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Estimates for the extremal sections of complex ℓ_p-balls

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Abstract

The problem of maximal hyperplane section of $B_p(\mathbb{C}^n)$ with $p \geq 1$ is considered, which is the complex version of central hyperplane section problem of $B_p(\mathbb{R}^n)$. The relation between the complex slicing problem and the complex isotropic constant of a body is established, an upper bound estimate for the volume of complex central hyperplane sections of normalized complex $\ell_p(\mathbb{C}^n)$-balls that does not depend on n and p is shown, which extends results of Oleszkiewicz and Pełczyński, Koldobsky and Zymonopoulou, and Meyer and Pajor.

MSC: 52A21; 46B07

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1 Introduction

Let $B_p(\mathbb{R}^n)$ and $B_p(\mathbb{C}^n)$ denote the unit balls of the real and complex n-dimensional ℓ_p spaces, $\ell_p(\mathbb{R}^n)$ and $\ell_p(\mathbb{C}^n)$, respectively. We denote by $\lambda_n(K)$ the n-dimensional Lebesgue measure of a compact set K. We write $r_{n,p} = \lambda_n(B_p(\mathbb{R}^n))^{-1/n}$ and $c_{n,p} = \lambda_{2n}(B_p(\mathbb{C}^n))^{-1/2n}$ such that $\lambda_n(r_{n,p}B_p(\mathbb{R}^n)) = 1$ and $\lambda_{2n}(c_{n,p}B_p(\mathbb{C}^n)) = 1$, respectively.

The extremal volume of central hyperplane section of $B_p(\mathbb{R}^n)$ is studied by various authors (see, *e.g.*, [1–8]). Especially, the left-hand side of the following inequalities is known due to Meyer and Pajor [6] for all $p \geq 2$ and $p = 1$, and Schmuckenschläger [8] for all $1 < p < 2$. The right-hand side of the following inequalities is due to Ma and the third named author [7], and it shows an upper bound estimate for the volume of central hyperplane sections of normalized ℓ_p-balls that does not depend on n and p.

Theorem 1.1 *Let $n \in \mathbb{N}$, $n \geq 2$, $p \geq 1$, and H any central hyperplane in $\mathbb{R}^n$. Then*

$$1 \leq \lambda_{n-1}\bigl(r_{n,p}B_p\bigl(\mathbb{R}^n\bigr) \cap H\bigr) \leq \sqrt{\pi e}.$$

Moreover, the minimum occurs for $B_\infty(\mathbb{R}^n)$ if ξ has only one non-zero coordinate where ξ is the normal vector of H.

Note that Theorem 1.1 is proved by determining the extremal value of the isotropic constant of $B_p(\mathbb{R}^n)$ together with the well-known relation between the slicing problem and the isotropic constant of a body. Motivated by this idea, we define a new quantity, called the complex isotropic constant, and establish its relation to the complex slicing problem. Thus, the complex version of Theorem 1.1 (see Theorem 1.2) can be proved by estimating the extremal value of the complex isotropic constant of $B_p(\mathbb{C}^n)$.

A noteworthy fact is that the extremal volume of complex central hyperplane section of $B_p(\mathbb{C}^n)$ has not been studied until recent years. Other results concerning convex bodies in a complex vector space as ambient space can be found in [9–19]. Especially, in [17], Oleszkiewicz and Pełczyński proved that $1 \leq \lambda_{2n-2}(c_{n,\infty}B_\infty(\mathbb{C}^n) \cap H_\xi) \leq 2$, where H_ξ is the complex central hyperplane (see Section 2.1 for the definition). Furthermore, they showed that the minimal sections are the ones orthogonal to vectors with only one non-zero coordinate, and the maximal sections are orthogonal to vectors of the form $e_j + \sigma e_k$, where $j \neq k$, e_j and e_k are standard basic vectors, and $\sigma \in \mathbb{C}$, $|\sigma| = 1$. In [16], Koldobsky and Zymonopoulou studied the extremal sections of $B_p(\mathbb{C}^n)$, for $0 < p \leq 2$ and showed that the minimum corresponds to hyperplanes orthogonal to vectors $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ with $|\xi_1| = \cdots = |\xi_n|$ and the maximum corresponds to hyperplanes orthogonal to vectors with only one non-zero coordinate. Moreover, a result of Meyer and Pajor [6, Corollary 2.5] states the following. Suppose $p \geq 2$, then $\lambda_{2n-2}(c_{n-1,p}B_p(\mathbb{C}^n) \cap H_\xi) \geq 1$; Suppose $1 \leq p \leq 2$, then $\lambda_{2n-2}(c_{n-1,p}B_p(\mathbb{C}^n) \cap H_\xi) \leq 1$, where $c_{n-1,p} = \lambda_{2n-2}(B_p(\mathbb{C}^{n-1}))^{-1/(2n-2)}$. Recently, Koldobsky and König [13] considered minimal volume of slabs for the complex cube.

The case $p \leq 2$ of the following theorem follows directly from the work of Koldobsky and Zymonopoulou [16]. The following theorem extends their results to $p > 2$, and it also shows an upper bound estimate for the volume of complex central hyperplane sections of normalized complex ℓ_p-balls that does not depend on n and p.

Theorem 1.2 *Let $n \in \mathbb{N}$, $n \geq 2$, $p \geq 1$, and H_ξ any complex hyperplane in $\mathbb{R}^{2n}$. Then*

$$1 \leq \lambda_{2n-2}\big(c_{n,p}B_p\big(\mathbb{C}^n\big) \cap H_\xi\big) \leq 3e.$$

Moreover, the minimum occurs for $B_\infty(\mathbb{C}^n)$ if ξ has only one non-zero coordinate.

