(w_n,u_n,v_n)\big)\\
&\quad\leq\varphi\big(d\big(G(x,y,z),G(u_n,v_n,w_n)\big)+d\big(G(y,z,x),G(v_n,w_n,u_n)\big)\\
&\qquad+d\big(G(z,x,y),G(w_n,u_n,v_n)\big)\big).
\end{aligned}\tag{33}$$

Using the property that $\varphi(t)<t$ and repeating this process, we get

$$\begin{aligned}
&d\big(G(x,y,z),G(u_{n+1},v_{n+1},w_{n+1})\big)+d\big(G(y,z,x),G(v_{n+1},w_{n+1},u_{n+1})\big)\\
&\qquad+d\big(G(z,x,y),G(w_{n+1},u_{n+1},v_{n+1})\big)\\
&\quad\leq\varphi^n\big(d\big(G(x,y,z),G(u_1,v_1,w_1)\big)+d\big(G(y,z,x),G(v_1,w_1,u_1)\big)\\
&\qquad+d\big(G(z,x,y),G(w_1,u_1,v_1)\big)\big)\quad\text{for all } n.
\end{aligned}\tag{34}$$

From $\varphi(t) < t$ and $\lim_{r\to t^+}\varphi(r) < t$, it follows that $\lim_{n\to\infty}\varphi^n(t) = 0$ for each $t > 0$. Therefore, from (34) we have

$$\lim_{n\to\infty}\big(d\big(G(x,y,z),G(u_{n+1},v_{n+1},w_{n+1})\big) + d\big(G(y,z,x),G(v_{n+1},w_{n+1,}u_{n+1})\big) + d\big(G(z,x,y),G(w_{n+1},u_{n+1},v_{n+1})\big)\big) = 0. \tag{35}$$

This implies that

$$\begin{aligned}&\lim_{n\to\infty} d\big(G(x,y,z),G(u_{n+1},v_{n+1},w_{n+1})\big) = 0\\ &\lim_{n\to\infty} d\big(G(y,z,x),G(v_{n+1},w_{n+1,}u_{n+1})\big) = 0 \quad\text{and}\\ &\lim_{n\to\infty} d\big(G(z,x,y),G(w_{n+1},u_{n+1},v_{n+1})\big) = 0.\end{aligned} \tag{36}$$

Similarly, we show that

$$\begin{aligned}&\lim_{n\to\infty} d\big(G\big(x^*,y^*,z^*\big),G(u_{n+1},v_{n+1},w_{n+1})\big) = 0,\\ &\lim_{n\to\infty} d\big(G\big(y^*,z^*,x^*\big),G(v_{n+1},w_{n+1,}u_{n+1})\big) = 0 \quad\text{and}\\ &\lim_{n\to\infty} d\big(G\big(z^*,x^*,y^*\big),G(w_{n+1},u_{n+1},v_{n+1})\big) = 0.\end{aligned} \tag{37}$$

From (36) and (37), we have

$$\begin{aligned}&G(x,y,z) = G\big(x^*,y^*,z^*\big),\qquad G(y,z,x) = G\big(y^*,z^*,x^*\big) \quad\text{and}\\ &G(z,x,y) = G\big(z^*,x^*,y^*\big).\end{aligned} \tag{38}$$

Now let the pair $\{F,G\}$ be commuting, we shall prove that F and G have a unique tripled fixed point. Since

$$F(x,y,z) = G(x,y,z),\qquad F(y,z,x) = G(y,z,x) \quad\text{and}\quad F(z,x,y) = G(z,x,y) \tag{39}$$

and F and G commutes, we have

$$\begin{aligned}G\big(G(x,y,z),G(y,z,x),G(z,x,y)\big) &= G\big(F(x,y,z),F(y,z,x),F(z,x,y)\big)\\ &= F\big(G(x,y,z),G(y,z,x),G(z,x,y)\big),\\ G\big(G(y,z,x),G(z,x,y),G(x,y,z)\big) &= G\big(F(y,z,x),F(z,x,y),F(x,y,z)\big)\\ &= F\big(G(y,z,x),G(z,x,y),G(x,y,z)\big) \quad\text{and}\\ G\big(G(z,x,y),G(x,y,z),G(y,z,x)\big) &= G\big(F(z,x,y),F(x,y,z),F(y,z,x)\big)\\ &= F\big(G(z,x,y),G(x,y,z),G(y,z,x)\big).\end{aligned} \tag{40}$$