Our problem is different from the extremal volume of central hyperplane section of $B_p(\mathbb{R}^{2n})$ problem in two aspects. First, $B_p(\mathbb{C}^n) \neq B_p(\mathbb{R}^{2n})$ except $p = 2$; Secondly, we do only $(2n-2)$-dimensional sections, sections by subspaces coming from complex hyperplanes, rather than all $(2n-2)$-dimensional sections, and $(2n-1)$-dimensional sections in real case.

2 Notations and preliminaries

2.1 Background on complex vector space

Throughout this paper, we denote their real scalar product by $\langle x, y\rangle$ and the Euclidean norm of x by $\|x\| = \sqrt{\langle x, x\rangle}$ for $x, y \in \mathbb{R}^n$. For $x, y \in \mathbb{C}^n$, their complex scalar product by $\langle x, y\rangle_c$ and the modulus of x by $\|x\| = \sqrt{\langle x, x\rangle_c}$.

Origin-symmetric convex bodies in $\mathbb{C}^n$ are the unit balls of norms on $\mathbb{C}^n$. We denote by $\|\cdot\|_K$ the norm corresponding to the body K:

$$K = \big\{z \in \mathbb{C}^n : \|\cdot\|_K \leq 1\big\}.$$

We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the standard mapping, that is, for

$$\begin{aligned} &\xi = (\xi_1, \dots \xi_n) = (\xi_{11} + i\xi_{12}, \dots, \xi_{n1} + i\xi_{n2}) \in \mathbb{C}^n, \\ &(\xi_{11} + i\xi_{12}, \dots, \xi_{n1} + i\xi_{n2}) \overset{\tau}{\mapsto} (\xi_{11}, \xi_{12}, \dots, \xi_{n1}, \xi_{n2}). \end{aligned} \tag{2.1}$$

Since norms on $\mathbb{C}^n$ satisfy the equality

$$\|\lambda z\| = |\lambda| \|z\|, \quad \forall z \in \mathbb{C}^n, \forall \lambda \in \mathbb{C},$$

origin-symmetric complex convex bodies correspond to those origin-symmetric convex bodies K in $\mathbb{R}^{2n}$ that are invariant with respect to any coordinate-wise two-dimensional rotation, namely for each $\theta \in [0, 2\pi]$ and each $(\xi_{11}, \xi_{12}, \dots \xi_{n1}, \xi_{n2}) \in \mathbb{R}^{2n}$

$$\|\xi\|_K = \left\| R_\theta(\xi_{11}, \xi_{12}), \dots, R_\theta(\xi_{n1}, \xi_{n2}) \right\|_K, \tag{2.2}$$

where R_θ stands for the counterclockwise rotation of $\mathbb{R}^2$ by the angle θ with respect to the origin. We define the map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$:

$$\xi = (\xi_{11}, \xi_{12}, \dots, \xi_{n1}, \xi_{n1}) \to \big(R_\theta(\xi_{11}, \xi_{12}), \dots, R_\theta(\xi_{n1}, \xi_{n2})\big)$$

as $\widetilde{R_\theta}$ for each $\theta \in [0, 2\pi]$.

For $\xi \in \mathbb{C}^n$, $|\xi| = 1$, denote by

$$H_\xi = \left\{ z \in \mathbb{C}^n : \langle z, \xi \rangle_c = \sum_{k=1}^{n} z_k \overline{\xi_k} = 0 \right\}$$

the complex hyperplane through the origin, perpendicular to ξ. Under the standard mapping from $\mathbb{C}^n$ to $\mathbb{R}^{2n}$ the hyperplane H_ξ turns into a $(2n-2)$-dimensional subspace of $\mathbb{R}^{2n}$ orthogonal to the vectors

$$\xi = (\xi_{11}, \xi_{12}, \dots, \xi_{n1}, \xi_{n2}) \quad \text{and} \quad \xi^\dagger = (-\xi_{12}, \xi_{11}, \dots, -\xi_{n2}, \xi_{n1}).$$

The orthogonal two-dimensional subspace $H_\xi^\perp$ has an orthonormal basis $\xi, \xi^\dagger$.

Let $B_p(\mathbb{C}^n)$ be the $\ell_p(\mathbb{C}^n)$-balls, when viewed as a subset of $\mathbb{R}^{2n}$:

$$\begin{aligned} B_p(\mathbb{C}^n) &= \left\{ (x_{11} + ix_{12}, \dots, x_{n1} + ix_{n2}) \in \mathbb{C}^n : \sum_{j=1}^{n} \left(x_{j1}^2 + x_{j2}^2\right)^{p/2} \le 1 \right\} \\ &= \left\{ (x_{11}, x_{12}, \dots, x_{n1}, x_{n2}) \in \mathbb{R}^{2n} : \sum_{j=1}^{n} \left(x_{j1}^2 + x_{j2}^2\right)^{p/2} \le 1 \right\}, \end{aligned}$$

if $0 < p < \infty$, and

$$\begin{aligned} B_\infty(\mathbb{C}^n) &= \left\{ (x_{11} + ix_{12}, \dots, x_{n1} + ix_{n2}) \in \mathbb{C}^n : \max_{1 \le j \le n} \left(x_{j1}^2 + x_{j2}^2\right) \le 1 \right\} \\ &= \left\{ (x_{11}, x_{12}, \dots, x_{n1}, x_{n2}) \in \mathbb{R}^{2n} : \max_{1 \le j \le n} \left(x_{j1}^2 + x_{j2}^2\right) \le 1 \right\}. \end{aligned}$$

If $p \geq 1$, $B_p(\mathbb{C}^n)$ is R_θ-invariant convex body in $\mathbb{R}^{2n}$. Here a convex body is a compact convex set with non-empty interior.

2.2 Complex isotropic bodies

First, noting that a subset $K \subset \mathbb{C}^n$ is called a complex convex body means that K is a convex body in $\mathbb{R}^{2n}$ under the map (2.1). An important notion in asymptotic convex geometry is the quantity called isotropic constant (see, *e.g.*, [20–28]).

Recall that a convex body K in $\mathbb{R}^n$ is called isotropic with the isotropic constant $L_K > 0$ if $\lambda_n(K) = 1$, the origin is the center of mass of K, and

$$\int_K \langle x, y\rangle^2 \, dx = L_K^2 \|y\|^2 \tag{2.3}$$

for every $y \in \mathbb{R}^n$.

Inspired by the interplay of (real) slicing problem and isotropic constant (see, *e.g.*, [23, 27]), we will consider the complex slicing problem via the quantity called complex isotropic constant.

Definition A body $K \subset \mathbb{R}^{2n}$ is called complex isotropic with the complex isotropic constant $\mathbb{L}_K > 0$, if $\lambda_{2n}(K) = 1$, the origin is the center of mass of K, and

$$\int_K \left(\langle x, y\rangle^2 + \left\langle x^\dagger, y\right\rangle^2\right) dx = \mathbb{L}_K^2 \|y\|^2 \tag{2.4}$$

for all $y \in \mathbb{R}^{2n}$.

This definition is natural since we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the mapping τ and

$$\begin{aligned} \left|\langle x, y\rangle_c\right|^2 &= \left\langle \tau(x), \tau(y)\right\rangle^2 + \left\langle \left(\tau(x)\right)^\dagger, \tau(y)\right\rangle^2 \\ &= \left\langle \tau(x), \tau(y)\right\rangle^2 + \left\langle \left(\tau(x)\right), \tau(y)^\dagger\right\rangle^2. \end{aligned} \tag{2.5}$$

Equivalently,

$$\int_K \left(x \otimes x + x^\dagger \otimes x^\dagger\right) dx = \mathbb{L}_K^2 I_{2n}, \tag{2.6}$$

where I_{2n} denotes the identity operator on $\mathbb{R}^{2n}$, and $x \otimes x$ is the rank 1 linear operator on $\mathbb{R}^{2n}$ that takes y to $\langle x, y\rangle x$. More precisely, (2.6) means

$$\int_K (x_{i1}x_{j1} + x_{i2}x_{j2}) \, dx = 0 \quad \text{for } 1 \leq i \neq j \leq n, \tag{2.7}$$

$$\int_K (x_{i1}x_{j2} - x_{i2}x_{j1}) \, dx = 0 \quad \text{for } 1 \leq i, j \leq n \tag{2.8}$$

and

$$\int_K \left(x_{i1}^2 + x_{i2}^2\right) dx = \mathbb{L}_K^2 \quad \text{for } 1 \leq i \leq n. \tag{2.9}$$

Summing (2.9) with $i = 1, \dots, n$, we have

$$\int_K \|x\|^2\,dx = n\mathbb{L}_K^2. \tag{2.10}$$

Now, we show the relation between real isotropic bodies and complex isotropic bodies when we consider the convex body defined in $\mathbb{R}^{2n}$.

Theorem 2.1 *Let $K \subset \mathbb{R}^{2n}$ be a convex body with $\lambda_{2n}(K) = 1$, center of mass at the origin. Then*

(i) *if K is (real) isotropic, then K is complex isotropic;*
(ii) *there exists a complex isotropic convex body K such that K is not (real) isotropic;*
(iii) *if K is complex isotropic and R_θ-invariant for every $\theta \in [0, 2\pi]$, then K is (real) isotropic.*

Proof (i) From (2.5) and the definition of isotropy (2.3), we have

$$\begin{aligned}
&\int_K \left(\langle x, y\rangle^2 + \left\langle x^\dagger, y\right\rangle^2\right) d\lambda_{2n}(x) \\
&\quad = \int_K \left(\langle x, y\rangle^2 + \left\langle x, y^\dagger\right\rangle^2\right) d\lambda_{2n}(x) \\
&\quad = \int_K \langle x, y\rangle^2\,d\lambda_{2n}(x) + \int_K \left\langle x, y^\dagger\right\rangle^2 d\lambda_{2n}(x) \\
&\quad = L_K^2\|y\|^2 + L_K^2\left\|y^\dagger\right\|^2 = 2L_K^2\|y\|^2.
\end{aligned}$$

From (2.4), we also have $\mathbb{L}_K^2 = 2L_K^2$.

(ii) We take K as

$$K = A \times \underbrace{B_2\left(\mathbb{R}^2\right) \times \cdots \times B_2\left(\mathbb{R}^2\right)}_{n-1},$$

where

$$A = \operatorname{conv}\{P, Q, R\},$$

with $P = (-\frac{1}{2}, \frac{3}{2})$, $Q = (-\frac{1}{2}, -\frac{3}{2})$, $R = (1, 0)$. Then A is a right triangle and the origin is the center of mass of A. Therefore, K is convex body and the origin is the center of mass of K. Moreover, it follows that $\lambda_2(A) = 9/4$,

$$\int_A x_{11}^2\,dx_{11}\,dx_{12} = \int_{-\frac{1}{2}}^{1}\int_{x_{11}-1}^{-x_{11}+1} x_{11}^2\,dx_{12}\,dx_{11} = \frac{9}{32} \tag{2.11}$$

and

$$\int_A x_{12}^2\,dx_{11}\,dx_{12} = \int_{-\frac{1}{2}}^{1}\int_{x_{11}-1}^{-x_{11}+1} x_{12}^2\,dx_{12}\,dx_{11} = \frac{27}{32}. \tag{2.12}$$

In this example, it is easy to verify that (2.7) and (2.8) are true.