Denote $G(x,y,z) = a$, $G(y,z,x) = b$, and $G(z,x,y) = c$. Then, by (39) and (40), one gets

$$G(a,b,c) = F(a,b,c),\qquad G(b,c,a) = F(b,c,a) \quad\text{and}\quad G(c,a,b) = F(c,a,b). \tag{41}$$

Therefore, (a,b,c) is a tripled point of coincidence of F and G. Then, by (38) with $x^* = a$, $y^* = b$, and $z^* = c$, it follows that

$$\begin{aligned} &a = G(x,y,z) = G(a,b,c), \qquad b = G(y,z,x) = G(b,c,a) \quad \text{and} \\ &c = G(z,x,y) = G(c,a,b). \end{aligned} \tag{42}$$

Thus (a,b,c) is a tripled fixed point of G, by (39) and (42), (a,b,c) is also a tripled fixed point of F. To prove the uniqueness, assume (p,q,r) form another tripled fixed point of F and G. Then (p,q,r) is a tripled point of coincidence of F and G. Using (38) and (42), we have

$$\begin{aligned} &p = G(p,q,r) = G(a,b,c) = a, \qquad q = G(q,r,p) = G(b,c,a) = b \quad \text{and} \\ &r = G(r,p,q) = G(c,a,b) = c. \end{aligned}$$ □

Next, we give some application of our results to a tripled point of coincidence theorems with F is G-increasing with respect to $\preceq$ and G has the mixed monotone property.

Corollary 3.4 *Let $(X,\preceq)$ be a partially ordered set and let d be a metric on X such that (X,d) is a complete metric space. Assume that $F,G : X \times X \times X \to X$ are two generalized compatible mappings such that F is G-increasing with respect to $\preceq$, G is continuous and has the mixed monotone property. Suppose that for any $x,y,z \in X$, there exist $u,v,w \in X$ such that $F(x,y,z) = G(u,v,w)$, $F(y,z,x) = G(v,w,u)$, and $F(z,x,y) = G(w,u,v)$. Suppose that there exists $\varphi \in \Phi$ such that the following holds*:

$$\begin{aligned} &d\big(F(x,y,z),F(u,v,w)\big) + d\big(F(y,z,x),F(v,w,u)\big) + d\big(F(z,x,y),F(w,u,v)\big) \\ &\quad \le \varphi\big(d\big(G(x,y,z),G(u,v,w)\big) + d\big(G(y,z,x),G(v,w,u)\big) + d\big(G(z,x,y),G(w,u,v)\big)\big) \end{aligned}$$

for all $x,y,z,u,v,w \in X$ with $G(x,y,z) \preceq G(u,v,w)$, $G(y,z,x) \succeq G(v,w,u)$, and $G(z,x,y) \preceq G(w,u,v)$.

Also suppose that either

(a) *F is continuous or*

(b) *X has the following properties: for any two sequences $\{x_n\}$ and $\{y_n\}$ we have*

  (i) *if the non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*

  (ii) *if the non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.*

If there exist $(x_0,y_0,z_0) \in X \times X \times X$ with

$$\begin{aligned} &G(x_0,y_0,z_0) \preceq F(x_0,y_0,z_0), \qquad G(y_0,z_0,x_0) \succeq F(y_0,z_0,x_0) \quad \text{and} \\ &G(z_0,x_0,y_0) \preceq F(z_0,x_0,y_0), \end{aligned}$$

then there exist $(x,y,z) \in X \times X \times X$ such that $G(x,y,z) = F(x,y,z)$, $G(y,z,x) = F(y,z,x)$, and $G(z,x,y) = F(z,x,y)$, that is, F and G have a tripled point of coincidence.