In order to verify (2.9), we divide it into two cases. For the case that $2 \le i \le n$,

$$\begin{aligned}
&\int_K \left(x_{i1}^2 + x_{i2}^2\right) d\lambda_{2n}(x) \\
&\quad = \pi^{n-2}\lambda_2(A)\int_{B_2(\mathbb{R}^2)} \left(x_{i1}^2 + x_{i2}^2\right) dx_{i1}\, dx_{i2} \\
&\quad = \pi^{n-2}\frac{9}{4}\cdot 2\pi \int_0^1 r^3\, dr = \frac{9}{8}\pi^{n-1}.
\end{aligned}$$

For the case that $i = 1$, together with (2.11), (2.12), we obtain

$$\begin{aligned}
&\int_K \left(x_{11}^2 + x_{12}^2\right) d\lambda_{2n}(x) \\
&\quad = \pi^{n-1}\int_A \left(x_{11}^2 + x_{12}^2\right) dx_{11}\, dx_{12} \\
&\quad = \pi^{n-1}\cdot\left(\frac{9}{32} + \frac{27}{32}\right) = \frac{9}{8}\pi^{n-1}.
\end{aligned}$$

Therefore, K is complex isotropic. However, K is not (real) isotropic in view of (2.11) and (2.12).

(iii) From the assumption that K is a complex isotropic R_θ-invariant convex body in $\mathbb{R}^{2n}$, we have $\widetilde{R_{\pi/2}}K = K$. Note that $\widetilde{R_{\pi/2}}x = x^\dagger$ and $|\det(\widetilde{R_{\pi/2}})| = 1$, together with (2.4), we obtain

$$\begin{aligned}
\mathbb{L}_K^2\|y\|^2 &= \int_K \left(\langle x, y\rangle^2 + \langle x^\dagger, y\rangle^2\right) d\lambda_{2n}(x) \\
&= \int_K \langle x, y\rangle^2\, d\lambda_{2n}(x) + \int_K \langle x^\dagger, y\rangle^2\, d\lambda_{2n}(x) \\
&= \int_K \langle x, y\rangle^2\, d\lambda_{2n}(x) + \int_{\widetilde{R_{\pi/2}}K} \langle \widetilde{R_{\pi/2}}x, y\rangle^2\, d\lambda_{2n}(x) \\
&= \int_K \langle x, y\rangle^2\, d\lambda_{2n}(x) + \int_K \langle x, y\rangle^2\, d\lambda_{2n}(x) \\
&= 2\int_K \langle x, y\rangle^2\, d\lambda_{2n}(x).
\end{aligned}$$

From (2.3), it follows that $\mathbb{L}_K^2 = 2L_K^2$. □

By Theorem 2.1, the class of complex isotropic bodies is larger than ones of real isotropic bodies.

3 The complex slicing problem

Observe that $(K \cap \operatorname{span}\{H_\xi, \xi\}) \cap (H_\xi + t\xi) = K \cap (H_\xi + t\xi)$, together with Brunn's theorem (see, *e.g.*, [29, p.18]), we have the following lemma.

Lemma 3.1 *Let K be an origin-symmetric convex body in $\mathbb{R}^{2n}$, and $f(t) = \lambda_{2n-2}(K \cap (H_\xi + t\xi))$ for $t \in \mathbb{R}$. Then $f(t)^{1/(2n-2)}$ is concave and $f(t)$ is decreasing for $t \ge 0$. Moreover, $f(t) \le f(0)$.*

Lemma 3.2 *Let K be a R_θ-invariant body in $\mathbb{R}^{2n}$, then $\lambda_{2n-2}(K \cap (H_\xi + (r_1\xi + r_2\xi^\dagger)))$ is constant for all $\sqrt{r_1^2 + r_2^2} = t$.*

Proof Since $\widetilde{R_\theta} H_\xi = H_\xi$ and $\widetilde{R_\theta} K = K$ for $\theta \in [0, 2\pi]$, we obtain

$$\widetilde{R_\theta}\bigl[K \cap \bigl(H_\xi + \bigl(r_1\xi + r_2\xi^\dagger\bigr)\bigr)\bigr] = K \cap \bigl(H_\xi + \bigl(r_1\widetilde{R_\theta}\xi + r_2\widetilde{R_\theta}\xi^\dagger\bigr)\bigr). \tag{3.1}$$

Obviously, there exists a map $\widetilde{R_\theta}$ such that

$$r_1\xi + r_2\xi^\dagger = t\widetilde{R_\theta}\xi$$

for any $r_1, r_2 > 0$ satisfies $\sqrt{r_1^2 + r_2^2} = t$.

Together with (3.1), we obtain

$$\begin{aligned}
&V\bigl(K \cap \bigl(H_\xi + \bigl(r_1\xi + r_2\xi^\dagger\bigr)\bigr)\bigr) \\
&\quad = V\bigl(K \cap (H_\xi + t\widetilde{R_\theta}\xi)\bigr) \\
&\quad = V\bigl(\widetilde{R_\theta}\bigl(K \cap (H_\xi + t\xi)\bigr)\bigr) \\
&\quad = V\bigl(K \cap (H_\xi + t\xi)\bigr)
\end{aligned}$$

for any $r_1, r_2 > 0$ satisfies $\sqrt{r_1^2 + r_2^2} = t$. □

Lemma 3.3 *Let K be a R_θ-invariant complex isotropic body in $\mathbb{R}^{2n}$ with $\lambda_{2n}(K) = 1$, then $\lambda_{2n}(K) = 2\pi\int_0^\infty tf(t)\,dt$ and $\mathbb{L}_K = 2\pi\int_0^\infty t^3 f(t)\,dt$, where $f(t) = \lambda_{2n-2}(K \cap (H_\xi + t\xi))$.*

Proof From (2.4) and Lemma 3.2, we have

$$\begin{aligned}
\mathbb{L}_K^2 &= \int_K \bigl(\langle x, \xi\rangle^2 + \bigl\langle x, \xi^\dagger\bigr\rangle^2\bigr)\,dx \\
&= 2\pi\int_0^\infty t^3\lambda_{2n-2}\bigl(K \cap (H_\xi + t\xi)\bigr)\,dt = 2\pi\int_0^\infty t^3 f(t)\,dt.
\end{aligned}$$

A similar argument shows $\lambda_{2n}(K) = 2\pi\int_0^\infty tf(t)\,dt$. □

The following lemma is given by Milman and Pajor in [27, Lemma 2.1].

Lemma 3.4 *Let $\varphi : \mathbb{R}^n \to \mathbb{R}_+$ be a measurable function such that $\|\varphi\|_\infty = 1$ and let L be a symmetric convex body in $\mathbb{R}^n$. Then the function*

$$F(p) = \left(\int_{\mathbb{R}^n} \|x\|_L^p \varphi(x)\,dx \Big/ \int_L \|x\|_L^p\,dx\right)^{1/(n+p)}$$

is an increasing function of p on $(-n, +\infty)$.

Theorem 3.5 *Let K be a R_θ-invariant complex isotropic convex body in $\mathbb{R}^{2n}$ with $\lambda_{2n}(K) = 1$, then*

$$\mathbb{L}_K \geq \frac{1}{\sqrt{2\pi}}\frac{1}{\lambda_{2n-2}(K \cap H_\xi)^{1/2}}. \tag{3.2}$$

Proof Let $f(t) = \lambda_{2n-2}(K \cap (H_\xi + t\xi))$. From Lemma 3.2, it follows that $f(t)$ is an even function. Note that K is R_θ-invariant implies K is origin symmetric. From Lemma 3.1, we have $\|f(t)\|_\infty = f(0)$. Taking $\varphi(t) := f(t)/f(0)$ and $L := [-1,1] \subset \mathbb{R}$ in Lemma 3.4, combining with Lemma 3.3, we get

$$F(3) = \left(\frac{\int_{\mathbb{R}} |t|^3 f(t)/f(0)\,dt}{\int_{-1}^{1} |t|^3\,dt}\right)^{1/4} = \left(\frac{2\mathbb{L}_K^2}{\pi \lambda_{2n-2}(K \cap H_\xi)}\right)^{1/4}$$

and

$$F(1) = \left(\frac{\int_{\mathbb{R}} |t| f(t)/f(0)\,dt}{\int_{-1}^{1} |t|\,dt}\right)^{1/2} = \left(\frac{1}{\pi \lambda_{2n-2}(K \cap H_\xi)}\right)^{1/2}.$$

From the comparison $F(3) \geq F(1)$, we have

$$\mathbb{L}_K^2 \geq \frac{1}{2\pi \lambda_{2n-2}(K \cap H_\xi)}. \tag{3.3}$$

□

Remark If

$$f(t) = \begin{cases} f(0), & \text{if } -a \leq t \leq a \text{ for some } a > 0, \\ 0, & \text{otherwise,} \end{cases}$$

we have equality in (3.2).

The following lemma is due to Marshall *et al.* [30]. A simple proof is given in [27, Lemma 2.6].

Lemma 3.6 *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function and let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\Phi(0) = 0$ and such that Φ and $\Phi(t)/t$ are increasing. Then*

$$G(p) = \left(\frac{\int_0^\infty h(\Phi(t)) t^p\,dt}{\int_0^\infty h(t) t^p\,dt}\right)^{1/(p+1)}$$

is a decreasing function of p on $(-1, \infty)$ (provided the integrals in $G(p)$ are well defined).

Theorem 3.7 *Let K be a R_θ-invariant complex isotropic convex body in $\mathbb{R}^{2n}$ with $\lambda_{2n}(K) = 1$, then*

$$\mathbb{L}_K \leq \sqrt{\frac{3}{\pi}} \frac{1}{\lambda_{2n-2}(K \cap H_\xi)^{1/2}}.$$

Proof Let $f(t) = \lambda_{2n-2}(K \cap (H_\xi + t\xi))$, $t \geq 0$. Let $h(t) = (1-t)^{2n-2}$ for $0 \leq t \leq 1$, $h(t) = 0$ for $t > 1$ and set $\Phi(t) = 1 - (f(t)/f(0))^{1/(2n-2)}$. Note that K is R_θ-invariant implies K is origin symmetric. From Lemma 3.1, $\Phi(t)$ is convex and all hypotheses of Lemma 3.6 are satisfied, so by Lemma 3.3 we get

$$G(3) = \left(\frac{\int_0^\infty t^3 f(t)/f(0)\,dt}{\int_0^1 t^3 (1-t)^{2n-2}\,dt}\right)^{1/4} = \left(\frac{n(n+1)(2n+1)(2n-1)\mathbb{L}_K^2}{3\pi \lambda_{2n-2}(K \cap H_\xi)}\right)^{1/4}$$

and

$$G(1) = \left(\frac{\int_0^\infty tf(t)/f(0)\,dt}{\int_0^1 t(1-t)^{2n-2}\,dt} \right)^{1/2} = \left(\frac{n(2n-1)}{\pi \lambda_{2n-2}(K \cap H_\xi)} \right)^{1/2}.$$

From the comparison $G(3) \le G(1)$, we have

$$\mathbb{L}_K^2 \le \frac{3n(2n-1)}{\pi(n+1)(2n+1)\lambda_{2n-2}(K \cap H_\xi)}.$$

Note that

$$\frac{n(2n-1)}{(n+1)(2n+1)} \tag{3.4}$$

is increasing when the positive integer n is increasing. Thus, the maximum of (3.4) occurs as n tends to infinity. □

4 Extremum of the complex isotropic constant of $B_p(\mathbb{C}^n)$

The following lemma is proved in the spirit of the real counterpart (see, *e.g.*, [29, p.32]).