Proof We define the subset $M \subseteq X^6$ by

$$M = \big\{(x,y,z,u,v,w) \in X^6 : x \preceq u, y \succeq v \text{ and } z \preceq w\big\}.$$

From Example 2.24, M is an (G,F)-closed set which satisfies the transitive property. For all $x,y,z,u,v,w \in X$ with $G(x,y,z) \preceq G(u,v,w)$, $G(y,z,x) \succeq G(v,w,u)$, and $G(z,x,y) \preceq G(w,u,v)$, we have

$$\bigl(G(x,y,z),G(y,z,x),G(z,x,y),G(u,v,w),G(v,w,u),G(w,u,v)\bigr) \in M.$$

By (5), we get

$$\begin{aligned}&d\bigl(F(x,y,z),F(u,v,w)\bigr) + d\bigl(F(y,z,x),F(v,w,u)\bigr) + d\bigl(F(z,x,y),F(w,u,v)\bigr)\\&\quad \le \varphi\bigl(d\bigl(G(x,y,z),G(u,v,w)\bigr) + d\bigl(G(y,z,x),G(v,w,u)\bigr) + d\bigl(G(z,x,y),G(w,u,v)\bigr)\bigr).\end{aligned}$$

Since $x_0,y_0,z_0 \in X \times X \times X$ with

$$\begin{aligned}&G(x_0,y_0,z_0) \preceq F(x_0,y_0,z_0), \qquad G(y_0,z_0,x_0) \succeq F(y_0,z_0,x_0) \quad \text{and}\\&G(z_0,x_0,y_0) \preceq F(z_0,x_0,y_0),\end{aligned} \tag{43}$$

we have

$$\bigl(G(x_0,y_0,z_0),G(y_0,z_0,x_0),G(z_0,x_0,y_0),F(x_0,y_0,z_0),F(y_0,z_0,x_0),F(z_0,x_0,y_0)\bigr) \in M.$$

If assumption (a) holds, F is continuous. By assumption (a) of Theorem 3.1, we have $G(x,y,z) = F(x,y,z)$, $G(y,z,x) = F(y,z,x)$, and $G(z,x,y) = F(z,x,y)$.

Next, if assumption (b) holds, since F is G-increasing with respect to $\preceq$, using (43) and (6), we can show that

$$\begin{aligned}&G(x_n,y_n,z_n) \preceq G(x_{n+1},y_{n+1},z_{n+1}), \qquad G(y_n,z_n,x_n) \succeq G(y_{n+1},z_{n+1},x_{n+1}) \quad \text{and}\\&G(z_n,x_n,y_n) \preceq G(z_{n+1},x_{n+1},y_{n+1}) \quad \text{for all } n.\end{aligned}$$

Therefore

$$\begin{aligned}&\bigl(G(x_n,y_n,z_n),G(y_n,z_n,x_n),G(z_n,x_n,y_n),\\&\quad G(x_{n+1},y_{n+1},z_{n+1}),G(y_{n+1},z_{n+1},x_{n+1}),G(z_{n+1},x_{n+1},y_{n+1})\bigr) \in M.\end{aligned}$$

For any three sequences $\{G(x_n,y_n,z_n)\}_{n=1}^{\infty}$, $\{G(y_n,z_n,x_n)\}_{n=1}^{\infty}$, and $\{G(z_n,x_n,y_n)\}_{n=1}^{\infty}$ such that $\{G(x_n,y_n,z_n)\}_{n=1}^{\infty}$ is a non-decreasing sequence in X with $G(x_n,y_n,z_n) \to x$, $\{G(y_n,z_n,x_n)\}_{n=1}^{\infty}$ is a non-increasing sequence in X with $G(y_n,z_n,x_n) \to y$ and $\{G(z_n,x_n,y_n)\}_{n=1}^{\infty}$ is a non-decreasing sequence in X with $G(z_n,x_n,y_n) \to z$. Using assumption (b), we have

$$G(x_n,y_n,z_n) \preceq x, \qquad G(y_n,z_n,x_n) \succeq y \quad \text{and} \quad G(z_n,x_n,y_n) \preceq z \quad \text{for all } n.$$