Lemma 4.1 *Let* $0 < p \le \infty$, *then*

$$\lambda_{2n}\big(B_p(\mathbb{C}^n)\big) = \frac{\pi^n(\Gamma(1+\frac{2}{p}))^n}{\Gamma(1+\frac{2n}{p})}.$$

Proof Obviously, $\lambda_{2n}(B_\infty(\mathbb{C}^n)) = \pi^n$. We only need to consider the case that $0 < p < \infty$. Note that we identify $\ell_p(\mathbb{C}^n)$ with the real $2n$-dimensional space equipped with the norm

$$\|x\|_p = \big[\big(x_{11}^2 + x_{12}^2\big)^{p/2} + \cdots + \big(x_{n1}^2 + x_{n2}^2\big)^{p/2}\big]^{1/p}.$$

Then

$$\begin{aligned} \int_{\mathbb{R}^{2n}} e^{-\|x\|_p^p}\,d\lambda_{2n}(x) &= \prod_{i=1}^n \left(\int_{\mathbb{R}^2} e^{-(x_{i1}^2 + x_{i2}^2)^{p/2}}\,dx_{i1}\,dx_{i2} \right) \\ &= \left(\pi \Gamma\left(1 + \frac{2}{p}\right) \right)^n. \end{aligned}$$

On the other hand, we compute the same integral in polar coordinates and the polar formula for the volume:

$$\begin{aligned} \int_{\mathbb{R}^{2n}} e^{-\|x\|_p^p}\,d\lambda_{2n}(x) &= \int_{S^{2n-1}} \int_0^\infty e^{-r^p\|\theta\|_p^p} r^{2n-1}\,dr\,d\theta \\ &= \frac{\Gamma(2n/p)}{p} \int_{S^{2n-1}} \|\theta\|_p^{-2n}\,d\theta \\ &= \Gamma\left(1 + \frac{2n}{p}\right)\lambda_{2n}\big(B_p\big(\mathbb{C}^n\big)\big). \end{aligned}$$

Comparing these two expressions for the same integral, we get the result. □

Theorem 4.2 *Let $1 \le p \le \infty$, then $c_{n,p}B_p(\mathbb{C}^n)$ is a complex isotropic convex body in $\mathbb{R}^{2n}$. Furthermore, its complex isotropic constant is*

$$\mathbb{L}^2_{B_p(\mathbb{C}^n)} = \frac{\Gamma(1+\frac{2n}{p})^{1+1/n}\Gamma(1+\frac{4}{p})}{2\pi\Gamma(1+\frac{2n+2}{p})\Gamma(1+\frac{2}{p})^2}.$$

Proof Assume that $1 \le p < \infty$. It is easy to verify that (2.7) and (2.8) are true for $B_p(\mathbb{C}^n)$. Now, we only need to verify (2.9) for $B_p(\mathbb{C}^n)$. Actually, we can explicitly calculate $\mathbb{L}_{B_p(\mathbb{C}^n)}$ by using Lemma 4.1, to find that

$$\begin{aligned}\mathbb{L}^2_{B_p(\mathbb{C}^n)} &= \frac{1}{\lambda_{2n}(B_p(\mathbb{C}^n))^{1+\frac{1}{n}}}\int_{B_p(\mathbb{C}^n)}\left(x_{11}^2+x_{12}^2\right)dx\\ &= \frac{2\pi}{\lambda_{2n}(B_p(\mathbb{C}^n))^{1+\frac{1}{n}}}\int_0^1 r^3\left(1-r^p\right)^{\frac{2n-2}{p}}\lambda_{2n-2}\left(B_p\left(\mathbb{C}^{n-1}\right)\right)dr\\ &= \frac{\Gamma(1+\frac{2n}{p})^{1+1/n}\Gamma(1+\frac{4}{p})}{2\pi\Gamma(1+\frac{2n+2}{p})\Gamma(1+\frac{2}{p})^2}.\end{aligned}$$

Thus, $c_{n,p}B_p(\mathbb{C}^n)$ $(1 \le p < \infty)$ is complex isotropic.

Similarly, $\pi^{-1/2}B_\infty(\mathbb{C}^n)$ is complex isotropic and $\mathbb{L}^2_{B_\infty(\mathbb{C}^n)} = 1/2\pi$. □

Remark From Theorem 2.1(iii), $c_{n,p}B_p(\mathbb{C}^n)$ is in fact isotropic since it is R_θ-invariant. However, the complex isotropic constant of $B_p(\mathbb{C}^n)$ is more useful as the argument in Section 3.

Next, we determine the extreme value of $\mathbb{L}_{B_p(\mathbb{C}^n)}$ for $p \ge 1$. The approach that we adopted is similar to the real case due to Ma and the third named author [7].

Lemma 4.3 *Let $p > 0$. Then, for each given positive integer n,*

$$F(p) = \frac{\Gamma(1+\frac{2n}{p})^{1+1/n}\Gamma(1+\frac{4}{p})}{\Gamma(1+\frac{2n+2}{p})\Gamma(1+\frac{2}{p})^2}$$

is a decreasing function for $0 < p < 2$ and an increasing function for $p \ge 2$.

Proof Making the change of variables $q = 2/p$, we have

$$G(q) = F\left(\frac{2}{q}\right) = \frac{\Gamma(1+nq)^{1+1/n}\Gamma(1+2q)}{\Gamma(1+(n+1)q)\Gamma(1+q)^2}. \tag{4.1}$$

It follows that

$$\frac{d\ln G(q)}{dq} = 2\left(\psi(1+2q)-\psi(1+q)\right)-(n+1)\left(\psi\left(1+(n+1)q\right)-\psi(1+nq)\right),$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. Now,

$$\begin{aligned}\psi\left(1+(n+1)q\right)-\psi(1+nq) &= \int_0^\infty \frac{1}{z}\left(\frac{1}{(1+z)^{1+nq}}-\frac{1}{(1+z)^{1+(n+1)q}}\right)dz\\ &= -\frac{1}{(n+1)q}\int_1^\infty \frac{t-1}{t^{\frac{1}{q}}-1}\,d\frac{1}{t^{n+1}},\end{aligned} \tag{4.2}$$

where we use the following integral representation for the function ψ:

$$\psi(x) = \int_0^\infty \frac{1}{z}\left(e^{-z} - \frac{1}{(1+z)^x}\right) dz,$$

and a change of variable $t = (1+z)^q$.