Therefore, we have $(G(x_n,y_n,z_n),G(y_n,z_n,x_n),G(z_n,x_n,y_n),x,y,z) \in M$, for all $n \ge 1$, and so assumption (b) of Theorem 3.1 holds. Now, since all the hypotheses of Theorem 3.1 hold, F and G have a tripled point of coincidence. The proof is completed. □

Corollary 3.5 *In addition to the hypotheses of Corollary* 3.4, *suppose that, for every* $(x,y,z),(x^*,y^*,z^*) \in X^3$, *there exist* $(u,v,w) \in X^3$, *comparable to* (x,y,z) *and* (x^*,y^*,z^*). *Then F and G have a unique tripled point of coincidence.*

Proof We define the subset $M \subseteq X^6$ by

$$M = \{(x,y,z,u,v,w) \in X^6 : x \preceq u, y \succeq v \text{ and } z \preceq w\}.$$

From Example 2.24, M is an (G,F)-closed set which satisfies the transitive property. Thus, the proof of the existence of a tripled point of coincidence is straightforward by following the same lines as in the proof of Corollary 3.4.

Next, we show the uniqueness of a tripled point of coincidence of F and G.

Since for all $(x,y,z),(x^*,y^*,z^*) \in X^3$, there exist $(u,v,w) \in X^3$ such that

$$G(x,y,z) \preceq G(u,v,w), \qquad G(y,z,x) \succeq G(v,w,u), \qquad G(z,x,y) \preceq G(w,u,v) \quad \text{and}$$
$$G(x^*,y^*,z^*) \preceq G(u,v,w), \qquad G(y^*,z^*,x^*) \succeq G(v,w,u), \qquad G(z^*,x^*,y^*) \preceq G(w,u,v).$$

We can conclude that

$$\big(G(x,y,z),G(y,z,x),G(z,x,y),G(u,v,w),G(v,w,u),G(w,u,v)\big) \in M \quad \text{and}$$
$$\big(G(x^*,y^*,z^*),G(y^*,z^*,x^*),G(z^*,x^*,y^*),G(u,v,w),G(v,w,u),G(w,u,v)\big) \in M.$$

Therefore, since all the hypotheses of Theorem 3.3 hold, F and G have a unique tripled point of coincidence. The proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

Acknowledgements
This research was supported by Chiang Mai University and the authors would like to express sincere appreciation to Prof. Suthep Suantai for very helpful suggestions and many kind comments.

Received: 26 March 2014 Accepted: 29 May 2014 Published: 23 June 2014

References
1. Ran, ACM, Reurings, MCB: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**, 1435-1443 (2004)
2. Nieto, JJ, Rodriguez-Lopez, R: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equation. Order **22**, 223-239 (2005)
3. Nieto, JJ, Rodriguez-Lopez, R: Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equation. Acta Math. Sin. Engl. Ser. **23**(12), 2205-2212 (2007)
4. Agarwal, RP, El-Gebeily, MA, O'Regan, D: Generalized contractions in partially ordered metric spaces. Appl. Anal. **87**, 1-8 (2008)
5. Guo, D, Lakshmikantham, V: Coupled fixed points of nonlinear operators with applications. Nonlinear Anal. **11**, 623-632 (1987)
6. Bhaskar, TG, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. TMA **65**, 1379-1393 (2006)
7. Lakshmikantham, V, Ćirić, L: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. TMA **70**, 4341-4349 (2009)
8. Alsulami, SM, Alotaibi, A: Tripled coincidence points for monotone operators in partially ordered metric spaces. Int. Math. Forum **7**, 1811-1824 (2012)

9. Mursaleen, M, Mohiuddine, SA, Agawal, RP: Coupled fixed point theorems for α-ψ-contractive type mappings in partially ordered metric spaces. Fixed Point Theory Appl. **2012**, 228 (2012)
10. Mursaleen, M, Mohiuddine, SA, Agawal, RP: Corrigendum to 'Coupled fixed point theorems for α-ψ-contractive type mappings in partially ordered metric spaces'. Fixed Point Theory Appl. **2013**, 127 (2013)
11. Berzig, M, Samet, B: An extension of coupled fixed point's concept in higher dimension and applications. Comput. Math. Appl. **63**, 1319-1334 (2012)
12. Samet, B, Vetro, C: Coupled fixed point F-invariant set and fixed point of N-order. Ann. Funct. Anal. **1**, 46-56 (2010)
13. Amini-Harandi, A: Coupled and tripled fixed point theory in partially ordered metric spaces with application to initial value problem. Math. Comput. Model. **57**(9-10), 2343-2348 (2013)
14. Hussain, N, Latif, A, Shah, MH: Coupled and tripled coincidence point results without compatibility. Fixed Point Theory Appl. **2012**, 77 (2012)
15. Hussain, N, Abbas, M, Azam, A, Ahmad, J: Coupled coincidence point results for a generalized compatible pair with applications. Fixed Point Theory Appl. **2014**, 62 (2014). doi:10.1186/1687-1812-2014-62
16. Berinde, V: Coupled fixed point theorems for φ-contractive mixed monotone mappings in partially ordered metric spaces. Nonlinear Anal., Theory Methods Appl. **75**(6), 3218-3228 (2012)
17. Berinde, V: Coupled coincidence point theorems for mixed monotone nonlinear operators. Comput. Math. Appl. **64**(6), 1770-1777 (2012)
18. Sintunavarat, W, Kumam, P, Cho, YJ: Coupled fixed point theorems for nonlinear contractions without mixed monotone property. Fixed Point Theory Appl. **2012**, 170 (2012)
19. Sintunavarat, W, Radenović, S, Golubović, Z, Kuman, P: Coupled fixed point theorems for F-invariant set. Appl. Math. Inf. Sci. **7**(1), 247-255 (2013)
20. Batra, R, Vashistha, S: Coupled coincidence point theorems for nonlinear contractions under (F, g)-invariant set in cone metric spaces. J. Nonlinear Sci. Appl. **6**, 86-96 (2013)
21. Roldán, A, Martínez-Moreno, J, Roldán, C: Multidimensional fixed point theorems in partially ordered complete metric spaces. J. Math. Anal. Appl. **396**(2), 536-545 (2012)
22. Roldán, A, Martínez-Moreno, J, Roldán, C, Karapınar, E: Multidimensional fixed point theorems in partially ordered complete partial metric spaces under (ψ, φ)-contractivity conditions. Abstr. Appl. Anal. **2013**, Article ID 634371 (2013)
23. Kutbi, MA, Roldán, A, Sintunavarat, W, Martínez-Moreno, J, Roldán, C: F-Closed sets and coupled fixed point theorems without the mixed monotone property. Fixed Point Theory Appl. **2013**, 330 (2013). doi:10.1186/1687-1812-2013-330
24. Karapınar, E, Roldán, A, Martínez-Moreno, J, Roldán, C: Meir-Keeler type multidimensional fixed point theorems in partially ordered metric spaces. Abstr. Appl. Anal. **2013**, Article ID 406026 (2013)
25. Luong, NV, Thuan, NX: Coupled fixed points in partially ordered metric spaces and application. Nonlinear Anal. **74**, 983-992 (2011)
26. Mohiuddine, SA, Alotaibi, A: On coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces. Abstr. Appl. Anal. **2012**, Article ID 897198 (2012)
27. Sintunavarat, W, Kumam, P: Coupled coincidence and coupled common fixed point theorems in partially ordered metric spaces. Thai J. Math. **10**(3), 551-563 (2012)
28. Agarwal, RP, Sintunavarat, W, Kumam, P: Coupled coincidence point and common coupled fixed point theorems with lacking the mixed monotone property. Fixed Point Theory Appl. **2013**, 22 (2013). doi:10.1186/1687-1812-2013-22
29. Kaushik, P, Kumar, S, Kumam, P: Coupled coincidence point theorems for α-ψ-contractive type mappings in partially ordered metric spaces. Fixed Point Theory Appl. **2013**, 325 (2013)