Let $n = 1$ in (4.2), then

$$\psi(1+2q) - \psi(1+q) = -\frac{1}{2q}\int_1^\infty \frac{t-1}{t^{\frac{1}{q}}-1}\, d\frac{1}{t^2}.$$

Thus,

$$\frac{d\ln G(q)}{dq} = \frac{1}{q}\int_1^\infty \frac{t-1}{t^{\frac{1}{q}}-1}\, d\left(\frac{1}{t^{n+1}} - \frac{1}{t^2}\right).$$

Then it follows that

$$\left.\frac{d\ln G(q)}{dq}\right|_{q=1} = \int_1^\infty d\left(\frac{1}{t^{n+1}} - \frac{1}{t^2}\right) = 0$$

and, if $0 < q < 1$,

$$\begin{aligned}\frac{d\ln G(q)}{dq} &= \frac{1}{q}\int_1^\infty \frac{t-1}{t^{\frac{1}{q}}-1}\, d\left(\frac{1}{t^{n+1}} - \frac{1}{t^2}\right)\\ &= \int_1^\infty \frac{(1-q)t^{\frac{1}{q}} + qt^0 - t^{\frac{1}{q}-1}}{q^2(t^{\frac{1}{q}}-1)^2}\left(\frac{1}{t^{n+1}} - \frac{1}{t^2}\right) dt\\ &< 0,\end{aligned}$$

where we use the arithmetic-geometric mean inequality, *i.e.*, $(1-\lambda)x + \lambda y \ge x^{1-\lambda}y^\lambda$.

Similarly, if $q > 1$, we have

$$\begin{aligned}\frac{d\ln G(q)}{dq} &= \frac{1}{q}\int_1^\infty \frac{t-1}{t^{\frac{1}{q}}-1}\, d\left(\frac{1}{t^{n+1}} - \frac{1}{t^2}\right)\\ &= \int_1^\infty \frac{(1-\frac{1}{q})t^{\frac{1}{q}} + \frac{1}{q}t^{\frac{1}{q}-1} - 1}{q(t^{\frac{1}{q}}-1)^2}\left(\frac{1}{t^2} - \frac{1}{t^{n+1}}\right) dt\\ &> 0.\end{aligned}$$

The result follows from (4.1). □

The following lemma can be simply deduced as in [23, p.4].

Lemma 4.4 *For every complex isotropic convex body K in $\mathbb{R}^{2n}$,*

$$\mathbb{L}_K \ge \mathbb{L}_{B_2(\mathbb{C}^n)}.$$

Recall that

$$\mathbb{L}^2_{B_p(\mathbb{C}^n)} = \frac{\Gamma(1+\frac{2n}{p})^{1+1/n}\Gamma(1+\frac{4}{p})}{2\pi\Gamma(1+\frac{2n+2}{p})\Gamma(1+\frac{2}{p})^2} \tag{4.3}$$

for $p \geq 1$. Thus, from Lemma 4.4, we have

$$\mathbb{L}_{B_p(\mathbb{C}^n)} \geq \mathbb{L}_{B_2(\mathbb{C}^n)} = \left(\frac{(n!)^{\frac{1}{n}}}{\pi(n+1)}\right)^{\frac{1}{2}} \geq \frac{1}{\sqrt{\pi e}}. \tag{4.4}$$

From Lemma 4.3 and (4.3), it follows that

$$\mathbb{L}_{B_p(\mathbb{C}^n)} \leq \max\{\mathbb{L}_{B_1(\mathbb{C}^n)}, \mathbb{L}_{B_\infty(\mathbb{C}^n)}\} \tag{4.5}$$

for $p \geq 1$. The following lemma is given by Gao [3].

Lemma 4.5 *Let* $0 \leq x \leq 1$ *and* $y > 0$. *For fixed* x, *the function*

$$f(x,y) = \left(1 + \frac{2}{y}\right) \ln\Gamma(1 + xy) - \ln\Gamma\bigl(1 + (y+2)x\bigr) \tag{4.6}$$

is a decreasing function of y *for* $y > 0$ *when* $0 \leq x \leq 1/2$ *and for* $y \geq 2$ *when* $1/2 < x \leq 1$.

If we set $x = 1/p$ and $y = 2n$ in (4.6) for $p \geq 1$, $n \geq 1$, by (4.3), it follows that

$$\mathbb{L}_{B_p(\mathbb{C}^n)} \leq \mathbb{L}_{B_p(\mathbb{C})}.$$

Combining with (4.5), we have

$$\mathbb{L}_{B_p(\mathbb{C}^n)} \leq \mathbb{L}_{B_1(\mathbb{C})} = \mathbb{L}_{B_\infty(\mathbb{C}^n)} = \frac{1}{\sqrt{2\pi}}. \tag{4.7}$$

Together with (4.4) and (4.7), we have the following theorem.

Theorem 4.6 *Let* $1 \leq p \leq \infty$, *then*

$$\frac{1}{\sqrt{\pi e}} \leq \mathbb{L}_{B_p(\mathbb{C}^n)} \leq \frac{1}{\sqrt{2\pi}}.$$

Now we complete the proof of our main result.