30. Berinde, V, Borcut, M: Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. Nonlinear Anal. **74**, 4889-4897 (2011)
31. Berinde, V, Borcut, M: Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces. Appl. Math. Comput. **218**(10), 5929-5936 (2012)
32. Aydi, H, Abbas, M, Sintunavarat, W, Kumam, P: Tripled fixed point of W-compatible mappings in abstract metric spaces. Fixed Point Theory Appl. **2012**, 134 (2012). doi:10.1186/1687-1812-2012-134
33. Borcut, M: Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces. Appl. Math. Comput. **218**, 7339-7346 (2012)
34. Choudhury, BS, Karapınar, E, Kundu, A: Tripled coincidence point theorems for nonlinear contractions in partially ordered metric spaces. Int. J. Math. Math. Sci. **2012**, Article ID 329298 (2012)
35. Aydi, H, Karapınar, E, Postolache, M: Tripled coincidence points for weak φ-contractions in partially ordered metric spaces. Fixed Point Theory Appl. **2012**, 44 (2012)
36. Abbas, M, Aydi, H, Karapınar, E: Tripled fixed points of multi-valued nonlinear contraction mappings in partially ordered metric spaces. Abstr. Appl. Anal. **2011**, Article ID 812690 (2011)
37. Abbas, M, Ali, B, Sintunavarat, W, Kumam, P: Tripled fixed point and tripled coincidence point theorems in intuitionistic fuzzy normed spaces. Fixed Point Theory Appl. **2012**, 187 (2012). doi:10.1186/1687-1812-2012-187
38. Aydi, H, Karapınar, E, Vetro, C: Meir-Keeler type contractions for tripled fixed points. Acta Math. Sci. **32**(6), 2119-2130 (2012)
39. Aydi, H, Karapınar, E: New Meir-Keeler type tripled fixed point theorems on ordered partial metric spaces. Math. Probl. Eng. **2012**, Article ID 409872 (2012)
40. Aydi, H, Abbas, M: Tripled coincidence and fixed point results in partial metric spaces. Appl. Gen. Topol. **13**(2), 193-206 (2012)
41. Aydi, H, Karapınar, E: Tripled fixed points in ordered metric spaces. Bull. Math. Anal. Appl. **4**(1), 197-207 (2012)
42. Aydi, H, Karapınar, E, Shatanawi, W: Tripled coincidence point results for generalized contractions in ordered generalized metric spaces. Fixed Point Theory Appl. **2013**, 34 (2013)
43. Charoensawan, P: Tripled fixed points theorems for ϕ-contractive mixed monotone operators on partially ordered metric spaces. Appl. Math. Sci. **6**, 5229-5239 (2012)
44. Charoensawan, P: Tripled coincidence point theorems for a ϕ-contractive mapping in a complete metric space without the mixed g-monotone property. Fixed Point Theory Appl. **2013**, 252 (2013)
45. Mohiuddine, SA, Alotaibi, A: Some results on a tripled fixed point for nonlinear contractions in partially ordered G-metric spaces. Fixed Point Theory Appl. **2012**, 179 (2012)

46. Cho, YJ, Gupta, A, Karapınar, E, Kumam, P, Sintunavarat, W: Tripled best proximity point theorem in metric space. Math. Inequal. Appl. **16**(4), 1197-1216 (2013)
47. Kumam, P, Moreno, JM, Hierro, AFL, Hierro, CRL: Berinde-Borcut tripled fixed point theorem in partially ordered (intuitionistic) fuzzy normed spaces. Arch. Inequal. Appl. **2014**, 47 (2014)
48. Wairojjana, N, Sintunavarat, W, Kumam, P: Common tripled fixed points for W-compatible mappings along with CLR_g property in abstract metric spaces. J. Inequal. Appl. **2014**, 133 (2014)
49. Karapınar, E, Roldán, A, Shahzad, N, Sintunavarat, W: Discussion of coupled and tripled coincidence point theorems for ϕ-contractive mappings without the mixed g-monotone property. Fixed Point Theory Appl. **2014**, 92 (2014)

doi:10.1186/1029-242X-2014-245
Cite this article as: Charoensawan and Thangthong: **(G,F)-Closed set and tripled point of coincidence theorems for generalized compatibility in partially metric spaces.** *Journal of Inequalities and Applications* 2014 **2014**:245.