Proof of Theorem 1.2 Combining with Theorem 3.5, Theorem 3.7, and Theorem 4.6, we obtain

$$1 \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\mathbb{L}_{B_p(\mathbb{C}^n)}} \leq \lambda_{2n-2}\bigl(c_{n,p}B_p\bigl(\mathbb{C}^n\bigr) \cap H_\xi\bigr)^{1/2} \leq \sqrt{\frac{3}{\pi}} \frac{1}{\mathbb{L}_{B_p(\mathbb{C}^n)}} \leq \sqrt{3e}.$$

The equality of the left inequality in this theorem holds for $B_\infty(\mathbb{C}^n)$ from the remark after Theorem 3.5 and (4.7). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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References
1. Ball, KM: Cube slicing in $\mathbb{R}^n$. Proc. Am. Math. Soc. **97**, 465-473 (1986)
2. Bastero, J, Galve, F, Peña, A, Romance, M: Inequalities for the gamma function and estimates for the volume of sections of B_p^n. Proc. Am. Math. Soc. **130**, 183-192 (2002)
3. Gao, P: A note on the volume of sections of B_p^n. J. Math. Anal. Appl. **326**, 632-640 (2007)
4. Hensley, D: Slicing the cube in $\mathbb{R}^n$ and probability. Proc. Am. Math. Soc. **73**, 95-100 (1979)
5. Koldobsky, A: An application of the Fourier transform to sections of star bodies. Isr. J. Math. **106**, 157-164 (1998)
6. Meyer, M, Pajor, A: Sections of the unit ball of l_p^n. J. Funct. Anal. **80**, 109-123 (1998)
7. Ma, D, He, B: Estimates for the extremal sections of B_p^n-balls. J. Math. Anal. Appl. **376**, 725-731 (2011)
8. Schmuckenschläger, M: Volume of intersections and sections of the unit ball of ℓ_p^n. Proc. Am. Math. Soc. **126**(5), 1527-1530 (1998)
9. Abardia, J: Difference bodies in complex vector spaces. J. Funct. Anal. **263**, 3588-3603 (2012)
10. Abardia, J, Bernig, A: Projection bodies in complex vector spaces. Adv. Math. **227**, 830-846 (2011)
11. Abardia, J, Gallego, E, Solanes, G: Gauss-Bonnet theorem and Crofton type formulas in complex space forms. Isr. J. Math. **187**, 287-315 (2012)
12. Koldobsky, A: Stability of volume comparison for complex convex bodies. Arch. Math. **97**, 91-98 (2011)
13. Koldobsky, A, König, H: Minimal volume of slabs in the complex cube. Proc. Am. Math. Soc. **140**, 1709-1717 (2012)
14. Koldobsky, A, König, H, Zymonopoulou, M: The complex Busemann-Petty problem on sections of convex bodies. Adv. Math. **218**, 352-367 (2008)
15. Koldobsky, A, Paouris, G, Zymonopoulou, M: Complex intersection bodies. J. Lond. Math. Soc. **88**, 538-562 (2013)
16. Koldobsky, A, Zymonopoulou, M: Extremal sections of complex l_p-balls, $0 < p < 2$. Stud. Math. **159**, 185-194 (2003)
17. Oleszkiewicz, K, Pełczyński, A: Polydisc slicing in $\mathbb{C}^n$. Stud. Math. **142**, 281-294 (2000)
18. Zymonopoulou, M: The complex Busemann-Petty problem for arbitrary measures. Arch. Math. **91**, 436-449 (2008)
19. Zymonopoulou, M: The modified complex Busemann-Petty problem on sections of convex bodies. Positivity **13**, 717-733 (2009)
20. Bourgain, J: On the distribution of polynomials on high dimensional convex sets. In: Lindenstrauss, J, Milman, VD (eds.) Geometric Aspects of Functional Analysis. Lecture Notes in Math., vol. 1469, pp. 127-137 (1991)
21. Bourgain, J: On the isotropy constant problem for ψ_2-bodies. In: Milman, VD, Schechtman, G (eds.) Geometric Aspects of Functional Analysis. Lecture Notes in Math., vol. 1807, pp. 114-121 (2003)
22. Bourgain, J, Klartag, B, Milman, V: Symmetrization and isotropic constants of convex bodies. In: Geometric Aspects of Functional Analysis. Lecture Notes in Math., vol. 1850, pp. 101-115. Springer, Berlin (2004)
23. Giannopoulos, AA: Notes on isotropic convex bodies. Lecture notes, Warsaw (2003). http://users.uoa.gr/~apgiannop/
24. He, B, Leng, G: Isotropic bodies and Bourgain problem. Sci. China Ser. A **48**(5), 666-679 (2005)
25. Klartag, B, Milman, E: Centroid bodies and the logarithmic Laplace transform - a unified approach. J. Funct. Anal. **262**, 10-34 (2012)
26. König, H, Meyer, M, Pajor, A: The isotropy constants of the Schatten classes are bounded. Math. Ann. **312**, 773-783 (1998)
27. Milman, VD, Pajor, A: Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. In: Lindenstrauss, J, Milman, VD (eds.) Geometric Aspects of Functional Analysis. Lecture Notes in Math., vol. 1376, pp. 64-104 (1989)
28. Paouris, G: On the isotropic constant of non-symmetric convex bodies. In: Geometric Aspects of Functional Analysis. Lecture Notes in Math., vol. 1745, pp. 239-243. Springer, Berlin (2000)
29. Koldobsky, A: Fourier Analysis in Convex Geometry. Math. Surveys Monogr., vol. 116. Am. Math. Soc., Providence (2005)
30. Marshall, AW, Olkin, I, Proschan, F: Monotonicity of ratios of means and other applications of majorization. In: Shisha, O (ed.) Inequalities, pp. 177-190. Academic Press, New York (1967)

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